Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 573583, 6 pages
http://dx.doi.org/10.1155/2013/573583

Research Article

A New Iterative Method for Solving a System of Generalized Mixed Equilibrium Problems for a Countable Family of Generalized Quasi-ϕ-Asymptotically Nonexpansive Mappings

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Received 3 October 2012; Accepted 4 January 2013

Academic Editor: Satit Saejung

By using a specific way of choosing the indexes, we introduce an up-to-date iterative algorithm for approximating common fixed points of a countable family of generalized quasi-ϕ-asymptotically nonexpansive mappings and obtain a strong convergence theorem under some suitable conditions. As application, an iterative solution to a system of generalized mixed equilibrium problems is studied. The results extend those of other authors, in which the involved mappings consist of just finite families.

1. Introduction

Throughout this paper, we assume that E is a real Banach space with its dual E^*, C is a nonempty closed convex subset of E, and $J : E \to 2^{E^*}$ is the *normalized duality mapping* defined by

$$Jx = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \quad \forall x \in E. \tag{1}$$

In the sequel, we use $F(T)$ to denote the set of fixed points of a mapping T.

Definition 1. (1) [1] A mapping $T : C \to C$ is said to be *generalized quasi-ϕ-asymptotically nonexpansive* in the light of [1], if $F(T) \neq \emptyset$, and there exist nonnegative real sequences $\{\nu_n\}$ and $\{\mu_n\}$ with $\nu_n, \mu_n \to 0$ (as $n \to \infty$) such that

$$\phi(p, T^n x) \le (1+\nu_n)\phi(p, x) + \mu_n, \quad \forall n \ge 1, \; x \in C, \; p \in F(T), \tag{2}$$

where $\phi : E \times E \to \mathbb{R}$ denotes the *Lyapunov functional* defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \tag{3}$$

It is obvious from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2. \tag{4}$$

(2) A mapping $T : C \to C$ is said to be *uniformly L-Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \le L\|x - y\|, \quad \forall n \ge 1, \; x, y \in C. \tag{5}$$

Example 2. Let C be a unit ball in a real Hilbert space l^2, and let $T : C \to C$ be a mapping defined by

$$T(x_1, x_2, \ldots) = \left(0, x_1^2, a_2 x_2, a_3 x_3, \ldots\right), \tag{6}$$

where $\{a_i\}$ is a sequence in $(0, 1)$ satisfying $\prod_{i=2}^{\infty} a_i = 1/2$. It is shown by Goebel and Kirk [2] that

$$\phi(p, T^n y) \le (1+\nu_n)\phi(p, y) + \mu_n, \quad \forall n \ge 1, \; y \in C, \; p \in F(T), \tag{7}$$

where $\phi(x, y) = \|x - y\|^2$, $\nu_n = (2\prod_{i=2}^{n} a_i)^2 - 1$, for all $n \ge 1$, and $\{\mu_n\}$ is a nonnegative real sequence with $\mu_n \to 0$ as $n \to \infty$. This shows that the mapping T defined earlier is a generalized quasi-ϕ-asymptotically nonexpansive mapping.

Let $\theta : C \times C \to \mathbb{R}$ be a bifunction, $\psi : C \to \mathbb{R}$ a real valued function, and $B : C \to E^*$ a nonlinear mapping. The so-called *generalized mixed equilibrium problem* (GMEP) is to find a $u \in C$ such that

$$\theta(u, y) + \langle y - u, Bu \rangle + \psi(y) - \psi(u) \geq 0, \quad \forall y \in C, \tag{8}$$

whose set of solutions is denoted by $\Omega(\theta, B, \psi)$.

The equilibrium problem is a unifying model for several problems arising in physics, engineering, science optimization, economics, transportation, network and structural analysis, Nash equilibrium problems in noncooperative games, and others. It has been shown that variational inequalities and mathematical programming problems can be viewed as a special realization of the abstract equilibrium problems. Many authors have proposed some useful methods to solve the (*equilibrium problem*) EP, (*generalized equilibrium problem*) GEP, (*mixed equilibrium problem*) MEP, and GMEP. Concerning the weak and strong convergence of iterative sequences to approximate a common element of the set of solutions for the GMEP, the set of solutions to variational inequality problems and the set of common fixed points for relatively nonexpansive mappings, quasi-ϕ-nonexpansive mappings, and quasi-ϕ-asymptotically nonexpansive mappings have been studied by many authors in the setting of Hilbert or Banach spaces (e.g., see [3–16] and the references therein).

