Research Article

Dynamical Behaviors of the Stochastic Hopfield Neural Networks with Mixed Time Delays

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This paper investigates dynamical behaviors of the stochastic Hopfield neural networks with mixed time delays. The mixed time delays under consideration comprise both the discrete time-varying delays and the distributed time-delays. By employing the theory of stochastic functional differential equations and linear matrix inequality (LMI) approach, some novel criteria on asymptotic stability, ultimate boundedness, and weak attractor are derived. Finally, a numerical example is given to illustrate the correctness and effectiveness of our theoretical results.

1. Introduction

The well-known Hopfield neural networks were firstly introduced by Hopfield [1, 2] in early 1980s. Since then, both the mathematical analysis and practical applications of Hopfield neural networks have gained considerable research attention. The Hopfield neural networks have already been successfully applied in many different areas such as combinatorial optimization, knowledge acquisition, and pattern recognition, see, for example, [3–5]. In both the biological and artificial neural networks, the interactions between neurons are generally asynchronous, which give rise to the inevitable signal transmission delays. Also, in electronic implementation of analog neural networks, time delay is usually time-varying due to the finite switching speed of amplifiers. Note that continuously distributed delays have gained particular attention, since a neural network usually has a spatial nature due to the presence of an amount of parallel pathways of a variety of axon sizes and lengths.

Recently, it has been well recognized that stochastic disturbances are ubiquitous and inevitable in various systems, ranging from electronic implementations to biochemical systems, which are mainly caused by thermal noise and environmental fluctuations as well as different orders of ongoing events in the overall systems [6, 7]. Therefore, considerable

attentions have been paid to investigate the dynamics of stochastic neural networks, and many results on stochastic neural networks with delays have been reported in the literature, see, for example, [8–30] and references therein. Among which, some sufficient criteria on the stability of uncertain stochastic neural networks were derived in [8–10]. Almost sure exponential stability of stochastic neural networks was discussed in [11–15]. In [16–22], mean square exponential stability and pth moment exponential stability of stochastic neural networks were investigated; Some sufficient criteria on the exponential stability for impulsive stochastic neural networks were established in [23–26]. In [27], the stability of discrete-time stochastic neural networks was analyzed, while exponential stability of stochastic neural networks with Markovian jump parameters is investigated in [28–30]. These references mainly considered the stability of equilibrium point of stochastic neural networks. What do we study when the equilibrium point does not exist?

Except for stability property, boundedness and attractor are also foundational concepts of dynamical systems. They play an important role in investigating the uniqueness of equilibrium, global asymptotic stability, global exponential stability, the existence of periodic solution, its control and synchronization [31, 32], and so on. Recently, ultimate boundedness and attractor of several classes of neural networks

with time delays have been reported. Some sufficient criteria were derived in [33, 34], but these results hold only under constant delays. Following, in [35], the globally robust ultimate boundedness of integrodifferential neural networks with uncertainties and varying delays was studied. After that, some sufficient criteria on the ultimate boundedness of neural networks with both varying and unbounded delays were derived in [36], but the concerned systems are deterministic ones. In [37, 38], a series of criteria on the boundedness, global exponential stability, and the existence of periodic solution for nonautonomous recurrent neural networks were established. In [39–41], the ultimate boundedness and attractor of the stochastic Hopfield neural networks with time-varying delays were discussed. To the best of our knowledge, for stochastic neural networks with mixed time delays, there are few published results on the ultimate boundedness and weak attractor. Therefore, the arising questions about the ultimate boundedness, weak attractor, and asymptotic stability of the stochastic Hopfield neural networks with mixed time delays are important yet meaningful.

The left of the paper is organized as follows and some preliminaries are in Section 2, Section 3 presents our main results, a numerical example and conclusions will be in Sections 4 and 5, respectively.

2. Preliminaries

Consider the following stochastic Hopfield neural networks with mixed time delays:

$$
\begin{aligned}
dx(t) = \Big[& -Cx(t) + Af(x(t)) + Bf(x(t - \tau(t))) \\
& + D \int_{t-\tau(t)}^{t} g(x(s))\, ds + J \Big] dt \\
& + \left[\sigma_1 x(t) + \sigma_2 x(t - \tau(t)) \right] dw(t),
\end{aligned}
\tag{1}
$$

