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Research Article

On the Generalized Weighted Lebesgue Spaces of Locally Compact Groups

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Let G be a locally compact group with a fixed left Haar measure λ and Ω be a system of weights on G. In this paper, we deal with locally convex space $L^p(G,\Omega)$ equipped with the locally convex topology generated by the family of norms $(\|.\|_{p,\omega})_{\omega\in\Omega}$. We study various algebraic and topological properties of the locally convex space $L^p(G,\Omega)$. In particular, we characterize its dual space and show that it is a semireflexive space. Finally, we give some conditions under which $L^p(G,\Omega)$ with the convolution multiplication is a topological algebra and then characterize its closed ideals and its spectrum.

1. Introduction

Throughout this paper, let G denote a locally compact Hausdorff group with a fixed left Haar measure λ. By a weight function on G, we mean an arbitrary strictly positive measurable function on G, and, by a system of weights on G, a set of weight functions Ω such that given ω_1, ω_2 in Ω and $c > 0$, there is an $\nu \in \Omega$ such that $c\omega_i(x) \leq \nu(x)$ $(i = 1, 2)$ for locally almost all $x \in G$.

For a weight function ω and $1 \leq p < \infty$, let $L^p(G,\omega)$ denote the space of all complex-valued measurable functions f on G such that $f\omega \in L^p(G)$, the usual Lebesgue space on G with respect to λ; see [1] for more details. Then, $L^p(G,\omega)$ with the norm $\|\cdot\|_{p,\omega}$ defined by $\|f\|_{p,\omega} := \|f\omega\|_p$ for all $f \in L^p(G,\omega)$ is a Banach space. We also denote by $L^\infty(G,1/\omega)$ the space of all measurable complex-valued functions f on G such that $f/\omega \in L^\infty(G)$, the space defined in [1]. Then, $L^\infty(G,1/\omega)$ with the norm $\|\cdot\|_{\infty,\omega}$ defined by $\|f\|_{\infty,\omega} := \|f/\omega\|_\infty$ for all $f \in L^\infty(G,1/\omega)$ is a Banach space. Furthermore, for $1 \leq p < \infty$, the topological dual of $L^p(G,\omega)$ coincides with $L^q(G,1/\omega)$, where q is the exponential conjugate to p defined by $1/p + 1/q = 1$. In fact, the mapping T from $L^q(G, 1/\omega)$ to $L^p(G, \omega)$ defined by

$$\langle T(f), g\rangle = \int_G f(x)g(x)d\lambda(x) \tag{1.1}$$

is an isometric isomorphism; see for example [2]. For measurable functions f and g on G, the convolution multiplication

$$(f * g)(x) = \int_G f(y)g\left(y^{-1}x\right)d\lambda(y) \tag{1.2}$$

is defined at each point $x \in G$ for which this makes sense. The algebraic and topological properties of weighted L^p-spaces have been studied extensively; see for example [2–5].

Let $1 \leq p < \infty$ and Ω be a system of weights on G, we set

$$L^p(G, \Omega) = \bigcap_{\omega\in\Omega} L^p(G, \omega). \tag{1.3}$$

In this paper, we equip the space $L^p(G, \Omega)$ with the natural locally convex topology generated by the family of norms $\|\cdot\|_{p,\omega}$, where ω runs through Ω. For a similar study in other contexts, see [6–8]. We investigate certain algebraic and topological properties of the locally convex space $L^p(G, \Omega)$. Our results generalize and improve some interesting results of [5] and partially answer a question raised in [3].

2. Preliminaries and Some Basic Results

Let G be a locally compact Hausdorff group with a fixed left Haar measure λ and Ω be a system of weights on G. We equip $L^p(G, \Omega)$ with the locally convex topology generated by the family of norms $(\|\cdot\|_{p,\omega})_{\omega\in\Omega}$ and denote this topology by τ_Ω. So $(L^p(G, \Omega), \tau_\Omega)$ has a basis of closed absolutely convex neighbourhoods at the origin of the form

$$V_{p,\omega} = \left\{f \in L^p(G, \Omega) : \|f\|_{p,\omega} \leq 1\right\}, \quad (\omega \in \Omega). \tag{2.1}$$

Note that the topology τ_Ω is Hausdorff, because if $f \in L^p(G, \Omega)$ and $f \neq 0$, we have $\lambda(\{x \in G : f(x) \neq 0\}) > 0$. Put $E = \{x \in G : f(x) \neq 0\}$ and fix $\omega \in \Omega$. Then,

$$\|f\|_{p,\omega} = \left(\int_G (|f|\omega)^p d\lambda\right)^{1/p} \geq \left(\int_E (|f|\omega)^p d\lambda\right)^{1/p} > 0, \tag{2.2}$$

and thus τ_Ω is Hausdorff.

If Ω and Γ are two systems of weights on G and for every $\omega \in \Omega$, there is a $\nu \in \Gamma$ such that $\omega \leq \nu$ (pointwise locally almost everywhere on G), then we write $\Omega \leq \Gamma$. In the case which $\Gamma \leq \Omega$ and $\Omega \leq \Gamma$, we write $\Omega \sim \Gamma$.

Proposition 2.1. *Let Ω and Γ be two systems of weights on G and $T : G \to G$ be a measurable mapping such that $\Omega \leq \Gamma \circ T := \{\nu \circ T : \nu \in \Gamma\}$. If the Radon-Nikodym function $h = d(\lambda \circ T^{-1})/d\lambda$ belongs to $L^\infty(G)$, then the mapping $f \mapsto f \circ T$ is a continuous linear map from $(L^p(G,\Gamma), \tau_\Gamma)$ into $(L^p(G,\Omega), \tau_\Omega)$.*

Proof. Given $f \in L^p(G,\Gamma)$ and $\omega \in \Omega$, choose $\nu \in \Gamma$ such that $\omega \leq \nu \circ T$. Then we have

$$\begin{aligned}\|f \circ T\|_{p,\omega} &= \left(\int_G \left(|f \circ T(x)|\omega(x)\right)^p d\lambda(x)\right)^{1/p} \leq \left(\int_G \left(|f \circ T(x)|(\nu \circ T)(x)\right)^p d\lambda(x)\right)^{1/p} \\ &= \left(\int_G \left(|f(x)|\nu(x)\right)^p d\left(\lambda \circ T^{-1}\right)(x)\right)^{1/p} \leq \left(\int_G \left(|f(x)|\nu(x)\right)^p h(x) d\lambda(x)\right)^{1/p} \\ &\leq \|h\|_\infty \|f\|_{p,\nu} < \infty.\end{aligned} \tag{2.3}$$

Hence, $\omega(f \circ T) \in L^p(G)$. Since $\omega \in \Omega$ was arbitrary, $f \circ T \in L^p(G,\Omega)$. Continuity also follows from the above relations. □

The space of all bounded Borel measurable functions on G with compact support will be denoted by $B_c(G)$. Let us remark that if $B_c(G) \subseteq L^p(G,\Omega)$, then $B_c(G)$ is norm dense in $L^p(G,\omega)$ for any weight ω on G; see for example [9].