Inspired and motivated by the study mentioned earlier, in this paper, by using a specific way of choosing the indexes, we propose an up-to-date iteration scheme for approximating common fixed points of a countable family of generalized quasi-ϕ-asymptotically nonexpansive mappings and obtain a strong convergence theorem for solving a system of generalized mixed equilibrium problems. The results extend those of the authors, in which the involved mappings consist of just finite families.

2. Preliminaries

A Banach space E is *strictly convex* if the following implication holds for $x, y \in E$:

$$\|x\| = \|y\| = 1, \quad x \neq y \Longrightarrow \left\| \frac{x+y}{2} \right\| < 1. \tag{9}$$

It is also said to be *uniformly convex* if for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \qquad \|x - y\| \geq \epsilon \Longrightarrow \left\| \frac{x+y}{2} \right\| \leq 1 - \delta. \tag{10}$$

It is known that if E is uniformly convex Banach space, then E is reflexive and strictly convex. A Banach space E is said to be *smooth* if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{11}$$

exists for each $x, y \in S(E) := \{x \in E : \|x\| = 1\}$. In this case, the norm of E is said to be *Gâteaux differentiable*. The space E is said to have *uniformly Gâteaux differentiable norm* if for each $y \in S(E)$, the limit (11) is attained uniformly for $x \in S(E)$. The norm of E is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit (11) is attained uniformly for $y \in S(E)$. The norm of E is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit (11) is attained uniformly for $x, y \in S(E)$. Note that E (E^*, resp.) is uniformly convex $\Leftrightarrow$ E^* (E, resp.) is uniformly smooth.

Following Alber [17], the *generalized projection* $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = \arg \inf_{y \in C} \phi(y, x), \quad \forall x \in E. \tag{12}$$

Lemma 3 (see [17]). *Let E be a smooth, strictly convex, and reflexive Banach space, and let C be a nonempty closed convex subset of E. Then, the following conclusions hold:*

(1) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y)$ *for all* $x \in C$ *and* $y \in E$;

(2) *if* $x \in E$ *and* $z \in C$, *then* $z = \Pi_C x \Leftrightarrow \langle z - y, Jx - Jz \rangle \geq 0$, *for all* $y \in C$;

(3) *for* $x, y \in E$, $\phi(x, y) = 0$ *if and only if* $x = y$.

Remark 4. The following basic properties for a Banach space E can be found in Cioranescu [18].

(i) If E is uniformly smooth, then J is uniformly continuous on each bounded subset of E.

(ii) If E is reflexive and strictly convex, then J^{-1} is norm-weak continuous.

(iii) If E is a smooth, strictly convex, and reflexive Banach space, then the normalized duality mapping $J : E \to 2^{E^*}$ is single valued, one-to-one, and onto.

(iv) A Banach space E is uniformly smooth if and only if E^* is uniformly convex.

(v) Each uniformly convex Banach space E has the *Kadec-Klee property*; that is, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$, where $x_n \rightharpoonup x$ denotes that $\{x_n\}$ converges weakly to x.

Lemma 5 (see [19]). *Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, and let C be a nonempty closed convex subset of E. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in C such that $x_n \to p$ and $\phi(x_n, y_n) \to 0$, where ϕ is the function defined by* (3)*; then, $y_n \to p$.*

Lemma 6 (see [1]). *Let E and C be the same as those in Lemma 5. Let $T : C \to C$ be a closed and generalized quasi-ϕ-asymptotically nonexpansive mapping with nonnegative real sequences $\{\nu_n\}$ and $\{\mu_n\}$; then, the fixed point set $F(T)$ of T is a closed and convex subset of C.*

Lemma 7 (see [20]). *Let E be a real uniformly convex Banach space, and let $B_r(0)$ be the closed ball of E with center at the origin and radius $r > 0$. Then, there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\alpha x + \beta y\|^2 \leq \alpha \|x\|^2 + \beta \|y\|^2 - \alpha \beta g(\|x - y\|) \tag{13}$$

for all $x, y, \in B_r(0)$, and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$.