where $x = (x_1, \ldots, x_n)^T$ is the state vector associated with the neurons, $C = \text{diag}\{c_1, \ldots, c_n\}$, $c_i > 0$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when being disconnected from the network and the external stochastic perturbation; $A = (a_{ij})_{n \times n}$, $B = (b_{ij})_{n \times n}$ and $D = (d_{ij})_{n \times n}$ represent the connection weight matrix; $J = (J_1, \ldots, J_n)^T$, J_i denotes the external bias on the ith unit; f_j and g_j denote activation functions, $f(x(t)) = (f_1(x_1(t)), \ldots, f_n(x_n(t)))^T$, $g(x(t)) = (g_1(x_1(t)), \ldots, g_n(x_n(t)))^T$; $\sigma_1, \sigma_2 \in R^{n \times n}$ are the diffusion coefficient matrices; $w(t)$ is one-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$ generated by $\{w(s) : 0 \leq s \leq t\}$; there exists a positive constant τ such that the transmission delay $\tau(t)$ satisfies

$$
0 \leq \tau(t) \leq \tau.
\tag{2}
$$

The initial conditions are given in the following form:

$$
x(s) = \xi(s), \quad -\tau \leq s \leq 0, \ j = 1, \ldots, n,
\tag{3}
$$

where $\xi(s) = (\xi_1(s), \ldots, \xi_n(s))^T$ is $C([-\tau, 0]; R^n)$-valued function, $\mathcal{F}_0$-measurable R^n-valued random variable satisfying $\|\xi\|_\tau^2 = \sup_{-\tau \leq s \leq 0} E\|\xi(s)\|^2 < \infty$, $\|\cdot\|$ is the Euclidean norm, and $C([-\tau, 0]; R^n)$ is the space of all continuous R^n-valued functions defined on $[-\tau, 0]$.

Let $F(x_t, t) = -Cx(t) + Af(x(t)) + Bf(x(t - \tau(t))) + D \int_{t-\tau(t)}^{t} g(x(s))ds + J$, $G(x_t, t) = \sigma_1 x(t) + \sigma_2 x(t - \tau(t))$, where

$$
x_t = \{x(t + \theta) : -\tau \leq \theta \leq 0, t \geq 0\} = \varphi(\theta).
\tag{4}
$$

Then system (1) can be written by

$$
dx(t) = F(x_t, t)\, dt + G(x_t, t)\, dw(t).
\tag{5}
$$

Throughout this paper, the following assumption will be considered.

(A1) There exist constants l_i^-, l_i^+, m_i^- and m_i^+ such that

$$
\begin{aligned}
l_i^- &\leq \frac{f_i(x) - f_i(y)}{x - y} \leq l_i^+, \\
m_i^- &\leq \frac{g_i(x) - g_i(y)}{x - y} \leq m_i^+,
\end{aligned}
\quad \forall x, y \in R.
\tag{6}
$$

Remark 1. It follows from [42] that under the assumption (A1), system (1) has a global solution on $t \geq 0$. Moreover, under assumption 1, it is not difficult to prove that $F(x_t, t)$ and $G(x_t, t)$ satisfy the local Lipschitz condition in [43].

Remark 2. We note that assumption (A1) is less conservative than that in [8, 9, 39], since the constants l_i^-, l_i^+, m_i^- and m_i^+ are allowed to be positive, negative numbers, or zeros.

The notation $A > 0$ (resp., $A \geq 0$) means that matrix A is symmetric positive definite (resp., positive semidefinite). A^T denotes the transpose of the matrix A. $\lambda_{\min}(A)$ represents the minimum eigenvalue of matrix A. Denote by $C(R^n \times [-\tau, \infty); R^+)$ the family of continuous functions from $R^n \times [-\tau, \infty)$ to $R^+ = [0, \infty)$. Let $C^{2,1}(R^n \times [-\tau, \infty); R^+)$ be the family of all continuous nonnegative functions $V(x, t)$ defined on $R^n \times [-\tau, \infty)$ such that they are continuously twice differentiable in x and once in t. Given $V \in C^{2,1}(R^n \times [-\tau, \infty); R^+)$, we define the functional $\mathbb{L}V : C([-\tau, 0]; R^n) \times R^+ \to R$ by

$$
\begin{aligned}
\mathbb{L}V(\varphi, t) &= V_t(\varphi(0), t) + V_x(\varphi(0), t) F(\varphi, t) \\
&\quad + \frac{1}{2} \text{trace}\left[G^T(\varphi, t) V_{xx}(\varphi(0), t) G(\varphi, t) \right],
\end{aligned}
\tag{7}
$$

where $V_x(x, t) = (V_{x_1}(x, t), \ldots, V_{x_n}(x, t))$ and $V_{xx}(x, t) = (V_{x_i x_j}(x, t))_{n \times n}$.