Corollary 2.2. *Let Ω and Γ be two systems of weights on G. Then,*

(i) *If $\Omega \leq \Gamma$, then the induced topology τ_Ω on $L^p(G,\Gamma)$ is weaker than τ_Γ.*

(ii) *If $B_c(G) \subseteq L^p(G,\Gamma) \subseteq L^p(G,\Omega)$ and $\tau_\Omega \subseteq \tau_\Gamma$, then $\Omega \leq \Gamma$. In particular, $\Omega \sim \Gamma$ if and only if $L^p(G,\Gamma) = L^p(G,\Omega)$.*

Proof. (i) is trivial. For (ii), we observe that for any $\omega \in \Omega$, there is a $\nu \in \Gamma$ such that $V_{p,\nu} \subseteq V_{p,\omega} \cap L^p(G,\Gamma)$. So the identity map I from $(L^p(G,\Omega), \|\cdot\|_{p,\nu})$ into $(L^p(G,\omega), \|\cdot\|_{p,\omega})$ is continuous. Since $L^p(G,\Gamma)$ is dense in $(L^p(G,\nu), \|\cdot\|_{p,\nu})$, I can be extended continuously to a continuous linear mapping on $L^p(G,\nu)$. The extension map is again the identity map. So $L^p(G,\nu) \subseteq L^p(G,\omega)$. Hence, there exists a constant $c > 0$ such that $\omega(x) \leq c\,\nu(x)$ locally almost everywhere; see Lemma 2.1 in [10]. This proves that $\Omega \leq \Gamma$. □

Let us recall the definition of the projective limit of a family of locally convex spaces. Let $(\Lambda, \leq)$ be a partially ordered set and $\{X_\alpha : \alpha \in \Lambda\}$ be a family of locally convex spaces, and for $\alpha \leq \beta$, $f_{\alpha,\beta}$ be a linear map from X_β into X_α. Suppose that $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$ for all $\alpha \leq \beta \leq \gamma$ and $f_{\alpha\alpha}$ be the identity map on X_α for all $\alpha \in \Lambda$. Then, the projective limit of the family $(X_\alpha, f_{\alpha,\beta})$ is defined as

$$\lim_{\alpha}(X_\alpha, f_{\alpha,\beta}) = \left\{(x_\alpha) \in \prod_{\alpha\in\Lambda} X_\alpha \ : \ x_\alpha = f_{\alpha,\beta}(x_\beta), \ \text{whenever } \alpha \leq \beta\right\}; \tag{2.4}$$

for more details see for example [11].

Proposition 2.3. *Let Ω be a system of weights on G. Then $(L^p(G,\Omega), \tau_\Omega)$ is a complete space.*

Proof. We note that for any two weights $\omega, \nu \in \Omega$ with $\omega \leq \nu$, $L^p(G,\nu) \subseteq L^p(G,\omega)$. Let the mapping $I_{\omega,\nu} : L^p(G,\nu) \to L^p(G,\omega)$ be the canonical injection. Then, it is clear that $(L^p(G,\Omega), \tau_\Omega)$ is isomorphic to the projective limit system $\lim_\omega (L^p(G,\omega), I_{\omega,\nu})$ of the Banach spaces $(L^p(G,\omega), \|\cdot\|_{p,\omega})$, $\omega \in \Omega$, and, hence, is complete; see Lemma 3.2.1 in [12]. □

Proposition 2.4. *The locally convex space $(L^p(G,\Omega), \tau_\Omega)$ is normable if and only if the topology τ_Ω is generated by $\|\cdot\|_{p,\omega}$ for some $\omega \in \Omega$.*

Proof. If $L^p(G,\Omega)$ is normable, then it has a neighbourhood V of zero that is norm bounded with respect to $\|\cdot\|_{p,\omega}$ for every $\omega \in \Omega$. Hence, there is $\omega' \in \Omega$ so that $V_{p,\omega'} = \{f \in L^p(G,\Omega) : \|f\|_{p,\omega'} \leq 1\}$ is norm bounded in the space $(L^p(G,\nu), \|\cdot\|_{p,\nu})$ for every $\nu \in \Omega$. This implies that there is a positive constant c_ν so that $V_{p,\omega'} \subseteq c_\nu V_{p,\nu}$, and our claim is proved. The converse is clear. □

3. The Dual and Bidual of $L^p(G,\Omega), 1 \leq p < \infty$

In this section we deal with the dual space of $(L^p(G,\Omega), \tau_\Omega)$ and, among other things, characterize its equicontinuous subsets.

Theorem 3.1. *If $1 \leq p < \infty$ and $B_c(G) \subseteq L^p(G,\Omega)$, then the dual space of $(L^p(G,\Omega), \tau_\Omega)$ is $\Omega \cdot L^q(G) := \{\omega f : \omega \in \Omega, f \in L^q(G)\}$ with $1/p + 1/q = 1$.*

Proof. Let $h \in L^q(G, 1/\omega)$. We define the linear functional $F : L^p(G,\Omega) \to \mathbb{C}$ by $F(f) = \int_G fh\, d\lambda$, then $F \in (L^p(G,\Omega), \tau_\Omega)^*$.