3. Main Results

Theorem 8. *Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, C a nonempty closed convex subset of E, and $T_i : C \to C$, $i = 1, 2, \ldots$ a countable family of closed and generalized quasi-ϕ-asymptotically nonexpansive mappings with nonnegative real sequences $\{\nu_n^{(i)}\}$ and $\{\mu_n^{(i)}\}$ satisfying $\nu_n^{(i)} \to 0$ and $\mu_n^{(i)} \to 0$ (as $n \to \infty$ and for each $i \geq 1$), and each T_i is uniformly L_i-Lipschitz continuous. Let $\{\alpha_n\}$ be a sequence in $[0, \epsilon]$ for some $\epsilon \in (0, 1)$, and let $\{\beta_n\}$ be a sequence in $[0, 1]$ satisfying $0 < \liminf_{n\to\infty}\beta_n(1 - \beta_n)$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}
&x_1 \in C; \quad C_1 = C,\\
&y_n = J^{-1}\left[\alpha_n J x_n + (1 - \alpha_n) J z_n\right],\\
&z_n = J^{-1}\left[\beta_n J x_n + (1 - \beta_n) J T_{i_n}^{m_n} x_n\right],\\
&C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n) + \xi_n\},\\
&x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1,
\end{aligned} \tag{14}$$

where $\xi_n := \nu_{m_n}^{(i_n)} \sup_{p\in F}\phi(p, x_n) + \mu_{m_n}^{(i_n)}$, $\Pi_{C_{n+1}}$ is the generalized projection of E onto C_{n+1}, and i_n and m_n satisfy the positive integer equation: $n = i + (m - 1)m/2$, $m \geq i$ $(m \geq i, n = 1, 2, \ldots)$; that is, for each $n \geq 1$, there exist unique i_n and m_n such that

$$\begin{gathered}
i_1 = 1, \quad i_2 = 1, \quad i_3 = 2, \quad i_4 = 1, \quad i_5 = 2,\\
i_6 = 3, \quad i_7 = 1, \quad i_8 = 2, \ldots;\\
m_1 = 1, \quad m_2 = 2, \quad m_3 = 2, \quad m_4 = 3, \quad m_5 = 3,\\
m_6 = 3, \quad m_7 = 4, \quad m_8 = 4, \ldots.
\end{gathered} \tag{15}$$

If $F := \bigcap_{i=1}^{\infty} F(T_i)$ is nonempty and bounded, then $\{x_n\}$ converges strongly to $\Pi_F x_1$.

Proof. We divide the proof into several steps.

(I) F and C_n (for all $n \geq 1$) both are closed and convex subsets in C.

In fact, it follows from Lemma 6 that each $F(T_i)$ is a closed and convex subset of C, so is F. In addition, with C_1 $(=C)$ being closed and convex, we may assume that C_n is closed and convex for some $n \geq 2$. In view of the definition of ϕ, we have that

$$C_{n+1} = \{v \in C : \varphi(v) \leq a\} \cap C_n, \tag{16}$$

where $\varphi(v) = 2\langle v, Jx_n - Jy_n\rangle$, and $a = \|x_n\|^2 - \|y_n\|^2 + \xi_n$. This shows that C_{n+1} is closed and convex.

(II) F is a subset of $\bigcap_{n=1}^{\infty} C_n$.

It is obvious that $F \subset C_1$. Suppose that $F \subset C_n$ for some $n \geq 2$. Since E is uniformly smooth, E^* is uniformly convex. Then, for any $p \in F \subset C_n$, we have that

$$\begin{aligned}
\phi(p, y_n) &= \phi\left(p, J^{-1}\left[\alpha_n J x_n + (1 - \alpha_n) J x_n\right]\right)\\
&= \|p\|^2 - 2\langle p, \alpha_n J x_n + (1 - \alpha_n) J z_n\rangle\\
&\quad + \|\alpha_n J x_n + (1 - \alpha_n) J z_n\|^2\\
&\leq \|p\|^2 - 2\alpha_n\langle p, J x_n\rangle - 2(1 - \alpha_n)\langle p, J z_n\rangle\\
&\quad + \alpha_n\|x_n\|^2 + (1 - \alpha_n)\|z_n\|^2\\
&= \alpha_n\phi(p, x_n) + (1 - \alpha_n)\phi(p, z_n).
\end{aligned} \tag{17}$$