The following lemmas will be used in establishing our main results.

Lemma 3 (see [44]). *For any positive definite matrix $P > 0$, scalar $\gamma > 0$, vector function $f : [0, \gamma] \to R^n$ such that the integrations concerned are well defined, and the following inequality holds:*

$$
\left(\int_0^\gamma f(s)\, ds \right)^T P \left(\int_0^\gamma f(s)\, ds \right) \leq \gamma \int_0^\gamma f^T(s) P f(s)\, ds.
\tag{8}
$$

Lemma 4 (see [43]). *Suppose that system (5) satisfies the local Lipschitz condition and the following assumptions hold.*

(A2) *There are two functions $V \in C^{2,1}(R^n \times [-\tau, \infty); R^+)$ and $U \in C(R^n \times [-\tau, \infty); R^+)$ and two probability measures $\mu(\cdot)$ and $\overline{\mu}(\cdot)$ on $[-\tau, 0]$ such that*

$$\lim_{\|x\| \to \infty} \inf_{0 \le t \le \infty} V(x,t) = \infty, \qquad (9)$$

while for all $(\varphi, t) \in C([-\tau, 0]; R^n) \times R^+$,

$$\mathbb{L}V(\varphi, t) \le \alpha_1 - \alpha_2 V(\varphi(0), t)$$
$$+ \alpha_3 \int_{-\tau}^0 V(\varphi(\theta), t + \theta) \, d\overline{\mu}(\theta)$$
$$- U(\varphi(0), t)$$
$$+ \alpha \int_{-\tau}^0 U(\varphi(\theta), t + \theta) \, d\mu(\theta), \qquad (10)$$

where $\alpha_1 \ge 0$, $\alpha_2 > \alpha_3 \ge 0$ and $\alpha \in (0, 1)$.

(A3) *If there is a pair of positive constants c and p such that*

$$c\|x\|^p \le V(x,t), \quad \forall (x,t) \in R^n \times [-\tau, \infty). \qquad (11)$$

Then the unique global solution $x(t)$ to system (5) obeys

$$\limsup_{t \to \infty} E\|x(t)\|^p \le \frac{\alpha_1}{c\varepsilon}, \qquad (12)$$

where $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ while $\varepsilon_2 = -\ln(\alpha)/\tau$ and $\varepsilon_1 > 0$ is the unique root to the following equation:

$$\alpha_2 = \varepsilon_1 + \alpha_3 e^{\varepsilon_1 \tau}. \qquad (13)$$

If, furthermore, $\alpha_1 = 0$, then

$$\limsup_{t \to \infty} \frac{1}{t} \ln \left(E\|x(t)\|^p \right) \le -\varepsilon,$$
$$\limsup_{t \to \infty} \frac{1}{t} \ln (\|x(t)\|) \le -\frac{\varepsilon}{p} \quad \text{almost surely.} \qquad (14)$$

3. Main Results

Theorem 5. *Suppose that there exist some matrices $P > 0$, $U_i = \text{diag}\{u_{i1}, \ldots, u_{in}\} \ge 0$ $(i = 1, 2, 3)$ and positive constants $\gamma_1, \gamma_2, \lambda$ such that $\lambda^{-1} \tau \gamma_2^{-1} \in (0, 1)$ and*

$$\Sigma = \begin{pmatrix} \Delta_1 & 0 & PA + L_2 U_1 & PB & U_3 M_2 \\ * & \Delta_2 & 0 & L_2 U_2 & 0 \\ * & * & \Delta_3 & 0 & 0 \\ * & * & * & \Delta_4 & 0 \\ * & * & * & * & \Delta_5 \end{pmatrix} < 0, \qquad (15)$$

where $\Delta_1 = (\gamma_1 + 2\lambda)P + 2\sigma_1^T P\sigma_1 - PC - CP + U_1(\lambda I - 2L_1) + U_3(\lambda I - 2M_1)$, $\Delta_2 = 2\sigma_2^T P\sigma_2 + (\lambda I - 2L_1)U_2$, $\Delta_3 = 2(\lambda - 1)U_1$, $\Delta_4 = 2(\lambda - 1)U_2$, $\Delta_5 = 2(\lambda - 1)U_3 + \gamma_2 D^T PD$, $L_1 = \text{diag}\{l_1^- l_1^+, \ldots, l_n^- l_n^+\}$, $L_2 = \text{diag}\{l_1^- + l_1^+, \ldots, l_n^- + l_n^+\}$, $M_1 = \text{diag}\{m_1^- m_1^+, \ldots, m_n^- m_n^+\}$, $M_2 = \text{diag}\{m_1^- + m_1^+, \ldots, m_n^- + m_n^+\}$, $*$ means the symmetric terms.

