Research Article

Poincaré Map and Periodic Solutions of First-Order Impulsive Differential Equations on Moebius Stripe

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This paper is mainly concerned with the existence, stability, and bifurcations of periodic solutions of a certain scalar impulsive differential equations on Moebius stripe. Some sufficient conditions are obtained to ensure the existence and stability of one-side periodic orbit and two-side periodic orbit of impulsive differential equations on Moebius stripe by employing displacement functions. Furthermore, double-periodic bifurcation is also studied by using Poincaré map.

1. Introduction

Many systems in physics, chemistry, biology, and information science have impulsive dynamical behavior due to abrupt jumps at certain instants during the evolving processes. This complex dynamical behavior can be modeled by impulsive differential equations. The theory of impulsive differential systems has been developed by numerous mathematicians (see [1–9]). As to the stability theory and boundary value problems to impulsive differential equations, There have been extensive studies in this area. However, there are very few works on the qualitative theory of impulsive differential equations and impulsive semidynamical systems. Recently, Bonotto and Federson have given a version of the Poincaré-Bendixson Theorem for impulsive semidynamical systems in [10, 11]. As it is known, the method of Poincaré map plays an important role in the research of qualitative theory and is a natural means to study the existence of periodic solutions and its asymptotic stability. However, due to the complexity of the associated impulsive dynamic models, this approach has only been applied successfully to Raibert's one-legged-hopper (see [12–14]) predator-prey models (see [15–18]), and so forth. The bifurcation theory for ordinary differential equations or smooth systems appeared during the last decades (see, e.g., [19]); however, little is known about the bifurcation theory of impulsive differential equations due to its complexity (see [20]). In this paper, we mainly study

a certain scalar impulsive differential equations on Moebius stripe undergoing impulsive effects at fixed time:

$$
\begin{aligned}
\frac{dx}{dt} &= f(t, x), \quad t \neq \tau_k, \\
\Delta x|_{t=\tau_k} &= I_k(x), \quad k \in \mathbb{Z}^+,
\end{aligned} \tag{1}
$$

where $0 \leq \tau_k < \tau_{k+1}, k \in \mathbb{Z}^+$ are fixed with $\tau_k \to +\infty$ as, $k \to +\infty$, and $\Delta x|_{t=\tau_k} = x(\tau_k^+) - x(\tau_k)$. Hu and Han (see [20]) investigated the existence of periodic solutions and bifurcations of (1) under the assumptions that $f(t, x)$ and $I_k(x)$ are periodic; that is, the following assumption holds.

(H*) There exist a constant $T > 0$, a positive integer q, and two mutual coprime positive integers m and n such that

$$
f(t + T, x) = f(t, x), \quad \forall t \in \mathbb{R}^+, \; x \in \mathbb{R},
$$

$$
I_{k+q}(x) = I_k(x), \quad \tau_{k+q} - \tau_k = T, \quad \forall k \in \mathbb{Z}^+, \; x \in \mathbb{R}, \tag{2}
$$

$$
m(t_{k+q} - t_k) = nT, \quad k \geq 1.
$$

In this paper, we assume that the following conditions hold.

(H1) Assume that both $f(t, x)$ and $f_x(t, x)$ are continuous scalar functions on $\mathbb{R} \times \mathbb{R}$, $I_k(x) : \mathbb{R} \to \mathbb{R}, k \in \mathbb{Z}$ are odd, continuous functions; that is, $I_k(-x) = -I_k(x), k \in \mathbb{Z}^+$.

(H2) There exists a constant $T > 0$, a positive integer q such that

$$f(t + T, -x) = -f(t, x), \quad \forall t \in \mathbb{R}, \ x \in \mathbb{R},$$

$$I_{k+q}(x) = I_k(x), \quad \tau_{k+q} - \tau_k = T, \quad \forall k \in \mathbb{Z}^+, \ x \in \mathbb{R}. \tag{3}$$

From (H2), we have $f(t, x)$ is $2T$ periodic and $\tau_{k+q} - \tau_k = 2T$. Hence assumption (H*) holds naturally. However, we show some new and fruitful results of system (1) with the condition (H1)-(H2). For example, we obtain the existence and stability of $2T$ periodic solutions to system (1) by double-periodic bifurcation.

This paper is organized as follows. In Section 2, for the sake of self-containedness of the paper, we present some basic definitions of impulsive differential equations. In Section 3, we describe the scalar impulsive differential equations on Moebius stripe and define the Poincaré map. Then we prove several essential lemmas and give sufficient conditions to ensure the existence and stability of one-side and two-side orbits of impulsive differential equation on Moebius stripe. In Section 4, we are mainly concerned with the double-periodic bifurcation impulsive differential equations on Moebius stripe.

2. Preliminaries

For the sake of self-containedness of the paper, we present the basic definitions and notations of the theory of impulsive differential equations we need (see [1, 2, 8]). We also include some fundamental results which are necessary for understanding the theory.

Let $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{Z}^+$ be the sets of real numbers, integers, and positive integers, respectively. Denote by $\theta = \{\theta_i\}$ a strictly increasing sequence of real numbers such that the set $\mathfrak{A}$ of indexes i is an interval in $\mathbb{Z}$.

Definition 1. A function $\phi : \mathbb{R} \to \mathbb{R}^n$, $n \in \mathbb{R}$, is from the set $PC(\mathbb{R}, \theta)$ if

(i) it is left continuous;

(ii) it is continuous, except, possibly, points of θ, where it has discontinuities of the first kind.

The last definition means that if $\phi(t) \in PC(\mathbb{R}, \theta)$, then the right limit $\phi(\theta_i+) = \lim_{t \to \theta_i^+} \phi(t)$ exists and $\phi(\theta_i(-)) = \phi(\theta_i)$, where $\phi(\theta_i-) = \lim_{t \to \theta_i^-} \phi(t)$, for each $\theta_i \in \theta$.

Definition 2. A function $\phi : \mathbb{R} \to \mathbb{R}^n$ is from the set $PC^1(\mathbb{R}, \theta)$ if $\phi(t), \phi'(t) \in PC(\mathbb{R}, \theta)$, where the derivative at points of θ is assumed to be the left derivative.

In what follows, in this section, $J \in \mathbb{R}$ is an interval in $\mathbb{R}$. For simplicity of notation, θ is not necessary a subset of J.

Definition 3. The solution $\phi(t)$ is stable if to any $\varepsilon > 0$ and $t_0 \in J$ there corresponds $\delta(t_0, \varepsilon) > 0$ such that for any other solution $\psi(t)$ of (1) with $\|\phi(t_0) - \psi(t_0)\| < \delta(t_0, \varepsilon)$ we have $\|\phi(t) - \psi(t)\| < \varepsilon$ for $t \geq t_0$; the solution $\phi(t)$ is uniformly stable, if $\delta(t_0, \varepsilon)$ can be chosen independently of t_0.
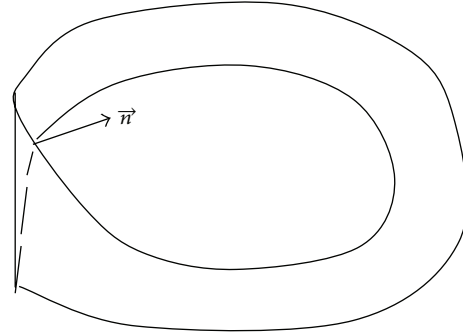

FIGURE 1: Moebius stripe.

Definition 4. The solution $\phi(t)$ is asymptotically stable if it is stable in the sense of Definition 3 and there exists a positive number $\kappa(t_0)$ such that if $\psi(t)$ is any other solution of (1) with $\|\phi(t_0) - \psi(t_0)\| < \kappa(t_0)$, then $\|\phi(t) - \psi(t)\| \to 0$ as $t \to \infty$; if $\kappa(t_0)$ can be chosen to be independent of t_0 and $\phi(t)$ is uniformly stable, then $\phi(t)$ is said to be uniformly asymptotically stable.