Furthermore, it follows from Lemma 7 that for any $p \in F$, we have that

$$\begin{aligned}
\phi(p, z_n) &= \phi\left(p, J^{-1}\left[\beta_n J x_n + (1 - \beta_n) J T_{i_n}^{m_n} x_n\right]\right)\\
&= \|p\|^2 - 2\left\langle p, \beta_n J x_n + (1 - \beta_n) J T_{i_n}^{m_n} x_n\right\rangle\\
&\quad + \left\|\beta_n J x_n + (1 - \beta_n) J T_{i_n}^{m_n} x_n\right\|^2\\
&\leq \|p\|^2 - 2\beta_n\langle p, J x_n\rangle - 2(1 - \beta_n)\\
&\quad \times \left\langle p, J T_{i_n}^{m_n} x_n\right\rangle + \beta_n\|x_n\|^2 + (1 - \beta_n)\left\|T_{i_n}^{m_n} x_n\right\|^2\\
&\quad - \beta_n(1 - \beta_n)\, g\left(\left\|J x_n - J T_{i_n}^{m_n} x_n\right\|\right)\\
&= \beta_n\phi(p, x_n) + (1 - \beta_n)\phi\left(p, T_{i_n}^{m_n} x_n\right)\\
&\quad - \beta_n(1 - \beta_n)\, g\left(\left\|J x_n - J T_{i_n}^{m_n} x_n\right\|\right)\\
&\leq \beta_n\phi(p, x_n) + (1 - \beta_n)\\
&\quad \times \left[\phi(p, x_n) + \nu_{m_n}^{(i_n)}\sup_{p\in F}\phi(p, x_n) + \mu_{m_n}^{(i_n)}\right]\\
&\quad - \beta_n(1 - \beta_n)\, g\left(\left\|J x_n - J T_{i_n}^{m_n} x_n\right\|\right)\\
&\leq \phi(p, x_n) + \nu_{m_n}^{(i_n)}\sup_{p\in F}\phi(p, x_n) + \mu_{m_n}^{(i_n)}\\
&\quad - \beta_n(1 - \beta_n)\, g\left(\left\|J x_n - J T_{i_n}^{m_n} x_n\right\|\right)\\
&= \phi(p, x_n) + \xi_n - \beta_n(1 - \beta_n)\, g\left(\left\|J x_n - J T_{i_n}^{m_n} x_n\right\|\right).
\end{aligned} \tag{18}$$

Substituting (18) into (17) and simplifying it, we have that

$$\phi(p, y_n) \leq \phi(p, x_n) + (1 - \alpha_n)\xi_n \leq \phi(p, x_n) + \xi_n. \tag{19}$$

This implies that $p \in C_{n+1}$, and so $F \subset C_{n+1}$.

(III) $x_n \to x^* \in C$ as $n \to \infty$.

In fact, since $x_n = \Pi_{C_n} x_1$, from Lemma 3 (2), we have that $\langle x_n - y, Jx_1 - Jx_n\rangle \geq 0$, for all $y \in C_n$. Again, since

$F \subset \bigcap_{n=1}^{\infty} C_n$, we have that $\langle x_n - p, Jx_1 - Jx_n \rangle \geq 0$, for all $p \in F$. It follows from Lemma 3 (1) that for each $p \in F$ and for each $n \geq 1$,

$$\phi(x_n, x_1) = \phi(\Pi_{C_n} x_1, x_1) \leq \phi(p, x_1) - \phi(p, x_n) \leq \phi(p, x_1), \tag{20}$$

which implies that $\{\phi(x_n, x_1)\}$ is bounded, so is $\{x_n\}$. Since for all $n \geq 1$, $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we have $\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$. This implies that $\{\phi(x_n, x_1)\}$ is nondecreasing; hence, the limit

$$\lim_{n \to \infty} \phi(x_n, x_1) \text{ exists.} \tag{21}$$

Since E is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x^* \in C$ as $i \to \infty$. Since C_n is closed and convex and $C_{n+1} \subset C_n$, this implies that C_n is weakly, closed and $x^* \in C_n$ for each $n \geq 1$. In view of $x_{n_i} = \Pi_{C_{n_i}} x_1$, we have that

$$\phi(x_{n_i}, x_1) \leq \phi(x^*, x_1), \quad \forall i \geq 1. \tag{22}$$

Since the norm $\|\cdot\|$ is weakly lower semicontinuous, we have that

$$\begin{aligned} \liminf_{i \to \infty} \phi(x_{n_i}, x_1) &= \liminf_{i \to \infty} \left(\|x_{n_i}\|^2 - 2 \langle x_{n_i}, Jx_1 \rangle + \|x_1\|^2 \right) \\ &\geq \|x^*\|^2 - 2 \langle x^*, Jx_1 \rangle + \|x_1\|^2 \\ &= \phi(x^*, x_1), \end{aligned} \tag{23}$$

and so

$$\phi(x^*, x_1) \leq \liminf_{i \to \infty} \phi(x_{n_i}, x_1) \leq \limsup_{i \to \infty} \phi(x_{n_i}, x_1) \leq \phi(x^*, x_1). \tag{24}$$