Then, the following results hold.

(i) *System (1) is stochastically ultimately bounded; that is, for any $\delta \in (0, 1)$, there exists a positive constant $C = C(\delta)$ such that the solution $x(t)$ of system (1) satisfies*

$$\limsup_{t \to \infty} P\{\|x(t)\| \le C\} \ge 1 - \delta. \qquad (16)$$

(ii) *If $\alpha_1 = 0$, where $\alpha_1 = \max(\gamma_3, 0)$, $\varepsilon > 0$ is the same as defined in Lemma 4,*

$$\gamma_3 = \lambda^{-1} J^T PJ$$
$$+ \sum_{i=1}^n (u_{1i} + u_{2i})$$
$$\times \left\{ -2f_i^2(0) + 2\lambda^{-1} f_i^2(0) \right.$$
$$\left. + \lambda^{-1}(l_i^+ + l_i^-)^2 f_i^2(0) \right\}$$
$$+ \sum_{i=1}^n u_{3i} \left\{ -2g_i^2(0) + 2\lambda^{-1} g_i^2(0) \right.$$
$$\left. + \lambda^{-1}(m_i^+ + m_i^-)^2 g_i^2(0) \right\}, \qquad (17)$$

then

$$\limsup_{t \to \infty} \frac{1}{t} \ln \left(E\|x(t)\|^2 \right) \le -\varepsilon,$$
$$\limsup_{t \to \infty} \frac{1}{t} \ln (\|x(t)\|) \le -\frac{\varepsilon}{2} \quad \text{almost surely.} \qquad (18)$$

Proof. Let the Lyapunov function $V(x,t) = x^T(t)Px(t)$. Applying Itô's formula in [42] to $V(t)$ along with system (1), one may obtain the following:

$$dV(x,t) = 2x^T(t)P[\sigma_1 x(t) + \sigma_2 x(t - \tau(t))] \, dw(t)$$
$$+ \mathbb{L}V(\varphi, t) \, dt, \qquad (19)$$

where

$$\mathbb{L}V(\varphi, t) = 2x^T(t)P\left[-Cx(t) + Af(x(t)) + Bf(x(t - \tau(t))) \right.$$
$$\left. + D\int_{t-\tau(t)}^t g(x(s)) \, ds + J \right]$$
$$+ [\sigma_1 x(t) + \sigma_2 x(t - \tau(t))]^T$$
$$\times P[\sigma_1 x(t) + \sigma_2 x(t - \tau(t))]$$
$$\le 2x^T(t)P[-Cx(t) + Af(x(t))$$
$$+ Bf(x(t - \tau(t)))]$$

$$+ 2\lambda x^T(t) Px(t)$$
$$+ \lambda^{-1} J^T PJ + \lambda^{-1} \left(\int_{t-\tau(t)}^{t} Dg(x(s))\,ds \right)^T$$
$$\times P\left(\int_{t-\tau(t)}^{t} Dg(x(s))\,ds \right)$$
$$+ 2\left[x^T(t) \sigma_1^T P\sigma_1 x(t) \right.$$
$$\left. + x^T(t-\tau(t)) \sigma_2^T P\sigma_2 x(t-\tau(t)) \right].$$

$$(20)$$

From Lemma 3, it follows that

$$\lambda^{-1} \left(\int_{t-\tau(t)}^{t} Dg(x(s))\,ds \right)^T P \left(\int_{t-\tau(t)}^{t} Dg(x(s))\,ds \right)$$
$$\leq \lambda^{-1} \left(\int_{t-\tau}^{t} Dg(x(s))\,ds \right)^T P \left(\int_{t-\tau}^{t} Dg(x(s))\,ds \right)$$
$$\leq \lambda^{-1}\tau \int_{t-\tau}^{t} g^T(x(s)) D^T PDg(x(s))\,ds$$
$$= \lambda^{-1}\tau \int_{-\tau}^{0} g^T(x(t+\theta)) D^T PDg(x(t+\theta))\,d\theta$$
$$= \lambda^{-1}\tau \int_{-\tau}^{0} g^T(\varphi(\theta)) D^T PDg(\varphi(\theta))\,d\theta.$$