Definition 5. The solution $\phi(t)$ is unstable if there exist numbers $\varepsilon_0 > 0$ and $t_0 \in J$ such that for any $\delta > 0$ there exists a solution $y_\delta(t)$, $\|\phi(t_0) - y_\delta(t_0)\| < \delta$, of (1) such that either it is not continuable to ∞ or there exists a moment t_1, $t_1 > t_0$ such that $\|\phi(t_1) - y_\delta(t_1)\| \geq \delta$.

For any $t_0 \in \mathbb{R}$, we assume that there exists a $k \in \mathbb{Z}^+$, such that $\tau_{k-1} < t_0 \leq \tau_k$; then the initial value problem (IVP) to first-order impulsive differential equations (1) is given as

$$\frac{dx}{dt} = f(t, x), \quad t \neq \tau_k,$$

$$\Delta x|_{t=\tau_k} = I_k(x), \quad k \in \mathbb{Z}^+, \tag{4}$$

$$x(t_0^+) = x_0.$$

In what followed, we use $x(t, t_0, x_0)$ to denote the solution of IVP (4).

In [20], Hu and Han investigated system (1) under the assumption (H*) and obtained the following stability results for the periodic solutions.

Theorem 6 (see [20]). *Let $x(t, t_0, x_0^*)$ be a periodic solution of system (1) with period T. If $0 < |P'(x_0^*)| < 1$ (>1), then it is uniformly asymptotically stable (unstable), where $P(x_0) = x(t_0 + nT^+, t_0, x_0)$ is the Poincaré map of system (1).*

3. Poincaré Map and Periodic Solutions

In this section, we describe the scalar impulsive differential equations on Moebius stripe and define the Poincaré map. Then we prove several essential lemmas and give sufficient conditions to ensure the existence and stability of one-side and two-side orbits (Figure 2) of impulsive differential equation on Moebius stripe.

Lemma 7. *Assume that conditions (H1), (H2) hold. Suppose that $x(t, t_0, x_0)$ is a solution of* (1) *satisfying initial value $x(t_0^+) = x_0$. Then $-x(t + T, t_0, x_0)$ is also a solution of* (1), *and*

$$-x\left(t + T, t_0, x_0\right) = x\left(t, t_0, -x\left(t_0 + T, t_0, x_0\right)\right), \quad t \in \mathbb{R}. \tag{5}$$

Proof. Let $\varphi(t) \equiv -x(t + T, t_0, x_0)$, $\psi(t) \equiv x(t, t_0, -x(t_0 + T, t_0, x_0))$. Then for $t \neq \tau_k, k \in \mathbb{Z}$, we have by (H2) that

$$
\begin{aligned}
\frac{d\varphi(t)}{dt} &= -\frac{dx\left(t + T, t_0, x_0\right)}{dt} \\
&= -f\left(t + T, x\left(t + T, t_0, x_0\right)\right) \\
&= f\left(t, -x\left(t + T, t_0, x_0\right)\right) = f\left(t, \varphi(t)\right).
\end{aligned} \tag{6}
$$

For $t = \tau_k, k \in \mathbb{Z}^+$, it follows from (H1), (H2) that $\tau_k + T = \tau_{k+q}, k \in \mathbb{Z}^+$ and

$$
\begin{aligned}
\Delta\varphi|_{t=\tau_k} &= -x\left(\tau_k + T^+, t_0, x_0\right) + x\left(\tau_k + T, t_0, x_0\right) \\
&= -x\left(\tau_{k+q}^+, t_0, x_0\right) + x\left(\tau_{k+q}, t_0, x_0\right) \\
&= -I_{k+q}\left(x\left(\tau_{k+q}, t_0, x_0\right)\right) \\
&= -I_k\left(x\left(\tau_{k+q}, t_0, x_0\right)\right) = -I_k\left(x\left(\tau_k + T, t_0, x_0\right)\right) \\
&= I_k\left(-x\left(\tau_k + T, t_0, x_0\right)\right) = I_k\left(\varphi(\tau_k)\right).
\end{aligned} \tag{7}
$$

Thus, we proved that $\varphi(t) \equiv -x(t + T, t_0, x_0)$ is a solution of (1). On the other hand, it is obvious that

$$\varphi(t)|_{t=t_0} = -x\left(t_0 + T, t_0, x_0\right) = \psi(t)|_{t=t_0}. \tag{8}$$

Hence, by uniqueness theorem we have that $\varphi(t) \equiv \psi(t), t \in \mathbb{R}$. This completes the proof. $\qquad\square$

Let D denotes the stripe area on the plain $\{(t, x) \mid (t, x) \in \mathbb{R} \times \mathbb{R}\}$ between two lines $t = t_0$ and $t = t_0 + T$; that is,

$$D = \left\{(t, x) \mid t_0 \leq t \leq t_0 + T, -\infty < x < +\infty\right\}. \tag{9}$$

Assume that $x(t, t_0, x_0)$ exists for all $t \in [t_0, +\infty)$. Define $L_0 = \{(t, x(t, t_0, x_0)) \mid t_0 \leq t \leq t_0 + T\}$. In general, we denote L_k $(k \geq 1)$ by

$$L_k = \left\{\left(t, x\left(t, t_0, -x_k\right)\right) \mid t_0 \leq t \leq t_0 + T\right\}, \tag{10}$$

where $x_k = x(t_0 + T^+, t_0, -x_{k-1}), t \geq t_0$.

It follows from Lemma 7 that L_k has the form

$$L_k = \left\{\left(t, (-1)^k x\left(t + kT, t_0, x_0\right)\right) \mid t_0 \leq t \leq t_0 + T\right\}. \tag{11}$$

We now introduce an equivalence relation $\sim$ on D such that for $(t, x), (t', x') \in D$

$$(t, x) \sim \left(t', x'\right) \quad \text{iff} \ \left|t - t'\right| = T, \ x = -x'. \tag{12}$$

Then we denote the corresponding quotient space by M_2. From geometric point of view, M_2 is obtained by considering
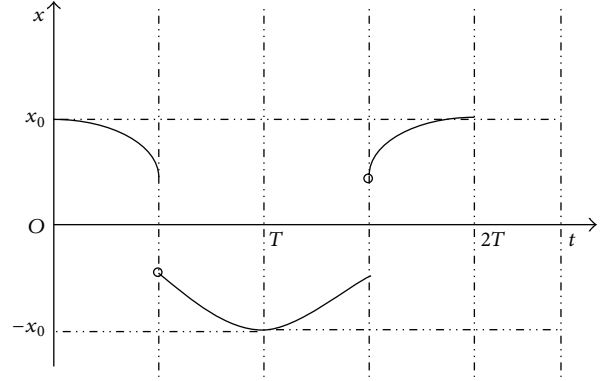

FIGURE 2: Figure of one-side and two-side orbits.

two elements (t_0, x) and $(t_0 + T, -x)$ on D as the same point (or sticking (t_0, x) and $(t_0 + T, -x)$ together). Thus M_2 is a surface with only one side or the well-known *Moebius stripe*. Obviously, by Lemma 7 the union

$$\bigcup_{k \in \mathbb{Z}^+} L_k = \bigcup_{k \in \mathbb{Z}^+} \left\{\left(t, (-1)^k x\left(t + kT, t_0, x_0\right)\right) \mid t_0 \leq t \leq t_0 + T\right\} \tag{13}$$

define a flow on M_2. From this point of view, we call (1) satisfying (H1) and (H2) an impulsive dynamical system on Moebius stripe (see Figure 1).

Definition 8 (Poincaré Map). Let $x(t, t_0, x_0)$ be the solution of (IVP) (4). Assume that there exists an interval J such that for any $x_0 \in J, x(t, t_0, x_0)$ exists on $[t_0, t_0 + T]$. A map $P : J \rightarrow \mathbb{R}$ is called a Poincaré map of system (1) if for any $x_0 \in J$

$$P(x_0) = -x\left(t_0 + T^+, t_0, x_0\right). \tag{14}$$

Definition 9. A closed curve $\gamma^+(x_0)$ is called a one-side periodic orbit on M_2 if $\gamma^+(x_0) = L_0$. And a closed curve $\gamma^+(x_0)$ is called a two-side periodic orbit on M_2 if $\gamma^+(x_0) = L_0 \cup L_1 \neq L_1$.