Conversely, let $F \in (L^p(G,\Omega), \tau_\Omega)^*$. First, we know that $B_c(G) \subseteq L^p(G,\Omega) \subseteq L^p(G,\omega)$ for every $\omega \in \Omega$. So there is a $\nu \in \Omega$ such that $|F(f)| \leq 1$ whenever $f \in \{h \in L^p(G,\Omega) : \|h\|_{p,\nu} \leq 1\}$. As F is bounded in the intersection of the unit ball of $(L^p(G,\nu), \|\cdot\|_{p,\nu})$ with $(L^p(G,\Omega), \|\cdot\|_{p,\nu})$, F is continuous on $L^p(G,\Omega)$ with the topology induced by the norm $\|\cdot\|_{p,\nu}$. Since $L^p(G,\Omega)$ is dense in $(L^p(G,\nu), \|\cdot\|_{p,\nu})$, F can be extended continuously to a continuous linear form on $L^p(G,\nu)$ which we denote by $\widetilde{F}$. Then, we have $\widetilde{F} \in (L^p(G,\nu), \|\cdot\|_{p,\nu})^*$, and hence there is a unique $h \in L^q(G, 1/\nu)$ so that

$$\widetilde{F}(f) = \int_G fh\, d\lambda \quad (f \in L^p(G,\nu)); \tag{3.1}$$

therefore, we obtain the following isomorphism:

$$\Phi : \bigcup_{\omega \in \Omega} L^q\left(G, \frac{1}{\omega}\right) \longrightarrow (L^p(G,\Omega), \tau_\Omega)^*, \tag{3.2}$$

defined by $\Phi(h) = F_h$, where $F_h(f) = \int_G fh\, d\lambda$ for all $f \in L^p(G,\Omega)$. □

Lemma 3.2. *Let Ω be a system of weights on G. For every $\omega \in \Omega$, define the mapping $T_\omega : L^p(G,\Omega) \to L^p(G)$ by $T_\omega(f) = f\omega$. Then, $V^\circ_{p,\omega} = T^*_\omega(B^\circ)$ for $\omega \in \Omega$, where B is the closed unit ball of $L^p(G)$ and B° denotes its polar.*

Proof. It is clear that T_ω is a well-defined continuous linear map. Also, $T_\omega(L^p(G,\Omega))$ is dense in $(L^p(G), \|\cdot\|_p)$. Therefore T^*_ω (the adjoint of T_ω) is weak* continuous and one to one linear map from $L^q(G)$ into $\Omega \cdot L^q(G)$, where $1/p + 1/q = 1$. Now, since B° is $\sigma(L^q(G), L^p(G))$-compact by the Alaoglu theorem and so $T_\omega^*(B^\circ)$ is $\sigma(\Omega \cdot L^q(G), L^p(G,\Omega))$-compact, while $T_\omega^*(B^\circ)$ is obviously convex. So we find that

$$\begin{aligned} V_{p,\omega} &= \{ f \in L^p(G,\Omega) : T_\omega(f) \in B \} = T_\omega^{-1}(B) = \{ f \in L^p(G,\Omega) : T_\omega(f) \in B^{\circ\circ} \} \\ &= \{ f \in L^p(G,\Omega) : |T_\omega^*(g)(f)| \le 1, \quad \text{for every } g \in B^\circ \} = T_\omega^*(B^\circ)^\circ. \end{aligned} \tag{3.3}$$

Form which it follows that

$$V_{p,\omega}^\circ = T_\omega^*(B^\circ)^{\circ\circ} = T_\omega^*(B^\circ). \tag{3.4}$$

□

We have the following characterization of the equicontinuous subsets of $\Omega \cdot L^q(G)$.

Theorem 3.3. *Let $1 \le p < \infty$ and M be a subset of $(L^p(G,\Omega), \tau_\Omega)^* = \Omega \cdot L^q(G)$. The following are equivalent.*

(a) *M is τ_Ω-equicontinuous.*

(b) *There are $\omega \in \Omega$ and an equicontinuous subset M' of $(L^p(G), \|\cdot\|_p)^* = L^q(G)$ so that $M \subseteq \omega \cdot M'$.*

(c) *There are $\omega \in \Omega$ and $\alpha > 0$ such that $\sup\{\|f/\omega\|_q : f \in M\} \le \alpha < \infty$ whenever $1/p + 1/q = 1$.*

Proof. ($a \Rightarrow b$) By (a), there is $\omega \in \Omega$ so that $M \subseteq V_{p,\omega}^\circ$, where $V_{p,\omega} = \{f \in L^p(G,\Omega) : \|f\|_{p,\omega} \le 1\}$. According to Lemma 3.2, we have $V_{p,\omega}^\circ = T_\omega^*(B^\circ)$, where B is the closed unit ball of $L^p(G)$. Hence $M \subseteq \omega B^\circ$.

($b \Rightarrow c$) There is $\alpha > 0$ so that $M' \subseteq \alpha B^\circ$ by (b). So $M \subseteq \alpha\omega B^\circ$, and $\sup_{f\in M}\|f/\omega\|_q \le \alpha$.

($c \Rightarrow a$) If $p = 1$, it is clear that

$$M \subseteq \left\{ f \in L^p(G,\Omega) : \int_G |f(x)|\omega(x)d\lambda(x) \le \frac{1}{\alpha} \right\}^\circ, \tag{3.5}$$

and if $1 < p < \infty$, by Hölder's inequality, for $h \in M$ and

$$f \in W = \left\{ f \in L^p(G,\Omega) : \|f\|_{p,\omega} \le \frac{1}{\alpha} \right\}, \tag{3.6}$$

we have

$$\left| \int_G hf\, d\lambda \right| \le \int_G \left| \frac{h}{\omega} \right| |f\omega| d\lambda \le \left\| \frac{h}{\omega} \right\|_q \|f\omega\|_p \le 1. \tag{3.7}$$

Hence, $M \subseteq W^\circ$, and this guarantees that M is τ_Ω-equicontinuous in both cases. □

Proposition 3.4. *Let Ω be a system of weights on G. Then, the set of extreme points of $V^{\circ}_{p,\omega}$ is the set $\{\omega f : f \in L^q(G), \|f\|_q = 1\}$ for $1 < p < \infty$, and $\{f \in L^{\infty}(G) : |f| = 1$ l.a.e. $\}$ for $p = \infty$.*

Proof. Fix $\omega \in \Omega$ and let $T_{\omega} : L^p(G,\Omega) \to L^p(G)$ be the map defined in Lemma 3.2. From Lemma 3.2, it follows that for any extreme point h of $V^{\circ}_{p,\omega}$, there is an extreme point f of B° so that $h = T^{*}_{\omega}(f) = f\omega$.