$$(21)$$

From (A1), it follows that for $i = 1,\ldots,n$,

$$0 \leq -2\sum_{i=1}^{n} u_{1i} \left[f_i(x_i(t)) - f_i(0) - l_i^+ x_i(t) \right]$$
$$\times \left[f_i(x_i(t)) - f_i(0) - l_i^- x_i(t) \right]$$
$$= -2\sum_{i=1}^{n} u_{1i} \left\{ f_i^2(x_i(t)) - (l_i^+ + l_i^-) x_i(t) f_i(x_i(t)) \right.$$
$$+ l_i^+ l_i^- x_i^2(t) + f_i^2(0) - 2f_i(0) f_i(x_i(t))$$
$$\left. + (l_i^+ + l_i^-) x_i(t) f_i(0) \right\}$$
$$= -2\sum_{i=1}^{n} u_{1i} \left\{ f_i^2(x_i(t)) - (l_i^+ + l_i^-) x_i(t) f_i(x_i(t)) \right.$$
$$\left. + l_i^+ l_i^- x_i^2(t) \right\}$$
$$+ \sum_{i=1}^{n} u_{1i} \left[-2f_i^2(0) + 4f_i(0) f_i(x_i(t)) \right.$$
$$\left. - 2(l_i^+ + l_i^-) x_i(t) f_i(0) \right]$$

$$\leq -2\sum_{i=1}^{n} u_{1i} \left\{ f_i^2(x_i(t)) - (l_i^+ + l_i^-) x_i(t) f_i(x_i(t)) \right.$$
$$\left. + l_i^+ l_i^- x_i^2(t) \right\}$$
$$+ \sum_{i=1}^{n} u_{1i} \left\{ -2f_i^2(0) + 2\left[\lambda f_i^2(x_i(t)) + \lambda^{-1} f_i^2(0) \right] \right.$$
$$\left. + \left[\lambda x_i^2(t) + \lambda^{-1}(l_i^+ + l_i^-)^2 f_i^2(0) \right] \right\}$$
$$= 2(\lambda-1) f^T(x(t)) U_1 f(x(t)) + 2f^T(x(t)) U_1 L_2 x(t)$$
$$+ x^T(t) U_1 (\lambda I - 2L_1) x(t)$$
$$+ \sum_{i=1}^{n} u_{1i} \left\{ -2f_i^2(0) + 2\lambda^{-1} f_i^2(0) + \lambda^{-1}(l_i^+ + l_i^-)^2 f_i^2(0) \right\}.$$

$$(22)$$

Similarly, one derives that

$$0 \leq -2\sum_{i=1}^{n} u_{2i} \left[f_i(x_i(t-\tau(t))) - f_i(0) - l_i^+ x_i(t-\tau(t)) \right]$$
$$\times \left[f_i(x_i(t-\tau(t))) - f_i(0) - l_i^- x_i(t-\tau(t)) \right]$$
$$\leq 2(\lambda-1) f^T(x(t-\tau(t))) U_2 f(x(t-\tau(t)))$$
$$+ 2f^T(x(t-\tau(t))) U_2 L_2 x(t-\tau(t))$$
$$+ x^T(t-\tau(t)) U_2 (\lambda I - 2L_1) x(t-\tau(t))$$
$$+ \sum_{i=1}^{n} u_{2i} \left\{ -2f_i^2(0) + 2\lambda^{-1} f_i^2(0) + \lambda^{-1}(l_i^+ + l_i^-)^2 f_i^2(0) \right\},$$

$$0 \leq -2\sum_{i=1}^{n} u_{3i} \left[g_i(x_i(t)) - g_i(0) - m_i^+ x_i(t) \right]$$
$$\times \left[g_i(x_i(t)) - g_i(0) - m_i^- x_i(t) \right]$$
$$\leq 2(\lambda-1) g^T(x(t)) U_3 g(x(t)) + 2g^T(x(t)) U_3 M_2 x(t)$$
$$+ x^T(t) U_3 (\lambda I - 2M_1) x(t)$$
$$+ \sum_{i=1}^{n} u_{3i} \left\{ -2g_i^2(0) + 2\lambda^{-1} g_i^2(0) + \lambda^{-1}(m_i^+ + m_i^-)^2 g_i^2(0) \right\}.$$