This implies that $\lim_{i \to \infty} \phi(x_{n_i}, x_1) = \phi(x^*, x_1)$, and so $\|x_{n_i}\| \to \|x^*\|$ as $i \to \infty$. Since $x_{n_i} \rightharpoonup x^*$, by virtue of *Kadec-Klee property* of E, we obtain that

$$\lim_{i \to \infty} x_{n_i} = x^*. \tag{25}$$

Since $\{\phi(x_n, x_1)\}$ is convergent, this, together with $\lim_{i \to \infty} \phi(x_{n_i}, x_1) = \phi(x^*, x_1)$, shows that $\lim_{n \to \infty} \phi(x_n, x_1) = \phi(x^*, x_1)$. If there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to y$ as $j \to \infty$, then, from Lemma 3 (1), we have that

$$\begin{aligned} \phi(x^*, y) &= \lim_{i,j \to \infty} \phi(x_{n_i}, x_{n_j}) = \lim_{i,j \to \infty} \phi(x_{n_i}, \Pi_{C_{n_j}} x_1) \\ &\leq \lim_{i,j \to \infty} \left(\phi(x_{n_i}, x_1) - \phi(\Pi_{C_{n_j}} x_1, x_1) \right) \\ &= \lim_{i,j \to \infty} \left(\phi(x_{n_i}, x_1) - \phi(x_{n_j}, x_1) \right) \\ &= \phi(x^*, x_1) - \phi(x^*, x_1) = 0; \end{aligned} \tag{26}$$

that is, $x^* = y$, and so

$$\lim_{n \to \infty} x_n = x^*. \tag{27}$$

(IV) x^* is some member of F.

Set $\mathscr{K}_i = \{k \geq 1 : k = i + (m-1)m/2, \ m \geq i, \ m \in \mathbb{N}\}$ for each $i \geq 1$. Note that $\nu_{m_n}^{(i_k)} = \nu_{m_k}^{(i)}$ and $\mu_{m_n}^{(i_k)} = \mu_{m_k}^{(i)}$ whenever $k \in \mathscr{K}_i$ for each $i \geq 1$. For example, by the definition of $\mathscr{K}_1$, we have that $\mathscr{K}_1 = \{1, 2, 4, 7, 11, 16, \ldots\}$, and $i_1 = i_2 = i_4 = i_7 = i_{11} = i_{16} = \cdots = 1$. Then, we have that

$$\xi_k = \nu_{m_k}^{(i)} \sup_{p \in F} \phi(p, x_k) + \mu_{m_k}^{(i)}, \quad \forall k \in \mathscr{K}_i. \tag{28}$$

Note that $\{m_k\}_{k \in \mathscr{K}_i} = \{i, i+1, i+2, \ldots\}$; that is, $m_k \uparrow \infty$ as $\mathscr{K}_i \ni k \to \infty$. It follows from (27) and (28) that

$$\lim_{\mathscr{K}_i \ni k \to \infty} \xi_k = 0. \tag{29}$$

Since $x_{n+1} \in C_{n+1}$, it follows from (14), (27), and (29) that

$$\phi(x_{k+1}, y_k) \leq \phi(x_{k+1}, x_k) + \xi_k \longrightarrow 0 \tag{30}$$

as $\mathscr{K}_i \ni k \to \infty$. Since $x_k \to x^*$ as $\mathscr{K}_i \ni k \to \infty$, it follows from (30) and Lemma 5 that

$$\lim_{\mathscr{K}_i \ni k \to \infty} y_k = x^*. \tag{31}$$

Note that $T_{i_k}^{m_k} = T_i^{m_k}$ whenever $k \in \mathscr{K}_i$ for each $i \geq 1$. From (17) and (18), for any $p \in F$, we have that

$$\begin{aligned} \phi(p, y_k) \leq{} & \phi(p, x_k) + \xi_k - (1 - \alpha_k) \beta_k (1 - \beta_k) g \\ & \times (\|Jx_k - JT_i^{m_k} x_k\|); \end{aligned} \tag{32}$$

that is,

$$\begin{aligned} & (1 - \alpha_k) \beta_k (1 - \beta_k) g (\|Jx_k - JT_i^{m_k} x_k\|) \\ & \quad \leq \phi(p, x_k) + \xi_k - \phi(p, y_k) \longrightarrow 0 \quad (\mathscr{K}_i \ni k \longrightarrow \infty). \end{aligned} \tag{33}$$