Conversely, let $\omega \in \Omega$ be arbitrary and let $h = \omega f$, where f is an extreme point of B°. Clearly, $h \in V^{\circ}_{p,\omega}$, and if $h = cg + (1-c)k$ for some $g, k \in V^{\circ}_{p,\omega}$ and $0 < c < 1$, then there are $m, n \in B^{\circ}$ such that $T^{*}_{\omega}(m) = g$ and $T^{*}_{\omega}(n) = k$. Thus, $T^{*}_{\omega}(f) = h = T^{*}_{\omega}(cm + (1-c)n)$ and since T^{*}_{ω} is one to one, $f = cm + (1-c)n$. However f is an extreme point of B°, which implies that $f = m = n$, and hence $h = g = k$, that is, h is an extreme point of $V^{\circ}_{p,\omega}$. Now the rest of the proof is easy to complete; see for example Section 2.14 in [13]. □

Let us recall that a locally convex space (E,τ) is called semireflexive if $(E,\tau)^{**} = E$.

Theorem 3.5. *Let Ω be a system of weights on G. Then $(L^p(G,\Omega), \tau_{\Omega})$ is semireflexive.*

Proof. If $F \in (L^p(G,\Omega),\tau_{\Omega})^{**}$, then the restriction of F to $L^q(G,1/\omega)$, for every $\omega \in \Omega$, belongs to $L^q(G,1/\omega)^{*}$, where $L^q(G,1/\omega)$ was considered with the induced strong topology on $(L^p(G,\Omega),\tau_{\Omega})^{*}$. Now if $\{h_{\alpha}\}_{\alpha\in I} \subseteq L^q(G,1/\omega)$ and $h_{\alpha} \to h$ for some $h \in L^q(G,1/\omega)$ in the norm $\|\cdot\|_{q,1/\omega}$, then for every weakly bounded set A in $L^p(G,\Omega)$,

$$\int_G f(h_{\alpha} - h)\,d\lambda \longrightarrow 0 \quad \text{uniformly on } A. \tag{3.8}$$

This means that $h_{\alpha} \to h$ in the strong topology of $(L^p(G,\Omega),\tau_{\Omega})^{*}$. Hence, for every $\omega \in \Omega$, there is a unique $f_{\omega} \in L^p(G,\omega)$ so that

$$F(h) = \int_G f_{\omega} h\,d\lambda \quad \text{on } L^q\left(G, \frac{1}{\omega}\right). \tag{3.9}$$

Now note that if $\omega, \nu \in \Omega$ with $\omega \le \nu$, then $L^p(G,\nu) \subseteq L^p(G,\omega)$ and $L^q(G,1/\omega) \subseteq L^q(G,1/\nu)$. Therefore for every $h \in L^q(G,1/\omega)$,

$$\int_G f_{\omega} h\,d\lambda = \int_G f_{\nu} h\,d\lambda, \tag{3.10}$$

and hence $f_{\omega} = f_{\nu}$ almost everywhere. This implies that

$$F \in \lim_{\omega}\left(L^p(G,\omega), I_{\omega,\nu}\right) = L^p(G,\Omega). \tag{3.11}$$

Conversely, if $f \in L^p(G,\Omega)$, then it is obvious that the linear form

$$F(h) = \int_G f h\,d\lambda \quad \left(h \in (L^p(G,\Omega),\tau_{\Omega})^{*}\right) \tag{3.12}$$

is continuous with respect to the strong topology on $(L^p(G,\Omega),\tau_\Omega)^*$. So the canonical imbedding $J : L^p(G,\Omega) \to (L^p(G,\Omega),\tau_\Omega)^{**}$ is onto. Hence $L^p(G,\Omega)$ is semireflexive. □

4. $L^p(G,\Omega)$ As a Topological Algebra

In this section, we study conditions on a system of weights Ω for that $L^p(G,\Omega)$ with the convolution multiplication to be a topological algebra. We commence with some definitions.

If f is a function on G, the left translate of f by $x \in G$ is the function given by $L_x f(y) = f(x^{-1}y)$. A subset $\mathcal{F}$ of functions on G is called left translation invariant if $L_x f \in \mathcal{F}$ for all $f \in \mathcal{F}$ and $x \in G$.

A weight function ω on a locally compact group G is called left moderate if

$$\ell(s) := \operatorname{ess\,sup}_t \frac{\omega(st)}{\omega(t)} < \infty, \tag{4.1}$$

for all $s \in G$. It is easy to see that $\ell(s) > 0$, $\ell(st) \le \ell(s)\ell(t)$; see [4] or [9]. Let us remark that any submultiplicative and any locally integrable left moderate measurable function is bounded and bounded away from zero on any compact subset of G; see Theorem 2.7 in [10]. In particular, ℓ is bounded on compact sets. The condition that ω is left moderate is equivalent to that the space $L^p(G,\omega)$ (for $1 \le p \le \infty$) being translation invariant; see for more details [4]. Observe that for $f \in L^p(G,\omega)$ and $x \in G$,

$$\begin{aligned} \|L_x f\|_{p,\omega} &= \left(\int_G \left(\left|f\left(x^{-1}t\right)\right|\omega(t)\right)^p d\lambda(t)\right)^{1/p} \\ &= \left(\int_G \left(|f(t)|\omega(xt)\right)^p d\lambda(t)\right)^{1/p} \\ &\le \left(\int_G \left(|f(t)|\ell(x)\omega(t)\right)^p d\lambda(t)\right)^{1/p} \\ &= \ell(x)\|f\|_{p,\omega}. \end{aligned} \tag{4.2}$$

Lemma 4.1. *Let Ω be a system of weights on G. Then $L^p(G,\Omega)$ is left translation invariant if and only if every element of Ω is left moderate.*

Proof. The "if" part is clear by the remarks above. For the converse, we need only to note that $L^p(G,\Omega)$ is dense in $(L^p(G,\omega),\|\cdot\|_{p,\omega})$ for $\omega \in \Omega$. □

Theorem 4.2. *Let Ω be a system of locally integrable left moderate weights on G and $f \in L^p(G,\Omega)$. Then, the map $x \mapsto L_x f$ from G into $(L^p(G,\Omega),\tau_\Omega)$ is continuous.*