$$(23)$$

Further from (20)–(23), one derives

$$\mathbb{L}V(\varphi,t) \leq 2x^T(t) P \left[-Cx(t) + Af(x(t)) + Bf(x(t-\tau(t))) \right]$$
$$+ 2\lambda x^T(t) Px(t)$$
$$+ \lambda^{-1}\tau \int_{-\tau}^{0} g^T(\varphi(\theta)) D^T PDg(\varphi(\theta))\,d\theta$$

$$+ 2 \left[x^T (t) \sigma_1^T P \sigma_1 x (t) \right.$$

$$+ x^T (t - \tau (t)) \sigma_2^T P \sigma_2 x (t - \tau (t)) \Big]$$

$$+ 2 (\lambda - 1) f^T (x (t)) U_1 f (x (t))$$

$$+ 2 f^T (x (t)) U_1 L_2 x (t)$$

$$+ x^T (t) U_1 \left(\lambda I - 2 L_1 \right) x (t)$$

$$+ 2 (\lambda - 1) f^T (x (t - \tau (t))) U_2 f (x (t - \tau (t)))$$

$$+ 2 f^T (x (t - \tau (t))) U_2 L_2 x (t - \tau (t))$$

$$+ x^T (t - \tau (t)) U_2 \left(\lambda I - 2 L_1 \right) x (t - \tau (t))$$

$$+ 2 (\lambda - 1) g^T (x (t)) U_3 g (x (t))$$

$$+ 2 g^T (x (t)) U_3 M_2 x (t)$$

$$+ x^T (t) U_3 \left(\lambda I - 2 M_1 \right) x (t)$$

$$+ \gamma_3 + \gamma_1 x^T (t) P x (t) - \gamma_1 V (\varphi (0), t)$$

$$+ \gamma_2 g^T (x (t)) D^T P D g (x (t))$$

$$- \gamma_2 g^T (\varphi (0)) D^T P D g (\varphi (0))$$

$$\leq \eta^T (t) \Sigma \eta (t) - \gamma_1 V (\varphi (0), t)$$

$$- \gamma_2 g^T (\varphi (0)) D^T P D g (\varphi (0)) + \gamma_3$$

$$+ \lambda^{-1} \tau \int_{-\tau}^{0} g^T (\varphi (\theta)) D^T P D g (\varphi (\theta)) \, d\theta$$

$$\leq \gamma_3 - \gamma_1 V (\varphi (0), t) - U (\varphi (0), t)$$

$$+ \lambda^{-1} \tau \gamma_2^{-1} \int_{-\tau}^{0} U (\varphi (\theta), t + \theta) \, d\theta,$$

$$(24)$$

where $\eta(t) = (x^T(t), x^T(t - \tau(t)), f^T(x(t)), f^T(x(t - \tau(t))),$ $g^T(x(t)))^T$, $\varphi(0) = x(t)$, $U(x, t) = \gamma_2 g^T(x) D^T P D g(x)$,

$$\gamma_3 = \lambda^{-1} J^T P J$$

$$+ \sum_{i=1}^{n} \left(u_{1i} + u_{2i} \right)$$

$$\times \left\{ -2 f_i^2 (0) + 2 \lambda^{-1} f_i^2 (0) + \lambda^{-1} (l_i^+ + l_i^-)^2 f_i^2 (0) \right\}$$

$$+ \sum_{i=1}^{n} u_{3i} \left\{ -2 g_i^2 (0) + 2 \lambda^{-1} g_i^2 (0) \right.$$

$$+ \lambda^{-1} (m_i^+ + m_i^-)^2 g_i^2 (0) \Big\}.$$

$$(25)$$

Let $\alpha_1 = \max(\gamma_3, 0)$, $\alpha_2 = \gamma_1$, $\alpha_3 = 0$, $\alpha = \lambda^{-1} \tau \gamma_2^{-1}$, $c = \lambda_{\min}(P)$. Then it follows from Lemma 4 that

$$\limsup_{t \to \infty} E \| x (t) \|^2 \leq \frac{\alpha_1}{\lambda_{\min}(P) \varepsilon} \leq \frac{\gamma_4}{\lambda_{\min}(P) \varepsilon}, \quad (26)$$

where $\varepsilon > 0$ is the same as defined in Lemma 4,

$$\gamma_4 = \lambda^{-1} J^T P J$$

$$+ \sum_{i=1}^{n} \left(u_{1i} + u_{2i} \right) \left\{ 2 \lambda^{-1} f_i^2 (0) + \lambda^{-1} (l_i^+ + l_i^-)^2 f_i^2 (0) \right\}$$