This, together with assumption conditions imposed on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, shows that $\lim_{\mathscr{K}_i \ni k \to \infty} g(\|Jx_k - JT_i^{m_k} x_k\|) = 0$. In view of property of g, we have that

$$\lim_{\mathscr{K}_i \ni k \to \infty} \|Jx_k - JT_i^{m_k} x_k\| = 0. \tag{34}$$

In addition, $Jx_k \to Jx^*$ implies that $\lim_{\mathscr{K}_i \ni k \to \infty} JT_i^{m_k} x_k = Jx^*$. From Remark 4 (ii), it yields that, as $\mathscr{K}_i \ni k \to \infty$,

$$T_i^{m_k} x_k \rightharpoonup x^*, \quad \forall i \geq 1. \tag{35}$$

Again, since for each $i \geq 1$, as $\mathscr{K}_i \ni k \to \infty$,

$$\left| \|T_i^{m_k} x_k\| - \|x^*\| \right| = \left| \|JT_i^{m_k} x_k\| - \|Jx^*\| \right| \leq \|JT_i^{m_k} x_k - Jx^*\| \longrightarrow 0. \tag{36}$$

This, together with (35) and the *Kadec-Klee property* of E, shows that

$$\lim_{\mathscr{K}_i \ni k \to \infty} T_i^{m_k} x_k = x^*, \quad \forall i \geq 1. \tag{37}$$

On the other hand, by the assumptions that for each $i \geq 1$, T_i is uniformly L_i-Lipschitz continuous, and noting again that $\{m_k\}_{k \in \mathcal{K}_i} = \{i, i+1, i+2, \ldots\}$, that is, $m_{k+1} - 1 = m_k$ for all $k \in \mathcal{K}_i$, we then have

$$\begin{aligned}\|T_i^{m_{k+1}} x_k - T_i^{m_k} x_k\| \leq\ & \|T_i^{m_{k+1}} x_k - T_i^{m_{k+1}} x_{k+1}\| \\ & + \|T_i^{m_{k+1}} x_{k+1} - x_{k+1}\| \\ & + \|x_{k+1} - x_k\| + \|x_k - T_i^{m_k} x_k\| \\ \leq\ & (L_i+1) \|x_{k+1} - x_k\| + \|T_i^{m_{k+1}} x_{k+1} - x_{k+1}\| \\ & + \|x_k - T_i^{m_k} x_k\|. \end{aligned} \tag{38}$$

From (37) and $x_k \to x^*$, we have that $\lim_{\mathcal{K}_i \ni k \to \infty} \|T_i^{m_{k+1}} x_k - T_i^{m_k} x_k\| = 0$, and $\lim_{\mathcal{K}_i \ni k \to \infty} T_i^{m_{k+1}} x_k = x^*$; that is, $\lim_{\mathcal{K}_i \ni k \to \infty} T_i (T_i^{m_{k+1}-1} x_k) = x^*$. It then follows that, for each $i \geq 1$,

$$\lim_{\mathcal{K}_i \ni k \to \infty} T_i (T_i^{m_k} x_k) = x^*. \tag{39}$$

In view of the closeness of T_i, it follows from (37) that $T_i x^* = x^*$, namely, for each $i \geq 1$, $x^* \in F(T_i)$, and, hence, $x^* \in F$.

(V) $x^* = \Pi_F x_1$, and so $x_n \to \Pi_F x_1$ as $n \to \infty$.

Put $u = \Pi_F x_1$. Since $u \in F \subset C_n$ and $x_n = \Pi_{C_n} x_1$, we have that $\phi(x_n, x_1) \leq \phi(u, x_1)$, for all $n \geq 1$. Then,

$$\phi(x^*, x_1) = \lim_{n \to \infty} \phi(x_n, x_1) \leq \phi(u, x_1), \tag{40}$$

which implies that $x^* = u$ since $u = \Pi_F x_1$, and, hence, $x_n \to x^* = \Pi_F x_1$.