Proof. Assume first that $f \in B_c(G)$ with $K = \operatorname{supp}(f)$. Let $x \in G$, $\omega \in \Omega$, and (x_α) be a net in G convergent to x. Choose a compact neighbourhood U of x, then $\operatorname{supp}(L_x f) \subseteq UK$ whenever $x \in U$. Let

$$k = \sup\{\omega(s) : s \in UK\} < \infty. \tag{4.3}$$

Choose α_0 such that $x_\alpha \in U$ for all $\alpha \leq \alpha_0$ and $\|L_{x_\alpha} f - L_x f\|_p \leq \epsilon/k$. Then

$$\begin{aligned}\|L_{x_\alpha} f - L_x f\|_{p,\omega} &= \left(\int_{UF} \left(\left|f\left({x_\alpha}^{-1}t\right) - f\left(x^{-1}t\right)\right|\omega(t)\right)^p d\lambda(t)\right)^{1/p} \\ &\leq k\left(\int_{UF} \left(\left|f\left({x_\alpha}^{-1}t\right) - f\left(x^{-1}t\right)\right|^p d\lambda(t)\right)^{1/p} \\ &= k\left\|L_{x_\alpha} f - L_x f\right\|_p \\ &\leq \epsilon,\end{aligned} \tag{4.4}$$

for all $\alpha \geq \alpha_0$.

Finally, let f be an arbitrary element of $L^p(G,\Omega)$ and $\epsilon > 0$. Let M be an upper bound for the function ℓ on the compact neighbourhood U of x; recall that ℓ is submultiplicative. For every $\omega \in \Omega$, there exists $g_\omega \in B_c(G)$ such that $\|f - g_\omega\|_{p,\omega} \leq \epsilon/3M$. By the first part, we can choose α_0 such that

$$\left\|L_{x_\alpha} g_\omega - L_x g_\omega\right\|_{p,\omega} \leq \frac{\epsilon}{3}, \quad x_\alpha \in U, \tag{4.5}$$

for all $\alpha \geq \alpha_0$. One can conclude that

$$\begin{aligned}\left\|L_{x_\alpha} f - L_x f\right\|_{p,\omega} &\leq \left\|L_{x_\alpha} f - L_{x_\alpha} g_\omega\right\|_{p,\omega} + \left\|L_{x_\alpha} g_\omega - L_x g_\omega\right\|_{p,\omega} + \left\|L_x g_\omega - L_x f\right\|_{p,\omega} \\ &\leq \ell(x_\alpha)\epsilon + \epsilon/3 + \ell(x)\epsilon \\ &\leq \frac{M\epsilon}{3M} + \frac{\epsilon}{3} + \frac{M\epsilon}{3M} = \epsilon,\end{aligned} \tag{4.6}$$

for all $\alpha \geq \alpha_0$. This finishes the proof. □

We now focus on some systems of weights for that $L^p(G,\Omega)$ to be an algebra under usual convolution

$$f * g(t) = \int_G f(s) g\left(s^{-1}t\right) d\lambda(s) \quad \left(f, g \in L^p(G,\Omega)\right), \tag{4.7}$$

whenever this integral makes sense. For $p = 1$, it is well known that $L^1(G,\omega)$ is a convolution algebra if and only if ω is weakly submultiplicative; that is, for all $x, y \in G$,

$$\omega(st) \leq c\omega(s)\omega(t), \tag{4.8}$$

for some $c > 0$.

For any two weight functions ω and ν on G, we set

$$\Phi_{[\omega,\nu]}(x) = \int_G \left(\frac{\omega(x)}{\nu(y)\nu(y^{-1}x)}\right)^q d\lambda(y). \tag{4.9}$$

In the case where $\omega = \nu$, we simply write $\Phi_\omega = \Phi_{[\omega,\omega]}$.

The following lemma is similar to Lemma 2.2 in [9].

Lemma 4.3. *Let $1 < p < \infty$ and Ω be a system of weights on G. If $L^p(G,\Omega)$ is a convolution algebra, then ω^p is locally integrable for each $\omega \in \Omega$.*

The next result gives a sufficient condition for that $L^p(G,\Omega)$ to be a convolution algebra.

Theorem 4.4. *Let $1 < p < \infty$ and Ω be a system of weights on G. If for every $\omega \in \Omega$, there is a $\nu \in \Omega$ such that $\Phi_{[\omega,\nu]} \in L^\infty(G)$, where q is the conjugate exponent to p, then the space $(L^p(G,\Omega), \tau_\Omega)$ is a complete locally convex algebra with continuous multiplication.*

Proof. We must show that

$$\|f * g\|_{p,\omega} \leq \|f\|_{p,\nu}\|g\|_{p,\nu} \tag{4.10}$$

for all $f, g \in L^p(G,\Omega)$. By Lemma 4.3, $B_c(G)$ is dense in $(L^p(G,\Omega), \|\cdot\|_{p,\omega})$, thus for any $\omega \in \Omega$, it suffices to show that

$$\|f * g\|_{p,\omega} \leq \|f\|_{p,\nu}\|g\|_{p,\nu}, \tag{4.11}$$

for all $f, g \in B_c(G)$. For this, let $f, g \in B_c(G)$. Writing

$$f * g(x) = \int_G f(y)g\left(y^{-1}x\right)\frac{\nu(y)\nu(y^{-1}x)}{\nu(y)\nu(y^{-1}x)}d\lambda(y), \tag{4.12}$$

and using Hölder's inequality, we obtain

$$\begin{aligned} &\left|f * g(x)\right| \\ &\quad \leq \left(\int_G (|f(y)|\nu(y))^p(|g(y^{-1}x)|\nu(y^{-1}x))^p d\lambda(y)\right)^{1/p}\left(\int_G \left(\frac{1}{\nu(y)\nu(y^{-1}x)}\right)^q d\lambda(y)\right)^{1/q}. \end{aligned} \tag{4.13}$$

This shows that

$$\begin{aligned}
&\left|\int_G \left(\left|f * g(x)\right|\omega(x)\right)^p d\lambda(x)\right| \\
&\quad \le \int_G \left(\int_G \left(\left|f(y)\right|\nu(y)\right)^p \left(\left|g\left(y^{-1}x\right)\right|\nu\left(y^{-1}x\right)\right)^p d\lambda(y)\right)\Phi_{[\omega,\nu]}(x)^{p/q} d\lambda(x) \\
&\quad \le \|f\|_{p,\nu}^p \|g\|_{p,\nu}^p \|\Phi_{[\omega,\nu]}\|_\infty^{p/q}.
\end{aligned} \tag{4.14}$$

Whence

$$\|f * g\|_{p,\omega} \le \|\Phi_{[\omega,\nu]}\|_\infty^{1/q} \|f\|_{p,\nu} \|g\|_{p,\nu}. \tag{4.15}$$

This completes the proof. □

The following corollary is a direct consequence of Theorem 4.4.