$$+ \sum_{i=1}^{n} u_{3i} \left\{ 2 \lambda^{-1} g_i^2 (0) + \lambda^{-1} (m_i^+ + m_i^-)^2 g_i^2 (0) \right\}.$$

$$(27)$$

Therefore, for any $\delta > 0$, it follows from Chebyshev's inequality that

$$\limsup_{t \to \infty} P \left\{ \| x (t) \| > \sqrt{\frac{\gamma_4}{\delta \lambda_{\min}(P) \varepsilon}} \right\}$$

$$\leq \limsup_{t \to \infty} \frac{E \| x (t) \|^2}{\gamma_4 / \delta \lambda_{\min}(P) \varepsilon}$$

$$\leq \delta.$$

$$(28)$$

If, furthermore, $\alpha_1 = 0$, then it follows from Lemma 4 that (ii) holds. □

Theorem 5 shows that there exists $t_0 > 0$ such that for any $t \geq t_0$, $P\{\|x(t)\| \leq C\} \geq 1 - \delta$. Let B_C denote by

$$B_C = \left\{ x \in R^n \mid \| x (t) \| \leq C, t \geq t_0 \right\}. \quad (29)$$

Clearly, B_C is closed, bounded, and invariant. Moreover,

$$\limsup_{t \to \infty} \inf_{y \in B_C} \| x (t) - y \| = 0, \quad (30)$$

with no less than probability $1 - \delta$, which means that B_C attracts the solutions infinitely many times with no less than probability $1 - \delta$, so we may say that B_C is a weak attractor for the solutions.

Theorem 6. *Suppose that all conditions of Theorem 5 hold, then there exists a weak attractor B_C for the solutions of system* (1).

Remark 7. Compared with [39–41], assumption (A1) is less conservative than that in [39] and system (1) includes mixed time delays, which is more complex than that in [39–41]. In addition, Lemma 4 is firstly used to investigate the dynamical behaviors of stochastic neural networks with mixed time delays. The bound for $\mathbb{L}V$ may be in a much weaker form. Our results do not only deal with the asymptotic moment estimation but also the path wise (almost sure) estimation.
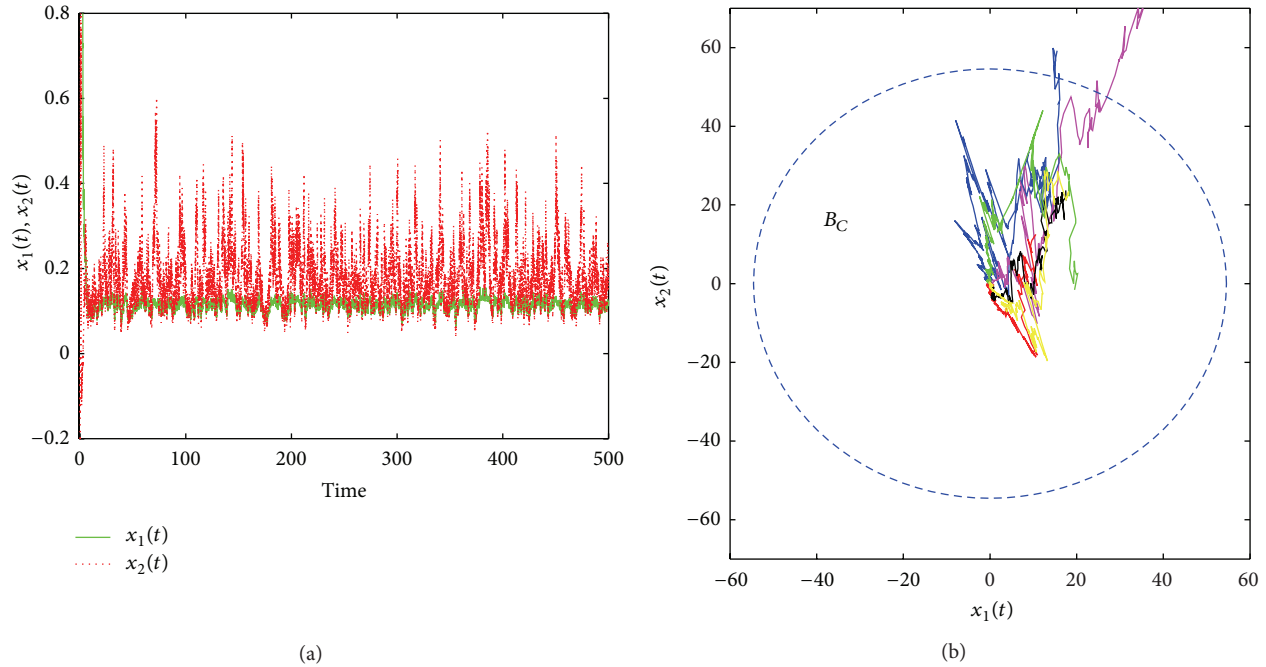
(a)