From Definitions 8 and 9, we can easily prove the following assertion.

Lemma 10. *One of following alternatives is valid:*

(i) *$\gamma^+(x_0)$ is a one-side periodic orbit;*

(ii) *x_0 is a fixed point of P; that is, $P(x_0) = x_0$;*

(iii) *$x(t + T, t_0, x_0) = -x(t, t_0, x_0), t \in \mathbb{R}$.*

Proof. We prove it from (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). Assume (i) is true; that is, $\gamma^+(x_0)$ is a one-side periodic orbit. Then by Definition 9 we have that

$$-x\left(t_0 + T^+, t_0, x_0\right) = x\left(t_0^+, t_0, x_0\right) = x_0, \tag{15}$$

that is, $P(x_0) = x_0$. Hence (ii) is valid.

Next, we suppose that (ii) is fulfilled; that is, $-x(t_0 + T^+, t_0, x_0) = x_0$. Then by Lemma 7 we know

$$x(t + T, t_0, x_0) = x(t, t_0, -x(t_0 + T^+, t_0, x_0))$$
$$= -x(t, t_0, x_0). \qquad (16)$$

Thus, (iii) is proved.

Finally, if (iii) is true, then $x(t_0 + T^+, t_0, x_0) = -x_0$. By the uniqueness of solution of IVP (4), we know

$$x(t, t_0, -x(t_0 + T^+, t_0, x_0)) = x(t, t_0, x_0),$$
$$t \in [t_0, t_0 + T]. \qquad (17)$$

Thus we obtain that $\gamma^+(x_0)$ is a one-side periodic orbit. The proof is completed. □

Similarly, as proof of Lemma 10, we have the following lemma.

Lemma 11. *One of following alternatives is valid:*

(i) $\gamma^+(x_0)$ *is a two-side periodic orbit;*

(ii) x_0 *is a 2-periodic point of P; that is, $P(x_0) \neq x_0$, $P^2(x_0) = x_0$;*

(iii) $x(t + 2T, t_0, x_0) = x(t, t_0, x_0)$, $t \in \mathbb{R}$. *And there exists a t_0, such that $x(t_0 + T^+, t_0, x_0) \neq -x_0$.*

Remark 12. From Lemmas 10 and 11, we see that a one-side periodic orbit must be a two-side periodic orbit since

$$P(x_0) = x_0 \quad \text{implies } P^2(x_0) = P(P(x_0)) = P(x_0) = x_0. \qquad (18)$$

Nevertheless, the converse is not true.

From Remark 12, we give the definition of stability of the mentioned orbits.

Definition 13. Let $\gamma^+(x_0)$ be a periodic orbit of system (1) (one-side or two-side). Then $\gamma^+(x_0)$ of system (1) is called stable (asymptotically stable or unstable) if $\gamma^+(x_0)$ as a $2T$ periodic solution is stable (asymptotically stable or unstable).

Theorem 14. *Assume $x(t, t_0 x_0)$ is the solution of IVP (4) and let $z(t) = \partial x(t, t_0, x_0)/\partial x_0$. Then $z(t)$ is a solution to the following IVP of impulsive differential equations:*

$$\frac{dz}{dt} = f_x(t, x) z, \quad t \neq t_k,$$
$$\Delta z|_{t=\tau_k} = I_k'(x), \quad k \in \mathbb{Z}^+, \qquad (19)$$
$$z(t_0) = 1.$$

Proof. Let $J = (t_0, +\infty)$ and $J_k = (\tau_{k-1}, \tau_k], k \in \mathbb{Z}^+$. Without losing generality, we assume that $t_0 \in J_j$ for some $j \geq 1$. The solution of IVP

$$\frac{dx}{dt} = f(t, x),$$
$$x(t_0^+) = x_0 \qquad (20)$$

can be expressed as

$$x(t, t_0, x_0) = x_0 + \int_{t_0}^t f(s, x(s, t_0, x_0)) ds. \qquad (21)$$

Differentiate between both sides of the above equation with respect to x_0, we have

$$\frac{\partial x(t, t_0, x_0)}{\partial x_0} = 1 + \int_{t_0}^t f_x(s, x(s, t_0, x_0)) \cdot \frac{\partial x(s, t_0, x_0)}{\partial x_0} ds. \qquad (22)$$

Let $z(t) = \partial x(t, t_0, x_0)/\partial x_0$, then for $t \in [t_0, \tau_j)$, $z(t)$ is the solution of IVP to ordinary differential equation

$$\frac{dz}{dt} = f_x(t, x) z, \quad t \neq t_k,$$
$$z(t_0) = 1. \qquad (23)$$

Thus

$$z(t) = \exp \int_{t_0}^t f_x(s, x(s, t_0, x_0)) ds. \qquad (24)$$

Since $z(t)$ is left continuous on $[t_0, \infty)$, we have

$$z(\tau_j) = \exp \int_{t_0}^{\tau_j} f_x(s, x(s, t_0, x_0)) ds. \qquad (25)$$

For $t \in J_{j+1}$, $x(t, t_0, x_0)$ is a solution of system

$$\frac{dx}{dt} = f(t, x),$$
$$x(\tau_j) = x_1, \qquad (26)$$

where $x_1 = x(\tau_{j^+}, t_0, x_0) = x(\tau_j, t_0, x_0) + I_j(x(\tau_j, t_0, x_0))$. Thus, we have

$$x(t, t_0, x_0) \equiv x(t, \tau_j, x_1)$$
$$= x_1 + \int_{\tau_j}^t f(s, x(s, \tau_j, x_1)) ds, \quad t \in J_{j+1}. \qquad (27)$$

Similarly, we have $t \in (\tau_j, \tau_{j+1})$,

$$\frac{\partial x(t, t_0, x_0)}{\partial x_1} = \frac{\partial x(t, \tau_j, x_1)}{\partial x_1}$$
$$= \exp \int_{\tau_j}^t f_x(s, x(s, \tau_j, x_1)) ds \qquad (28)$$
$$= \exp \int_{\tau_j}^t f_x(s, x(s, t_0, x_0)) ds.$$

Note

$$\frac{\partial x_1}{\partial x_0} = \frac{\partial \left[x \left(\tau_j, t_0, x_0 \right) + I_j \left(x \left(\tau_j, t_0, x_0 \right) \right) \right]}{\partial x \left(\tau_j, t_0, x_0 \right)}$$

$$\cdot \frac{\partial x \left(\tau_j, t_0, x_0 \right)}{\partial x_0} \tag{29}$$

$$= \left(1 + I_j' \left(x \left(\tau_j, t_0, x_0 \right) \right) \right) \frac{\partial x \left(\tau_j, t_0, x_0 \right)}{\partial x_0}.$$

We obtain for $t \in (\tau_j, \tau_{j+1})$ that

$$\frac{\partial x \left(t, t_0, x_0 \right)}{\partial x_0} = \frac{\partial x \left(t, \tau_j, x_1 \right)}{\partial x_0}$$

$$= \left(1 + I_j' \left(x \left(\tau_j, t_0, x_0 \right) \right) \right) \tag{30}$$

$$\cdot \exp \int_{t_0}^{t} f_x \left(s, x \left(s, t_0, x_0 \right) \right) ds.$$

Deducing in a similar way, we get

$$\frac{\partial x \left(t, t_0, x_0 \right)}{\partial x_0} = \prod_{t_0 < \tau_k \le t} \left(1 + I_k' \left(x \left(\tau_j, t_0, x_0 \right) \right) \right)$$

$$\cdot \exp \int_{t_0}^{t} f_x \left(s, x \left(s, t_0, x_0 \right) \right) ds, \tag{31}$$

where $t \in J$. Then the proof is completed. □

By Definitions 9 and (31), we conclude the following assertion.