Corollary 4.5. *Let $1 < p < \infty$ and Ω be a system of weights on G such that for every $\omega \in \Omega$, $\Phi_\omega \in L^\infty(G)$. Then $L^p(G, \Omega)$ is a complete locally multiplicative convex algebra.*

The next result provides us with a class of weights ω on the additive group $\mathbb{R}^n$ for which the usual weighted Lebesgue space $L^p(\mathbb{R}^n, \omega)$ becomes a Banach algebra.

Proposition 4.6. *Let $1 < p < \infty$ and n be a natural number. Let $\varpi : \mathbb{R}^n \to (0, +\infty)$ be a function such that*

(i) *If $\|x\| \le \|y\|$, then $\varpi(x) \le \varpi(y)$.*

(ii) *$\varpi^{-1} \in L^1(\mathbb{R}^n)$.*

(iii) *There exists a positive number M such that $\varpi(2x) \le M\varpi(x)$ for all $x \in \mathbb{R}^n$.*

Then $L^p(\mathbb{R}^n, \sqrt[q]{\varpi})$ is a Banach algebra, where q is the conjugate exponent to p.

Proof. For any $x \in \mathbb{R}^n$, let $A_x = \{y \in \mathbb{R}^n : 2\|y\| \ge \|x\|\}$ and observe that

$$\begin{aligned}
\varpi(y) &\ge \varpi\left(\frac{x}{2}\right), \quad \text{if } y \in A_x, \\
\varpi(x-y) &\ge \varpi\left(\frac{x}{2}\right), \quad \text{if } y \in \mathbb{R}^n \setminus A_x.
\end{aligned} \tag{4.16}$$

Hence, for $x \in \mathbb{R}^n$,

$$\begin{aligned}
\Phi_{\sqrt[q]{\varpi}}(x) &= \int_{\mathbb{R}^n} \frac{\varpi(x)}{\varpi(y)\varpi(x-y)} dy \\
&= \int_{\mathbb{R}^n \setminus A_x} \frac{\varpi(x)}{\varpi(y)\varpi(x-y)} dy + \int_{A_x} \frac{\varpi(x)}{\varpi(y)\varpi(x-y)} dy
\end{aligned}$$

$$\le \left(\frac{\varpi(x)}{\varpi(x/2)}\right)\left(\int_{\mathbb{R}^n\setminus A_x}\frac{1}{\varpi(y)}dy+\int_{A_x}\frac{1}{\varpi(x-y)}dy\right)$$
$$\le 2M\|\varpi\|_1. \tag{4.17}$$

Thus, $\Phi_{\sqrt[q]{\varpi}} \in L^\infty(\mathbb{R}^n)$, and now the result follows from Corollary 4.5. □

Example 4.7. Let $1 \le p < \infty$, q be the conjugate exponent to p, and $n \in \mathbb{N}$. Set

$$\omega(x) = \left(a + \|x\|^r\right)^{s/q} b^{(1/q)\ln(c+\|x\|^t)} \quad (x \in \mathbb{R}^n), \tag{4.18}$$

where $sr > n$, $b > 1$ and $a, c, t > 0$. Then $L^p(\mathbb{R}^n, \omega)$ is a Banach algebra.

We are going to prove the converse of Theorem 4.4. For this, we fix some notation. If f, g be two complex-valued functions on G, then $f \otimes g$ denotes the function on $G \times G$ given by $f \otimes g(x, y) = f(x)g(y)$ for all $x, y \in G$. Also for any two sets $\mathcal{F}$ and $\mathcal{K}$ of functions on G we set $\mathcal{F} \otimes \mathcal{K} = \{f \otimes g : f \in \mathcal{F}, g \in \mathcal{K}\}$. For a locally compact group G, note that the cartesian product $G \times G$ is a locally compact group by defining the product $(x, y)(s, t) = (xs, yt)$ for all $x, y, s, t \in G$.

We need the following easy lemma in the sequel.

Lemma 4.8. *Let $1 < p < \infty$ and Ω be a system of weights on G such that $B_c(G) \subseteq L^p(G, \Omega)$. Then $B_c(G) \otimes B_c(G)$ is dense in $(L^p(G \times G, \omega \otimes \omega), \|\cdot\|_{p,\omega\otimes\omega})$.*

Proof. Since $B_c(G)$ is norm dense in $L^p(G, \omega)$, then $B_c(G) \otimes B_c(G)$ is projective tensor norm dense in $L^p(G, \omega)\widehat{\otimes}_\pi L^p(G, \omega)$, where $\widehat{\otimes}$ is the projective tensor product. Hence $L^p(G, \omega) \otimes L^p(G, \omega)$ is π-dense in $L^p(G, \omega)\widehat{\otimes}_\pi L^p(G, \omega)$. On the other hand, it is known that $L^p(G, \omega)\widehat{\otimes}_\pi L^p(G, \omega)$ is isometric with $(L^p(G \times G, \omega \otimes \omega), \|\cdot\|_{p,\omega\otimes\omega})$. In fact, the linear map

$$\varrho : L^p(G, \omega)\widehat{\otimes}_\pi L^p(G, \omega) \longrightarrow L^p(G \times G, \omega \otimes \omega), \qquad \varrho(f \otimes g)(x, y) = f(x)g(y) \tag{4.19}$$

for all $f, g \in L^p(G, \omega)$ and $x, y \in G$, can be extended to a surjective isometry; for more details, see for example [14]. Now we conclude that $B_c(G) \otimes B_c(G)$ is $\|\cdot\|_{p,\omega\otimes\omega}$-dense in $L^p(G \times G, \omega \otimes \omega)$. □

The next theorem is our main result in this section.