(b)

FIGURE 1: Time trajectories (a) as well as the set B_C and several typical phase portraits (b) for the system in Example 8 (color online). Where initial values for $t < 0$ are chosen as $x(t) = (50, 80)$. For (b), only phase portraits for $t \geq 0$ are shown.

4. Numerical Example

In this section, a numerical example is presented to demonstrate the validity and effectiveness of our theoretical results.

Example 8. Consider the following stochastic Hopfield neural networks with mixed time delays:

$$dx(t) = \left[-Cx(t) + Af(x(t)) + Bf(x(t - \tau(t))) \right. $$

$$\left. + D \int_{t-\tau(t)}^{t} g(x(s)) \, ds + J \right] dt \qquad (31)$$

$$+ \left[\sigma_1 x(t) + \sigma_2 x(t - \tau(t)) \right] dw(t),$$

where $J = (0.01, 0.01)^T$, $\tau(t) = 0.2|\sin t|$,

$$A = \begin{pmatrix} -0.2 & 0.3 \\ 0.2 & -0.5 \end{pmatrix}, \qquad B = \begin{pmatrix} 0.5 & -1 \\ -1.4 & 0.8 \end{pmatrix},$$

$$C = \begin{pmatrix} 1.2 & 0 \\ 0 & 1.2 \end{pmatrix}, \qquad D = \begin{pmatrix} 0.3 & -0.1 \\ 0.1 & 0.4 \end{pmatrix}, \qquad (32)$$

$$\sigma_1 = \begin{pmatrix} 0.2 & 0.1 \\ 0.3 & 0.2 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0.1 & -0.2 \\ -0.2 & 0.3 \end{pmatrix},$$

and $f(x) = g(x) = \tanh(x)$, $w(t)$ is one-dimensional Brownian motion. Then $L_1 = M_1 = 0$, $L_2 = M_2 = \text{diag}\{1, 1\}$. By using the Matlab LMI Control Toolbox [45], for $\gamma_1 = \gamma_2 = \lambda = 0.5$, $\tau = 0.2$, based on Theorem 5, such system is stochastically

ultimately bounded when P, U_1, U_2 and U_3 are chosen as follows:

$$P = \begin{pmatrix} 0.3278 & 0.2744 \\ 0.2744 & 0.3567 \end{pmatrix}, \qquad U_1 = \begin{pmatrix} 0.0668 & 0 \\ 0 & 0.2255 \end{pmatrix},$$

$$U_2 = \begin{pmatrix} 0.1263 & 0 \\ 0 & 0.1218 \end{pmatrix}, \qquad U_3 = \begin{pmatrix} 0.0901 & 0 \\ 0 & 0.1031 \end{pmatrix}. \tag{33}$$

For $\delta = 0.01$, $\gamma_1 = \gamma_2 = \lambda = 0.5$, $\tau = 0.2$, we obtain $\varepsilon = 0.5$ and constant $C = C(\delta) = \sqrt{J^T P J / \lambda_{\min}(P) \delta \lambda \varepsilon} = \sqrt{0.1141/0.0675} = 1.3001$. Then $B_C = \{x \in R^2 \mid \|x(t)\| \leq 1.3001, t \geq t_0\}$, $P(x \in B_C) \geq 0.99$. For the system in Example 8 (color online), Figure 1(a) shows time trajectories, and Figure 1(b) shows the set B_C and several typical phase portraits, where initial value for $t < 0$ is chosen as $x(t) = (50, 80)$. In Figure 1(b), only phase portraits for $t \geq 0$ are shown. From Figure 1, one can easily find that these trajectories are almost all attracted by the set B_C.

5. Conclusion

In this paper, by using the theory of stochastic functional differential equations and linear matrix inequality, new results and sufficient criteria on the asymptotic stability, ultimate boundedness, and attractor of stochastic Hopfield neural networks with mixed time delays are established. A numerical example is also presented to demonstrate the correctness of our theoretical results.

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