Corollary 15. *Assume that conditions (H1), (H2) hold. Then*

$$P' \left(x_0 \right) = - \prod_{t_0 < \tau_k \le t_0 + T} \left(1 + I_k' \left(x \left(\tau_k, t_0, x_0 \right) \right) \right)$$

$$\cdot \exp \int_{t_0}^{t_0 + T} f_x \left(t, x \left(t, t_0, x_0 \right) \right) dt. \tag{32}$$

As usual, one uses the notion $P^2(x_0) = P(P(x_0))$. Then one has

$$\left[P^2 \left(x_0 \right) \right]' = \prod_{t_0 < \tau_k \le t_0 + 2T} \left(1 + I_k' \left(x \left(\tau_k, t_0, x_0 \right) \right) \right)$$

$$\cdot \exp \int_{t_0}^{t_0 + 2T} f_x \left(t, x \left(t, t_0, x_0 \right) \right) dt. \tag{33}$$

Definition 16. x_0 is called a hyperbolic fixed point of P if $x_0 = P(x_0)$ and $P'(x_0) \ne -1$; the corresponding one-side periodic orbit $\gamma^+(x_0)$ is called hyperbolic one-side periodic orbit. If $\gamma^+(x_0)$ is a two-side periodic orbit with $(P^2)'(x_0) \ne 1$, then we call $\gamma^+(x_0)$ a hyperbolic two-side periodic orbit.

Theorem 17. *Assume that the conditions (H1), (H2) hold. Let $\gamma^+(x_0)$ be a periodic orbit of system (1) and $I_k'(x(\tau_k, t_0, x_0)) \ne -1$. Then (i) $\int_{t_0}^{t_0 + 2T} f_x(t, x(t, t_0, x_0)) dt < - \sum_{t_0 < \tau_k \le t_0 + 2T} \ln |1 + I_k'(x(\tau_k, t_0, x_0))|$ implies $\gamma^+(x_0)$ is asymptotically stable,*

(ii) $\int_{t_0}^{t_0 + 2T} f_x(t, x(t, t_0, x_0)) dt > - \sum_{t_0 < \tau_k \le t_0 + 2T} \ln |1 + I_k'(x(\tau_k, t_0, x_0))|$ implies $\gamma^+(x_0)$ is unstable.

Proof. If $\gamma^+(x_0)$ is a two-side periodic orbit; that is, $x(t, t_0, x_0)$ is a $2T$ periodic solution of (1). Since both (H1) and (H2) hold, we know that (1) is a periodic impulsive differential equation. Then by (33) and Theorem 6, the conclusion is straightforward. □

Example 18. Consider the linear periodic impulsive differential equations on Moebius stripe as follows:

$$\frac{dx}{dt} = a(t) x + b(t), \quad t \ne \tau_k,$$

$$\Delta x|_{t=\tau_k} = c_k x \left(\tau_k \right), \quad k \in \mathbb{Z}^+, \tag{34}$$

where $\tau_k < \tau_{k+1}$ $(k \ge 1)$, $\tau_k \to +\infty$, $k \to +\infty$, $c_k \ne -1$ and there exists a constant $T > 0$, a positive integer q, such that the following conditions are satisfied:

($\widetilde{H}$1) $a(t + T) = a(t)$ and $b(t + T) = -b(t)$ for $t \in \mathbb{R}$;

($\widetilde{H}$2) $a(t)$ and $b(t)$ are continuous;

($\widetilde{H}$3) $c_{k+q} = c_k$, for all $k \in \mathbb{Z}^+$;

($\widetilde{H}$4) $\tau_{k+q} - \tau_k = T$, for all $k \in \mathbb{Z}^+$.

Assume that $x(t, t_0, x_0)$ is a one-side periodic solution of system (34), by the method of variation of constants formula (see [1]), we get

$$x \left(t, t_0, x_0 \right)$$

$$= \prod_{t_0 < \tau_k \le t} \left(1 + c_k \right)$$

$$\cdot \exp \int_{t_0}^{t} a(t) dt$$

$$\times \left[x_0 + \int_{t_0}^{t} \exp \left(- \int_{t_0}^{s} a(u) du \right) b(s) ds \right], \tag{35}$$

$$P \left(x_0 \right)$$

$$= - \prod_{t_0 < \tau_k \le t_0 + T} \left(1 + c_k \right)$$

$$\cdot \exp \int_{t_0}^{t_0 + T} a(t) dt$$

$$\times \left[x_0 + \int_{t_0}^{t_0 + T} \exp \left(- \int_{t_0}^{t} a(u) du \right) b(t) dt \right], \tag{36}$$

$$P^2(x_0)$$

$$= \prod_{t_0 < \tau_k \le t_0+2T} (1 + c_k)$$

$$\cdot \exp \int_{t_0}^{t_0+2T} a(t)\, dt$$

$$\times \left[x_0 + \int_{t_0}^{t_0+2T} \exp\left(-\int_{t_0}^{t} a(u)\, du \right) b(t)\, dt \right].$$

$$(37)$$

Let $A = \prod_{t_0 < \tau_k \le t_0+T} (1 + c_k) \cdot \exp \int_{t_0}^{t_0+T} a(t)\, dt$; therefore, we have the following theorem.

Theorem 19. *Suppose that ($\widetilde{H}$1–$\widetilde{H}$4) are satisfied, then*

   (i) *there exists a unique one-side periodic orbit for system (34) if $A \ne -1$, which is asymptotically stable (unstable) provided $0 < |A| < 1$ ($|A| > 1$),*

   (ii) *if $A^2 \ne 1$, (34) has no two-side periodic orbit. If $A = 1$ all the trajectories are two-side periodic orbits expect for a unique one-side periodic orbit.*

Proof. For the sake of convenience, we denote

$$B_1 = \int_{t_0}^{t_0+T} \exp\left(-\int_{t_0}^{t} a(u)\, du \right) b(t)\, dt,$$

$$B_2 = \int_{t_0}^{t_0+2T} \exp\left(-\int_{t_0}^{t} a(u)\, du \right) b(t)\, dt. \tag{38}$$

Then

$$P(x_0) = -A(x_0 + B_1), \qquad P^2(x_0) = A^2(x_0 + B_2). \tag{39}$$

Obviously, $P(x_0) = x_0$ has a unique solution for any $x_0 \in \mathbb{R}$ if $A \ne -1$, and $P^2(x_0) = x_0$ has a unique solution for any $x_0 \in \mathbb{R}$ if $A^2 \ne 1$. Observing that any two-side periodic orbit obtained under the assumption $A^2 \ne 1$ must be a one-side periodic orbit since $A^2 \ne 1$ implies $A \ne -1$, together with Remark 12, we have (34) has no two-side periodic orbit.

It follows from (36) that $P'(x_0) = -A$. Then by Theorem 6 we have the one-side orbit is asymptotically stable (unstable) provided $0 < |A| < 1$ ($|A| > 1$).

Next, let $A = 1$. By taking (36) and (37) into account, we have

$$P(x_0) = -x_0 - \int_{t_0}^{t_0+T} \exp\left(-\int_{t_0}^{t} a(u)\, du \right) b(t)\, dt$$

$$\equiv -x_0 - B_1,$$

$$P^2(x_0) = x_0 + \int_{t_0}^{t_0+2T} \exp\left(-\int_{t_0}^{t} a(u)\, du \right) b(t)\, dt$$

$$\equiv x_0 + B_2. \tag{40}$$

Suppose that P has a unique fixed point $x_0^* = -B_1/2$, from the above we have $-B_1/2 + B_2 = -B_1/2$, then $B_2 \equiv 0$ and

$P^2(x_0) = x_0$. So by taking Lemma 11, $\gamma^+(x_0)$ is a two-side periodic orbit if $x_0 \ne x_0^*$.

The proof is ended.   □

Remark 20. If $c_k \equiv 0$, $k \in \mathbb{Z}^+$, in (34); that is, (34) reduces to an ordinary differential equation. We see that $A = \exp \int_{t_0}^{t_0+T} a(t)\, dt$. Hence $A \ne -1$ holds automatically, and therefore (34) always has a unique one-side periodic orbit.