Theorem 4.9. *Let $1 < p < \infty$, G be σ-compact, and Ω be a system of weights on G. If the space $(L^p(G, \Omega), \tau_\Omega)$ is an algebra with continuous multiplication, then for every $\omega \in \Omega$ there exists a $\nu \in \Omega$ such that $\Phi_{[\omega,\nu]} \in L^\infty(G)$.*

Proof. Choose an arbitrary $\omega \in \Omega$. Then, by assumption, there exists some $\nu \in \Omega$ such that for every $f, g \in L^p(G, \Omega)$, $\|f * g\|_{p,\omega} \le \|f\|_{p,\nu}\|g\|_{p,\nu}$. Now for every $h \in L^q(G, 1/\omega)$,

$$F(f) = \int_G f(x)h(x)d\lambda(x) \quad (f \in B_c(G)) \tag{4.20}$$

defines a continuous linear functional on $B_c(G)$ with the norm $\|F\| = \|h\|_{q,1/\omega}$. Also for every $f, g \in B_c(G)$, $f * g \in B_c(G)$, and we have

$$\begin{aligned} F(f * g) &= \int_G f * g(x) h(x) d\lambda(x) = \int_G \left(\int_G f(y) g\left(y^{-1}x\right) d\lambda(y) \right) h(x) d\lambda(x) \\ &= \int_G \int_G f(y) g(x) h(yx) d\lambda(x) d\lambda(y) < \infty. \end{aligned} \tag{4.21}$$

Set $F(f \otimes g) = F(f * g) = \int_{G\times G} f(y)g(x)h(yx)\ d\lambda \times \lambda(x,y)$ for $f, g \in B_c(G)$. By Lemma 4.8, F can be extended to a $\|\cdot\|_{p.\nu\times\nu}$-continuous functional on $L^p(G \times G, \nu \otimes \nu)$. Since G is σ-compact, by Exercise 15.14 in [1], it follows that the function $(x, y) \mapsto h(yx)$ belongs to $L^q(G \times G, 1/(\nu \otimes \nu))$. But

$$\begin{aligned} \int_G \int_G \left(\frac{h(yx)}{\nu(x)\nu(y)} \right)^q d\lambda(x) d\lambda(y) &= \int_G \int_G \left(\frac{h(x)}{\nu(y)\nu(y^{-1}x)} \right)^q d\lambda(x) d\lambda(y) \\ &= \int_G \left(\frac{h(x)}{\omega(x)} \right)^q \Phi_{[\omega,\nu]}(x) d\lambda(x) < \infty. \end{aligned} \tag{4.22}$$

Since $(h/\omega)^q \in L^1(G)$ is arbitrary, we conclude that $\Phi_{[\omega,\nu]} \in L^\infty(G)$; see Section 14 in [15] or Theorem 20.15 in [1]. □

As an immediate consequence of Theorem 4.9, we obtain the following corollary that partially answers a question raised in [3].

Corollary 4.10. *Let ω be a weight on σ-compact group G and $1 < p < \infty$. Then $L^p(G, \omega)$ is a convolution algebra if and only if $\Phi_\omega \in L^\infty(G)$.*

5. Ideals and the Spectrum of the Algebra $L^P(G, \Omega)$

We commence this section with the following proposition.

Proposition 5.1. *Let $1 \leq p < \infty$ and let $L^p(G, \Omega)$ be a translation invariant algebra. Then*

(i) $(L^p(G, \Omega), \tau_\Omega)$ *has an approximate identity.*

(ii) $(L^p(G, \Omega), \tau_\Omega)$ *has a bounded approximate identity or an identity if and only if G is discrete.*

Proof. (i) Let U be a fixed relatively compact neighbourhood of the identity element e, and let $\mathcal{U}$ be the family of all neighbourhoods of e contained in U directed by reverse inclusion. Set $e_V := \chi_V/\lambda(V)$, and note that since elements of Ω are locally integrable, $e_V \in L^p(G, \Omega)$. Given

$\epsilon > 0$ and $\omega \in \Omega$, then by Theorem 4.2, there exists a neighbourhood W of the identity such that $\|f - L_t f\|_{p,\omega} < \epsilon$ for $t \in W$. Now, for $V \in \mathcal{U}$ with $V \subseteq W$, and $g \in L^q(G, 1/\omega)$, we have

$$\begin{aligned}
\left|\langle e_V * f - f, g\rangle\right| &= \left|\int_G (e_V * f - f)(x) g(x) d\lambda(x)\right| \\
&\leq \int_G \int_V \frac{\left|f(t^{-1}x) - f(x)\right|}{\lambda(V)} d\lambda(t) \left|g(x)\right| \, d\lambda(x) \leq \frac{1}{\lambda(V)} \int_V \langle \left|L_t f - f\right|, \left|g\right|\rangle \, d\lambda(t) \\
&\leq \sup_{t\in V} \left\|L_t f - f\right\|_{p,\omega} \left\|g\right\|_{q,1/\omega} < \epsilon \left\|g\right\|_{q,1/\omega}.
\end{aligned} \tag{5.1}$$

Hence, $\|e_V * f - f\|_{p,\omega} \leq \epsilon$ for all neighborhoods $V \subseteq W$, from which it follows that $e_V * f \to f$ in τ_Ω-topology.

(ii) Let $(e_\alpha)_\alpha$ be a bounded left approximate identity for $L^p(G, \Omega)$. Fix an $\omega \in \Omega$, then $\|e_\alpha\|_{p,\omega} \leq M$ for some positive number M. Let $f \in L^p(G, \omega)$. Since $L^p(G, \Omega)$ is dense in $L^p(G, \omega)$ with the norm $\|\cdot\|_{p,\omega}$, then given $\epsilon > 0$, there exists $g \in L^p(G, \Omega)$ such that $\|f - g\|_{p,\omega} \leq \epsilon/3(M + 1)$. Choose α_0 such that $\|e_\alpha * g - g\|_{p,\omega} \leq \epsilon/3$ for all $\alpha \geq \alpha_0$. Then it follows that

$$\begin{aligned}
\left\|e_\alpha * f - f\right\|_{p,\omega} &\leq \left\|e_\alpha * f - e_\alpha * g\right\|_{p,\omega} + \left\|e_\alpha * g - g\right\|_{p,\omega} + \left\|f - g\right\|_{p,\omega} \\
&\leq M \frac{\epsilon}{3(M+1)} + \frac{\epsilon}{3} + \frac{\epsilon}{3(M+1)} < \epsilon,
\end{aligned} \tag{5.2}$$

for all $\alpha \geq \alpha_0$. This means that $(L^p(G, \omega), \|\cdot\|_{p,\omega}$ has a bounded left approximate identity. But according to Theorem 4.2 in [9], this is equivalent to that G is discrete. □

The next theorem shows that closed ideals of the algebra $(L^p(G, \Omega), \tau_\Omega)$ are exactly translation invariant subspaces.