This completes the proof. □

4. Applications

Let E be a smooth, strictly convex, and reflexive Banach space, and let C be a nonempty closed convex subset of E. Let $\{B_i\}_{i=1}^{\infty} : C \to E^*$ be a sequence of β_i-inverse strongly monotone mappings, $\{\psi\}_{i=1}^{\infty} : C \to \mathbb{R}$ a sequence of lower semicontinuous and convex functions, and $\{\theta_i\}_{i=1}^{\infty} : C \times C \to \mathbb{R}$ a sequence of bifunctions satisfying the following conditions:

(A_1) $\theta(x, x) = 0$;

(A_2) θ is monotone; that is, $\theta(x, y) + \theta(y, x) \leq 0$;

(A_3) $\limsup_{t \downarrow 0} \theta(x + t(z - x), y) \leq \theta(x, y)$;

(A_4) the mapping $y \mapsto \theta(x, y)$ is convex and lower semicontinuous.

A system of generalized mixed equilibrium problems (GMEPs), for $\{\theta_i\}_{i=1}^{\infty}$, $\{B_i\}_{i=1}^{\infty}$, and $\{\psi_i\}_{i=1}^{\infty}$ is to find an $x^* \in C$ such that

$$\theta_i(x^*, y) + \langle y - x^*, B_i x^* \rangle + \psi_i(y) - \psi_i(x^*) \geq 0, \quad \forall y \in C,\ i \geq 1, \tag{41}$$

whose set of common solutions is denoted by $\Omega := \cap_{i=1}^{\infty} \Omega_i$, where Ω_i denotes the set of solutions to generalized mixed equilibrium problem for θ_i, B_i, and ψ_i.

Define a countable family of mappings $\{T_{r,i}\}_{i=1}^{\infty} : E \to C$ with $r > 0$ as follows:

$$T_{r,i}(x) = \left\{ z \in C : \tau_i(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0,\ \forall y \in C \right\}, \quad \forall i \geq 1, \tag{42}$$

where $\tau_i(x, y) = \theta_i(x, y) + \langle y - x, B_i x \rangle + \psi_i(y) - \psi_i(x)$, for all $x, y \in C$, $i \geq 1$. It has been shown by Zhang [15] that

(1) $\{T_{r,i}\}_{i=1}^{\infty}$ is a sequence of single-valued mappings;

(2) $\{T_{r,i}\}_{i=1}^{\infty}$ is a sequence of closed quasi-ϕ-nonexpansive mappings;

(3) $\bigcap_{i=1}^{\infty} F(T_{r,i}) = \Omega$.

Now, we have the following result.

Theorem 9. *Let E be the same as that in Theorem 8, and let C be a nonempty closed convex subset of E. Let $\{T_{r,i}\}_{i=1}^{\infty} : C \to C$ be a sequence of mappings defined by (42) with $F := \bigcap_{i=1}^{\infty} F(T_{r,i}) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0, \epsilon]$ for some $\epsilon \in (0, 1)$, and let $\{\beta_n\}$ be a sequence in $[0, 1]$ satisfying $0 < \liminf_{n \to \infty} \beta_n (1 - \beta_n)$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned} & x_1 \in C; \quad C_1 = C, \\ & y_n = J^{-1} [\alpha_n J x_n + (1 - \alpha_n) J z_n], \\ & z_n = J^{-1} [\beta_n J x_n + (1 - \beta_n) J T_{r,i_n} x_n], \\ & C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n)\}, \\ & x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{aligned} \tag{43}$$

where i_n satisfies the positive integer equation: $n = i + (m - 1)m/2$, and $m \geq i$ ($m \geq i$, $n = 1, 2, \ldots$). Then, $\{x_n\}$ converges strongly to $\Pi_F x_1$ which is some solution to the system of generalized mixed equilibrium problems for $\{T_{r,i}\}_{i=1}^{\infty}$.

Proof. Note that $\{T_{r,i}\}_{i=1}^{\infty}$ are quasi-ϕ-nonexpansive mappings; so, they are obviously generalized quasi-ϕ-asymptotically nonexpansive. Therefore, this conclusion can be obtained immediately from Theorem 8. □

Acknowledgments

The author is greatly grateful to the referees for their useful suggestions by which the contents of this paper are improved. This work was supported by the National Natural Science Foundation of China (Grant no. 11061037).

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