Corollary 21. *Let ($\widetilde{H}$1–$\widetilde{H}$4) be fulfilled and $A = 1$. Then*

$$B_2 = \int_{t_0}^{t_0+2T} \exp\left(-\int_{t_0}^{t} a(u)\, du \right) b(t)\, dt = 0. \tag{41}$$

Now we are in position to consider nonlinear impulsive system on Meobius stripe. To explore the uniqueness of one-side periodic orbit, we induce the following condition.

(H3) Operator $B_k : \mathbb{R} \to \mathbb{R}$, $B_k(x) = x + I_k(x)$ is strictly increasing, for all $k \in \mathbb{Z}^+$.

Theorem 22. *Suppose that conditions (H1)–(H3) hold, then*

   (i) *system (1) has at most one one-side periodic orbit;*

   (ii) *if any solution $x(t, t_0, x_0)$ of (1) with $|x_0| \le |P(0)|$ is well defined on $t \in [t_0, t_0 + T]$, then system (1) must has a unique one-side periodic orbit.*

Proof. We first prove that system (1) cannot have two one-side periodic orbits. Suppose $\gamma_1^+(x_0^*) : x = x(t, t_0, x_0^*)$, $t \in [t_0, t_0 + T]$ and $\gamma_2^+(x_0) : x = x(t, t_0, x_0)$, $t \in [t_0, t_0 + T]$ are two one-side periodic orbits system (1). Then

$$x(t_0 + T, t_0, x_0^*) = -x_0^*, \qquad x(t_0 + T, t_0, x_0) = -x_0. \tag{42}$$

Without losing generality, we assume $x_0 > x_0^*$, then it follows from uniqueness theorem of ordinary differential equations that $x = x(t, t_0, x_0)$ and $x(t, t_0, x_0^*)$ cannot intersect when t is not an impulsive time. Therefore we have

$$x(t, t_0, x_0) > x(t, t_0, x_0^*), \quad t_0 \le t \le \tau_1. \tag{43}$$

Note $B_k(x) = x + I_k(x)$ is strictly increasing, we get

$$x(\tau_1^+, t_0, x_0) > x(\tau^+, t_0, x_0^*). \tag{44}$$

In a similar way, we can prove that $x(t, t_0, x_0) > x(t, t_0, x_0^*)$, $t_0 \le t \le t_0 + T$. That is, the curve $\{(t, x) \mid x = x(t, t_0, x_0), t_0 \le t \le t_0 + T\}$ always stays above curve $\{(t, x) \mid x = x(t, t_0, x_0^*), t_0 \le t \le t_0 + T\}$. This contradicts (42). We put it in another way that

$$x_0 > x_0^* \Longrightarrow P(x_0) > P(x_0^*). \tag{45}$$

Thus, (1) has at most a one-side periodic orbit.

Further, let the solution $x(t, t_0, x_0)$ of system (1) be all defined on $t \in [t_0, t_0 + T]$. If $P(0) = 0$, the conclusion is proved. We assume that $P(0) > 0$, then we know $P(x_0) < P(0)$ if $0 < x_0 \le P(0)$. Note

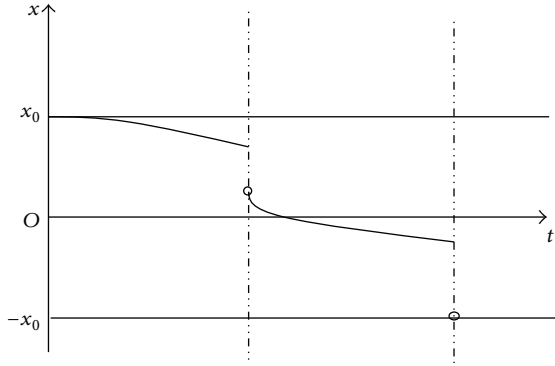$$P^2(0) - P(0) = P(P(0)) - P(0) < P(0) - P(0) = 0. \tag{46}$$

FIGURE 3: A one-side periodic orbit.

We obtain that $P(x_0) - x_0$ have opposite signs between $x_0 = 0$ and $x_0 = P(0)$, and then it follows from the continuity of P that there exists $x_0^* \in (0, P(0))$ such that $P(x_0^*) = x_0^*$. Similarly, we can prove P has a fixed point in the case of $P(0) < 0$. The proof is completed. □

Theorem 23. *Assume that conditions (H1)–(H3) hold. Furthermore, suppose there exists a positive number N such that*

$$f(t, N) \leq 0, \quad f(t, -N) \geq 0, \quad t \in [t_0, t_0 + T], \quad (47)$$

$$-2N \leq I_k(N) \leq 0, \quad \forall k \in \mathbb{Z}^+. \quad (48)$$

Then (1) *has a unique one-side periodic orbit.*

Proof. From (47) we have that $x(t, t_0, x_0)$ will stay inside $[-N, N]$ for $t \neq \tau_k, k \in \mathbb{Z}^+$. On the other hand, by (H3), we have that $-N + I_k(-N) \leq x(\tau_k) + I_k(x(\tau_k)) \leq N + I_k(N)$ for $-N \leq x(\tau_k) \leq N$. Then it follows from (48) that

$$-N \leq N + I_k(N) \leq N \leq N,$$
$$-N \leq -N + I_k(-N) = -N - I_k(N) \leq N \quad (49)$$

(see Figure 3). □

Thus,

$$|P(0)| = \left| -x(t_0 + T, 0) \right| = \left| x(t_0 + T, 0) \right| \leq N. \quad (50)$$

This implies that $P(x_0)$ is well defined for $|x_0| \leq |P(0)|$. By Theorem 22, we obtain that (1) has a unique one-side periodic orbit.

4. Double-Period Bifurcation

In this section, we mainly discuss the bifurcation on periodic orbits. If system (1) has a one-side periodic orbit, without losing generality, we may assume that $f(t, 0) = 0$; that is, $x = 0$ is the one-side periodic orbit. Actually, if $x(t)$ is a one-side periodic orbit, then we let $y = x - x(t)$; therefore

there exists a transformation of system (1) that

$$\frac{dy}{dt} = f(t, y + x(t)) - f(t, x(t)) \equiv g(t, y), \quad t \neq \tau_k,$$

$$\Delta y|_{t=\tau_k} = I_k(y + x(\tau_k)) - I_k(x(\tau_k)) \equiv h_k(y), \quad k \in \mathbb{Z}^+,$$
$$(51)$$

By (H2) and Lemma 10, we know $g(t + T, -y) = -g(t, y)$, $g(t, 0) = 0$, $h_{k+q}(y) = h_k(y)$, and $\tau_{k+q} - \tau_k = T$, for all $k \in \mathbb{Z}^+$.

Next, we consider the following perturbed system of system (1):

$$\frac{dx}{dt} = F(t, x, \varepsilon), \quad t \neq \tau_k,$$

$$\Delta x|_{t=\tau_k} = \widetilde{I}_k(x(\tau_k), \varepsilon), \quad k \in \mathbb{Z}^+,$$
$$(52)$$

where $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is C^3 with respect to x, continuously differentiable with respect to ε. $\widetilde{I}_k : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is C^3 ($k \in \mathbb{Z}^+$) with respect to x. Moreover, we suppose $F(t + T, -x, \varepsilon) = -F(t, x, \varepsilon)$, $\widetilde{I}_k(x, \varepsilon) = \widetilde{I}_{k+q}(x, \varepsilon)$, $\widetilde{I}_k(-x) = -\widetilde{I}_k(x)$, for all $k \in \mathbb{Z}^+$, where $\tau_{k+q} - \tau_k = T$. For $\varepsilon = 0$, we have $F(t, x, 0) = f(t, x)$, $\widetilde{I}_k(x, 0) = I_k(x)$. These assumptions mean (H1) and (H2) hold for F and $\widetilde{I}_k$, for all $k \in \mathbb{Z}^+$. Furthermore, assume that $x + \widetilde{I}_k(x, \varepsilon)$ is strictly increasing, then by Theorem 22 we have that system (52) has at most a one-side periodic orbit.