Theorem 5.2. *Let $1 \leq p < \infty$ and $L^p(G, \Omega)$ be a translation invariant algebra. Then a closed linear subspace of $L^p(G, \Omega)$ is an ideal in $L^p(G, \Omega)$ if and only if it is two-sided translation invariant.*

Proof. Suppose that I is a τ_Ω-closed two-sided translation invariant subspace of $L^p(G, \Omega)$. We have to show that $g * f \in I$ and $f * g \in I$ for all $f \in I$ and $g \in L^p(G, \Omega)$. Let $h \in L^q(G, 1/\omega)$, for some $\omega \in G$, such that $\int_G f(x) h(x) d\lambda(x) = 0$ for all $f \in I$. Then, for $f \in I$ and any $g \in L^p(G, \Omega)$,

$$\begin{aligned}
\int_G (g * f)(x) h(x) d\lambda(x) &= \int_G h(x) \left(\int_G g(y) f\left(y^{-1}x\right) d\lambda(y)\right) d\lambda(x) \\
&= \int_G g(y) \left(\int_G L_y f(x) h(x) d\lambda(x)\right) d\lambda(y) \\
&= 0.
\end{aligned} \tag{5.3}$$

Since $(L^p(G, \Omega), \tau_\Omega)^* = \Omega \cdot L^q(G)$, the Hahn-Banach theorem implies that $g * f \in I$ for all $f \in I$ and $g \in L^p(G, \Omega)$. Thus I is a left ideal, and using the right translation invariance of I, it is readily seen, in the same way, that I is also a right ideal.

Conversely, let I be a closed ideal of $(L^p(G,\Omega),\tau_\Omega)$, and $x \in G$. Let (e_α) be an approximate identity for $L^p(G,\Omega)$. Then for each $f \in L^p(G,\Omega)$, we have

$$\left\|L_x(e_\alpha) * f - L_x f\right\|_{p,\omega} \leq \ell(x)\left\|e_\alpha * f - f\right\|_{p,\omega} \longrightarrow 0. \tag{5.4}$$

Hence, $L_x(e_\alpha) * f \to L_x f$ in τ_Ω-topology. As I is a τ_Ω–closed left ideal, it follows that $L_x f \in I$; that is, I is left translation invariant. Similarly, it is shown that I is also right translation invariant. □

We denote by $\Delta(L^p(G,\Omega))$ the spectrum of $(L^p(G,\Omega),\tau_\Omega)$ consisting of all τ_Ω-continuous nonzero linear functionals Φ on $\mathrm{L}^p(G,\Omega)$ which are multiplicative; that is,

$$\Phi(f * g) = \Phi(f)\Phi(g) \quad (f,g \in L^p(G,\Omega)). \tag{5.5}$$

We conclude this work with the following result which is a characterization of the spectrum of $(L^p(G,\Omega),\tau_\Omega)$.

Proposition 5.3. *Let Ω be a system of weights on 6-compact group G. Then*

$$\Delta(L^p(G,\Omega)) = \left\{\Phi_\rho : \rho \in L^q\left(G,\frac{1}{\omega}\right),\ \omega \in \Omega\ ,\ \rho(xy) = \rho(x)\rho(y)\right\}, \tag{5.6}$$

where

$$\Phi_\rho(f) = \int_G f(x)\rho(x)d\lambda(x) \quad (f \in L^p(G,\Omega)). \tag{5.7}$$

Proof. Let $\rho \in L^q(G,1/\omega)$ for some $\omega \in \Omega$ such that $\rho(xy) = \rho(x)\rho(y)$ for almost all $x,y \in G$. Then, Φ_ρ is $\|\cdot\|_{p,\omega}$-continuous and so τ_Ω-continuous. Moreover, for $f,g \in L^p(G,\Omega)$,

$$\begin{aligned}\Phi_\rho(f * g) &= \int_G\int_G f(x)g(y)\rho(xy)d\lambda(x)d\lambda(y)\\ &= \int_G\int_G f(x)g(y)\rho(x)\rho(y)d\lambda(x)d\lambda(y)\\ &= \Phi_\rho(f)\Phi_\rho(g).\end{aligned} \tag{5.8}$$

That is, $\Phi_\rho \in \Delta(L^p(G,\Omega))$.

Conversely, let $\Phi \in \Delta(L^p(G,\Omega))$. Then Φ is bounded on a τ_Ω-neighbourhood of zero. Thus Φ is bounded on the set $\{f \in L^p(G,\Omega) : \|f\|_{p,\omega} < 1\} \cap L^p(G,\omega)$ for some $\omega \in \Omega$. Therefore Φ can be extended to an element $\overline{\Phi}$ in $(L^p(G,\omega),\|\cdot\|_{p,\omega})^*$. It follows that there exists a function $\rho \in L^q(G,1/\omega)$ such that

$$\overline{\Phi}(f) = \int_G f\rho\, d\lambda, \tag{5.9}$$

for all $f \in L^p(G,\omega)$. Since for $f, g \in B_c(G)$, $\Phi(f)\Phi(g) = \Phi(f * g)$, we infer that

$$\begin{aligned}\int_{G\times G} f(y)g(x)\rho(y)\rho(x)d\lambda\times\lambda(x,y) &= \int_G\int_G f(y)g(x)\rho(y)\rho(x)d\lambda(y)d\lambda(x)\\ &= \int_G\int_G f(y)g(x)\rho(yx)d\lambda(y)d\lambda(x)\\ &= \int_{G\times G} f(y)g(x)\rho(yx)d\lambda\times\lambda(x,y).\end{aligned} \tag{5.10}$$

By an argument similar to the proof of Theorem 4.9, we deduce that $\rho(xy) = \rho(x)\rho(y)$ for almost all $x, y \in G$. □

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