Suppose that (1) has a one-side periodic orbit and $f(t, 0) = 0$. Then by using implicit function theorem in the Poincaré map of system (52), we know that system (52) has a one-side periodic orbit when $|\varepsilon|$ is sufficiently small. Now let $x^*(t, \varepsilon)$ be the solution of system (52) and $y(t, \varepsilon) = x(t, \varepsilon) - x^*(t, \varepsilon)$. Then we can get a transformation of system (52):

$$\frac{dy}{dt} = F(t, y + x^*, \varepsilon) - F(t, x^*, \varepsilon) = G(t, y, \varepsilon), \quad t \neq \tau_k,$$

$$\Delta y|_{t=\tau_k} = \widetilde{I}_k(y + x^*(\tau_k, \varepsilon), \varepsilon)$$
$$- \widetilde{I}_k(x^*(\tau_k, \varepsilon), \varepsilon) = H_k(y, \varepsilon), \quad k \in \mathbb{Z}^+.$$
$$(53)$$

By Taylor's formula, we have

$$G(t, y, \varepsilon) = A_1(t, \varepsilon) y + A_2(t, \varepsilon) y^2$$
$$+ A_3(t, \varepsilon) y^3 + o(y^3), \quad (54)$$

$$H_k(y, \varepsilon) = B_{k1}(\varepsilon) y + B_{k3}(\varepsilon) y^3 + o(y^3),$$

where

$$A_i(t, \varepsilon) = \frac{1}{i!} \frac{\partial^i F}{\partial x^i}(t, x^*, \varepsilon),$$

$$B_{kj}(\varepsilon) = \frac{1}{j!} \frac{\partial^j \widetilde{I}_k}{\partial x^j}(x^*, \varepsilon), \quad (55)$$

$$A_i(t + T, \varepsilon) = (-1)^{i-1} A_i(t, \varepsilon),$$

for $k \geq 1$, $i = 1, 2, 3$; $j = 1, 3$.

If $\varepsilon = 0$, then $x^* = 0$. So $A_i(t, 0) = (1/i!)(\partial^i f/\partial x^i)(t, 0)$ and $B_{kj}(0) = (1/j!)I_k^{(j)}(0)$, for $k \geq 1$, $i = 1, 2, 3$, $j = 1, 3$.

Suppose that $y(t, y_0, \varepsilon)$ $(t \geq t_0)$ is the solution of system (53) with the initial value $y(t_0^+, y_0, \varepsilon) = y_0$, $P(y_0, \varepsilon)$ is the Poincaré map of system (53). Note

$$\widetilde{P}(y_0, \varepsilon) = y(t_0 + 2T^+, y_0, \varepsilon) = P^2(y_0, \varepsilon). \quad (56)$$

Without losing generality, let $x^*(t, \varepsilon) = 0$ is a nonhyperbolic solution. That is, $P(0, 0) = 0$ and $(\partial P/\partial y_0)(0, 0) = -1$.

Noting that $P(0, \varepsilon) = 0$, then by Taylor's formula, we have

$$P(y_0, \varepsilon) = \overline{A}_1(\varepsilon) y_0 + \overline{A}_2(\varepsilon) y_0^2 + \overline{A}_3(\varepsilon) y_0^3 + o(y_0^3), \quad (57)$$

where $\overline{A}_1(\varepsilon) = (\partial P/\partial y_0)(0, \varepsilon)$, $\overline{A}_2(\varepsilon) = (1/2)(\partial^2 P/\partial y_0^2)(0, \varepsilon)$, and $\overline{A}_3(\varepsilon) = (1/6)(\partial^3 P/\partial y_0^3)(0, \varepsilon)$.

Theorem 24. *Suppose that $f(t, 0) = 0$ and $x = 0$ is a one-side periodic orbit of system $(52)_{\varepsilon=0}$ with $P(0, 0) = 0$ and $(\partial P/\partial y_0)(0, 0) = -1$. Let $a_3^* = (1/6)\widetilde{P}'''(0, 0)$. If $a_3^* \neq 0$, then for $|\varepsilon|$ sufficiently small and $[\overline{A}_1(\varepsilon) + 1]a_3^* > 0$ (≤ 0) implies that system (52) has a unique (no) two-sides periodic orbit near $x = 0$, except for a one-side periodic orbit $x^*(t, \varepsilon)$.*

Proof. As before, we obtain that $\widetilde{P}'(0, \varepsilon) = [P'(0, \varepsilon)]^2 = \overline{A}_1^2(\varepsilon)$:

$$
\begin{aligned}
\widetilde{P}(y_0, \varepsilon) &= P(P(y_0, \varepsilon), \varepsilon) \\
&= \overline{A}_1(\varepsilon) P(y_0, \varepsilon) + \overline{A}_2(\varepsilon) P^2(y_0, \varepsilon) \\
&\quad + \overline{A}_3(\varepsilon) P^3(y_0, \varepsilon) + o(P^3(y_0, \varepsilon)) \\
&= \overline{A}_1^2(\varepsilon) y_0 + \left[\overline{A}_1^2(\varepsilon) + \overline{A}_1(\varepsilon)\right] \overline{A}_2(\varepsilon) y_0^2 \\
&\quad + \overline{A}_1 \left[\overline{A}_3(\varepsilon) + 2\overline{A}_2^2(\varepsilon) + \overline{A}_1^2(\varepsilon) \overline{A}_3(\varepsilon)\right] y_0^3 \\
&\quad + o(y_0^3).
\end{aligned}
\quad (58)
$$

By our assumption, we have $\overline{A}_1(\varepsilon) = -1 + \overline{A}_1'(0)\varepsilon + o(\varepsilon)$. Therefore,

$$
\begin{aligned}
\widetilde{d}(y_0, \varepsilon) &= \frac{P^2(y_0, \varepsilon) - y_0}{y_0} \\
&= d_0(\varepsilon) + d_1(\varepsilon) y_0 + d_2(\varepsilon) y_0^2 + o(y_0^2),
\end{aligned}
\quad (59)
$$

where

$$
\begin{aligned}
d_0(\varepsilon) &= \overline{A}_1^2(\varepsilon) - 1 = -2\left[\overline{A}_1(\varepsilon) + 1\right] + o\left(\left[\overline{A}_1(\varepsilon) + 1\right]\right), \\
d_1(\varepsilon) &= \left[\overline{A}_1^2(\varepsilon) + \overline{A}_1(\varepsilon)\right] \overline{A}_2(\varepsilon) \\
&= O\left(\overline{A}_1(\varepsilon) + 1\right) = -\overline{A}_1'(0) \overline{A}_2(0) \varepsilon + o(\varepsilon), \\
d_2(\varepsilon) &= \overline{A}_1(\varepsilon) \left[\overline{A}_3(\varepsilon) + 2\overline{A}_2^2(\varepsilon) + \overline{A}_1^2(\varepsilon) \overline{A}_3(\varepsilon)\right] \\
&= -2\left[\overline{A}_2^2(0) + \overline{A}_3(0)\right] + o(1) = a_3^* + o(1).
\end{aligned}
\quad (60)
$$

By the implicit function theorem, there exists a unique function $y_0 = y_1(\varepsilon)$, $y_1(0) = 0$ such that $(\partial\widetilde{d}/\partial y_0)(y_1(\varepsilon), \varepsilon) = 0$. Therefore, for $|\varepsilon|$ sufficiently small, there is a unique extremal point $y_0 = y_1(\varepsilon)$ near $x = 0$. Moreover, the function $\widetilde{d}(y_0, \varepsilon)$ takes its minimum (maximum) $\overline{\Delta}(\varepsilon) \equiv \widetilde{d}(y_1(\varepsilon), \varepsilon)$ only if $a_3^* > 0$ (< 0):

$$\overline{\Delta}(\varepsilon) = d_0(\varepsilon) + o(\varepsilon) = \overline{A}_1^2(\varepsilon) - 1 = -2\left[\overline{A}_1(\varepsilon) + 1\right] + o(\varepsilon). \quad (61)$$

Without loss of generality, we can let $a_3^* = d_2(0) > 0$ and then $y_0 = 0$ is the minimum point of $d(y_0, 0)$. So there exists $\varepsilon_0 > 0$, such that

$$\widetilde{d}(\pm\varepsilon_0, 0) > 0, \qquad \frac{\partial\widetilde{d}}{\partial y_0}(\varepsilon_0, 0) > 0, \qquad \frac{\partial\widetilde{d}}{\partial y_0}(-\varepsilon_0, 0) < 0, \quad (62)$$

and for $|y_0| \leq \varepsilon_0$, $(\partial^2\widetilde{d}/\partial y_0^2)(y_0, 0) > 0$ exists. Therefore, there exists a δ_0, such that, for $|\varepsilon| \leq \delta_0$, we have

$$\widetilde{d}(\pm\varepsilon_0, \varepsilon) > 0, \qquad \frac{\partial\widetilde{d}}{\partial y_0}(\varepsilon_0, \varepsilon) > 0, \qquad \frac{\partial\widetilde{d}}{\partial y_0}(-\varepsilon_0, \varepsilon) < 0. \quad (63)$$

For $|\varepsilon| \leq \delta_0$ and $|y_0| \leq \varepsilon_0$, we have

$$\frac{\partial^2\widetilde{d}}{\partial y_0^2}(y_0, \varepsilon) > 0. \quad (64)$$

From (64), for any $|\varepsilon| \leq \delta_0$, we have

$$\overline{\Delta}(\varepsilon) = \min_{|y_0|\leq\varepsilon_0} \widetilde{d}(y_0, \varepsilon), \quad -\varepsilon_0 < y_1(\varepsilon) < \varepsilon_0. \quad (65)$$

And for $y_0 \in (-\varepsilon_0, y_1(\varepsilon))(\in (y_1(\varepsilon), \varepsilon_0))$,

$$\frac{\partial\widetilde{d}}{\partial y_0}(y_0, \varepsilon) < 0 \ \ (> 0). \quad (66)$$

If $\overline{\Delta}(\varepsilon) > 0$, then for all $|\varepsilon| \leq \delta_0$ and $|x_0| < \varepsilon_0$, we have $0 < \overline{\Delta}(\varepsilon) \leq \widetilde{d}$.

If $\overline{\Delta}(\varepsilon) = 0$, then $y_0 = y_1(\varepsilon)$ is the unique solution of function $\widetilde{d}$.

If $\overline{\Delta}(\varepsilon) < 0$, then there exist a unique $y_1(\varepsilon)$ and a unique $y_2(\varepsilon)$, such that

$$\widetilde{d}(y_i(\varepsilon), \varepsilon) = 0, \qquad \frac{\partial\widetilde{d}}{\partial y_0}(y_i(\varepsilon), \varepsilon) \neq 0, \quad i = 1, 2. \quad (67)$$

Thus system (52) has two (no) two-side periodic orbits if $a_3^*\overline{\Delta}(\varepsilon) < 0$ (≥ 0). The conclusion is completed (see Figures 4, 5, and 6). $\square$
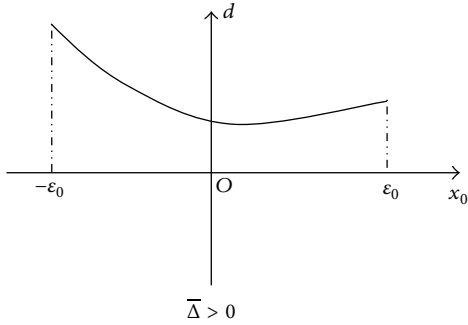
$\overline{\Delta} > 0$

FIGURE 4

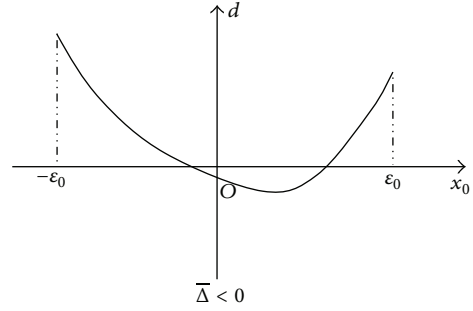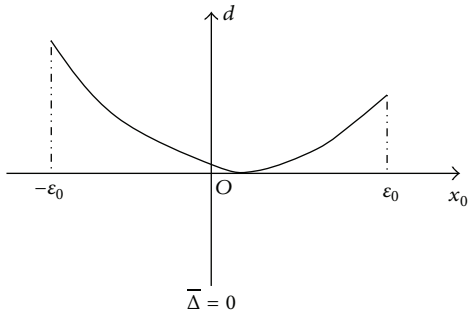$\overline{\Delta} < 0$

FIGURE 6

$\overline{\Delta} = 0$

FIGURE 5

Now we shall calculate $\overline{A}_1(0)$ and a_3^* in the simplest case, let $q = 1$. For $q > 1$ we can calculate them in the same way. In this case, $I_k \equiv I$ and $B_{ki}(y_0, \varepsilon) = B_i(y_0, \varepsilon)$, $i = 1, 3$. Suppose $y(t, y_0, \varepsilon)$ $(t \leq t_0)$ is the solution to system (53) with initial value $y(t_0+, y_0, \varepsilon) = y_0$. For $y(t, 0, \varepsilon) = 0$, let

$$y(t, y_0, \varepsilon) = \varphi_1(t, \varepsilon) y_0 + \varphi_2(t, \varepsilon) y_0^2 \qquad (68)$$
$$+ \varphi_3(t, \varepsilon) y_0^3 + o(y_0^3), \quad t \leq t_0.$$

Then for $t \in [t_0, t_0 + T]$, taking $y(t, y_0, \varepsilon)$ into system (53), we can obtain φ_1, φ_2, and φ_3 satisfying the following equations:

$$\varphi_1'(t, \varepsilon) = A_1(t, \varepsilon) \varphi_1(t, \varepsilon),$$
$$\varphi_2'(t, \varepsilon) = A_1(t, \varepsilon) \varphi_2(t, \varepsilon) + A_2(t, \varepsilon) \varphi_1^2(t, \varepsilon),$$
$$\varphi_3'(t, \varepsilon) = A_1(t, \varepsilon) \varphi_3(t, \varepsilon) + 2A_2(t, \varepsilon) \varphi_1(t, \varepsilon) \varphi_2(t, \varepsilon)$$
$$+ \varphi_3(t, \varepsilon) \varphi_1^3(t, \varepsilon).$$
$$(69)$$

For $y(0, y_0, \varepsilon) = y_0$, we know

$$\varphi_1(0, \varepsilon) = 1, \qquad \varphi_2(0, \varepsilon) = \varphi_3(0, \varepsilon) = 0. \qquad (70)$$

From (69) and (70), we have

$$\varphi_1(t, \varepsilon) = \exp \int_{t_0}^t A_1(u, \varepsilon) \, du,$$

$$\varphi_2(t, \varepsilon) = \varphi_1(t, \varepsilon) \int_{t_0}^t A_2(s, \varepsilon) \varphi_1(s, \varepsilon) \, ds,$$

$$\varphi_3(t, \varepsilon) = \varphi_1(t, \varepsilon)$$
$$\times \int_{t_0}^t \left[2A_2(s, \varepsilon) \varphi_2(s, \varepsilon) + A_3(s, \varepsilon) \varphi_1^2(s, \varepsilon) \right] ds.$$
$$(71)$$

For $t_0 < t < t_0 + T$, as we know, we get

$$y(t_0 + T^+, y_0, \varepsilon) = \left[1 + B_1(\varepsilon) \right] y(t_0 + T, \varepsilon)$$
$$+ B_2(\varepsilon) y^2(t_0 + T, \varepsilon)$$
$$+ B_3(\varepsilon) y^3(t_0 + T, \varepsilon)$$
$$+ o\left(y^3(t_0 + T, \varepsilon) \right)$$
$$= \varphi_1(t_0 + T^+, \varepsilon) y_0 + \varphi_2(t_0 + T^+, \varepsilon) y_0^2$$
$$+ \varphi_3(t_0 + T^+, \varepsilon) y_0^3 + o(y_0^3),$$
$$(72)$$

where

$$\varphi_1(t_0 + T^+, \varepsilon) = \left[1 + B_1(\varepsilon) \right] \varphi_1(t_0 + T, \varepsilon),$$
$$\varphi_2(t_0 + T^+, \varepsilon) = \left[1 + B_1(\varepsilon) \right] \varphi_2(t_0 + T, \varepsilon)$$
$$+ B_2(\varepsilon) \varphi_1^2(t_0 + T, \varepsilon),$$
$$\varphi_3(t_0 + T^+, \varepsilon) = \left[1 + B_1(\varepsilon) \right] \varphi_3(t_0 + T, \varepsilon)$$
$$+ 2B_2(\varepsilon) \varphi_1(t_0 + T, \varepsilon) \varphi_2(t_0 + T, \varepsilon)$$
$$+ B_3(\varepsilon) \varphi^3(t_0 + T, \varepsilon).$$
$$(73)$$

Clearly, $\overline{A}_1(\varepsilon) = -\varphi_1(t_0 + T^+, \varepsilon)$, $\overline{A}_2(\varepsilon) = -\varphi_2(t_0 + T^+, \varepsilon)$, and $\overline{A}_3(\varepsilon) = -\varphi_3(t_0 + T^+, \varepsilon)$.

Moreover, we know

$$\int_{t_0}^{t_0+T} A_2(s,\varepsilon)\,\varphi_2(s,\varepsilon)\,ds$$

$$= \int_{t_0}^{t_0+T} A_2(s,\varepsilon)\,\varphi_1(s,\varepsilon)$$

$$\times \left[\int_{t_0}^{s} A_2(u,\varepsilon)\,\varphi_1(u,\varepsilon)\,du \right] ds \qquad (74)$$

$$= \frac{1}{2}\left[\int_{t_0}^{t_0+T} A_2(u,\varepsilon)\,\varphi_1(u,\varepsilon)\,du \right]^2.$$

Denote $\Phi(\varepsilon) = \int_{t_0}^{t_0+T} A_2(t,\varepsilon)\varphi_1(t,\varepsilon)dt$ and $\phi_1(\varepsilon) = \varphi_1(t_0 + T,\varepsilon)$. Then,

$$\varphi_2(t_0+T,\varepsilon) = \phi_1(\varepsilon)\,\Phi(\varepsilon),$$

$$\varphi_3(t_0+T,\varepsilon) = \phi_1(\varepsilon)\,\Phi^2(\varepsilon) + \int_{t_0}^{t_0+T} A_3(s,\varepsilon)\,\varphi_1^2(s,\varepsilon)\,ds. \qquad (75)$$

Then we can obtain

$$\overline{A}_1(\varepsilon) = -[1+B_1(\varepsilon)]\exp\left(\int_{t_0}^{t_0+T} A_1(u,\varepsilon)\,du \right),$$

$$\overline{A}_2(\varepsilon) = -[1+B_1(\varepsilon)]\,\phi_1(\varepsilon)\,\Phi(\varepsilon),$$

$$\overline{A}_3(\varepsilon) = -[1+B_1(\varepsilon)]\,\phi_1(\varepsilon) \qquad (76)$$

$$\times \left[\Phi^2(\varepsilon) + \int_{t_0}^{t_0+T} A_3(s,\varepsilon)\,\varphi_1^2(s,\varepsilon)\,ds \right]$$

$$- B_3(\varepsilon)\,\phi_1^3(\varepsilon).$$

For $\overline{A}_1(0) = -1$, we can have $\phi_1(0) = 1/(1+B_1(0))$. Then

$$\overline{A}_2(0) = -\Phi(0), \qquad (77)$$

$$\overline{A}_3(0) = -\Phi^2(0) - \frac{B_3(0)}{\left[1+B_1(0)\right]^3} - \Delta(0), \qquad (78)$$

where $\Delta(\varepsilon) = \int_{t_0}^{t_0+T} A_3(s,\varepsilon)\varphi_1^2(s,\varepsilon)ds = \int_{t_0}^{t_0+T} A_3(s,\varepsilon)\exp[2\int_{t_0}^{s} A_1(u,\varepsilon)du]ds$. Therefore,

$$a_3^* = -2\left[\overline{A}_2^2(0) + \overline{A}_3(0) \right] = 2\Delta(0) + \frac{2B_3(0)}{\left[1+B_1(0)\right]^3}. \qquad (79)$$

By considering (76)–(79), we can easily have the following theorem when $q = 1$.

Theorem 25. *Suppose that* $f(t,0) = 0$ *and* $x = 0$ *is a one-side periodic solution of system* (52)$_{\varepsilon=0}$ *with* $P(0,0) = 0$ *and* $(\partial P/\partial y_0)(0,0) = -1$. *Let*

$$\overline{A}_1(\varepsilon) = -\left(1 + \widetilde{I}'(x^*,\varepsilon) \right)\exp\int_{t_0}^{t_0+T} F_x(t,x^*,\varepsilon)\,dt,$$

$$a_3^* = \frac{1}{6}\int_{t_0}^{t_0+2T} f_x'''(s,0)\,e^{2\int_{t_0}^{s} f_x(u,0)du}\,ds + \frac{2\widetilde{I}'''(0)}{\left[1+\widetilde{I}'(0)\right]^3}. \qquad (80)$$

If $a_3^* \neq 0$, *then for* $|\varepsilon|$ *sufficiently small,* $[\overline{A}_1(\varepsilon)+1]a_3^* > 0$ (≤ 0) *implies that system* (52) *has a unique (no) two-side periodic orbit of near* $x = 0$, *except for a one-side periodic orbit* $x^*(t,\varepsilon)$.

By virtue of Theorem 25, we can have the following conclusion.

Corollary 26. *(i) Let* $\overline{A}_1(0) = -1$, $a_3^* > 0$ *(<0). Then* $x = 0$ *is a nonhyperbolic one-side periodic orbit of system* (48) *($\varepsilon = 0$), which is asymptotically stable (unstable). (ii) Let* $\overline{A}_1(0) = -1$, $[\overline{A}_1(\varepsilon)+1]a_3^* > 0$, $0 < |\varepsilon| \ll 1$. *Then* $(\overline{A}_1(\varepsilon)+1) < 0$ *(>0), $x^*(t,\varepsilon)$ is a hyperbolic one-side periodic orbit of system* (48) *($\varepsilon = 0$), which is asymptotically stable (unstable. Moreover, the two-side periodic orbit is unstable (asymptotically stable) near* $x = 0$.

Finally, we give an example to illustrate it.

Example 27. Consider

$$\frac{dx}{dt} = \varepsilon^2 x + |\varepsilon|(\sin t)x^2 - (1-2\cos 2t)x^3, \quad t \neq k\pi,$$

$$\Delta x|_{t=k\pi} = -\varepsilon x(k\pi) + (b_3+\varepsilon)x^3(k\pi), \quad k \in \mathbb{Z}^+, \qquad (81)$$

where $|\varepsilon| > 0$. It is obvious that for $|\varepsilon| > 0$ sufficiently small, $x - \varepsilon x + (b_3+\varepsilon)x^3$ is strictly increasing, and then $x^*(t,\varepsilon) = 0$ is the unique π-periodic solution. By direct computation, we have $A_1(t,\varepsilon) = \varepsilon^2$, $A_2(t,\varepsilon) = |\varepsilon|(\sin t)$, $A_3(t,\varepsilon) = 2\cos 2t - 1$, $B_1(\varepsilon) = -\varepsilon$, and $B_3(\varepsilon) = b_3 + \varepsilon$. Therefore, $\overline{A}_1(\varepsilon) + 1 = (\varepsilon - 1)e^{\varepsilon^2\pi} + 1 = \varepsilon + o(\varepsilon)$, $a_3^* = 2b_3 - \pi$. It follows from Theorem 25 that system (81) has two (no) 2π-periodic solution of near $x = 0$ if $|\varepsilon|$ is sufficiently small and $a_3^*\varepsilon > 0$ (≤ 0).

Acknowledgments

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