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Research Article

Certain Properties of a Class of Close-to-Convex Functions Related to Conic Domains

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We aim to define a new class of close-to-convex functions which is related to conic domains. Many interesting properties such as sufficiency criteria, inclusion results, and integral preserving properties are investigated here. Some interesting consequences of our results are also observed.

1. Introduction

Let $\mathscr{A}$ be the class of functions f

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. Let f and g be analytic in E, and we say that f is subordinate to g, written as $f(z) \prec g(z)$ if there exists a Schwarz function w, which is analytic in E with $w(0) = 0$ and $|w(z)| < 1$ $(z \in E)$, such that $f(z) = g(w(z))$. In particular, when g is univalent, then the above subordination is equivalent to $f(0) = g(0)$ and $f(E) \subseteq g(E)$; see [1].

Kanas and Wisniowska [2, 3] introduced and studied the classes of k-uniformly convex functions denoted by $k\text{-}\mathscr{UCV}$ and the corresponding class $k\text{-}\mathscr{ST}$ related by the Alexander-type relation. Later, the class k-uniformly close-to-convex functions denoted by $k\text{-}\mathscr{UK}$ defined as

$$k\text{-}\mathscr{UK} = \left\{ f \in \mathscr{A} : \operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > k\left|\frac{zf'(z)}{g(z)} - 1\right|, \quad g(z) \in k\text{-}\mathscr{ST},\ z \in E \right\} \quad (2)$$

was considered by Acu [4]; for study details on these classes, we refer to [5–7]. All these above mentioned classes were generalized to the classes $\mathscr{SD}(k,\delta)$, $\mathscr{CD}(k,\delta)$, and $\mathscr{KD}(k,\beta,\delta)$ by Shams et al. [8] and Srivastava et al. [9], respectively. The classes $\mathscr{SD}(k,\delta)$ and $\mathscr{CD}(k,\delta)$ are defined as

$$\begin{gathered}\mathscr{SD}(k,\delta) = \left\{ f \in \mathscr{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \delta,\ z \in E \right\},\\ \mathscr{CD}(k,\delta) = \left\{ f \in \mathscr{A} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| + \delta,\ z \in E \right\},\end{gathered} \quad (3)$$

where $k \geq 0$, $0 \leq \beta, \delta < 1$. The class $\mathscr{KD}(k,\beta,\delta)$ known as k-uniformly close-to-convex functions of order β type δ

is the class of all those functions $f \in \mathcal{A}$ which satisfies the following condition:

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) \geq k\left|\frac{f'(z)}{g'(z)} - 1\right| + \beta, \qquad (k \geq 0,\ 0 \leq \beta,\ \delta < 1;\ z \in E), \tag{4}$$

for some $g \in \mathcal{CD}(k, \delta)$.

Motivated by the work of Noor et al. [10–13], we define the following.

Definition 1. Let $f \in \mathcal{A}$. Then, f is in the class $\mathcal{TD}(k, \alpha, \beta, \gamma, \delta)$ if and only if, for $\alpha \geq 0$, $0 \leq \beta, \gamma, \delta < 1$,

$$\operatorname{Re}\left(H(\alpha, \beta, \gamma; f, g)\right) \geq k\left|H(\alpha, \beta, \gamma; f, g) - 1\right|, \qquad (k \geq 0;\ z \in E), \tag{5}$$

for some $g \in \mathcal{CD}(k, \delta)$, where

$$H(\alpha, \beta, \gamma; f, g) = \frac{1-\alpha}{1-\beta}\left[\frac{f'(z)}{g'(z)} - \beta\right] + \frac{\alpha}{1-\gamma}\left[\frac{\left(zf'(z)\right)'}{g'(z)} - \gamma\right]. \tag{6}$$

Special Cases

(i) $(k, 0, \beta, \gamma, \delta) = \mathcal{KD}(k, \beta, \delta)$; see [9].

(ii) $\mathcal{TD}(1, 0, \beta, \gamma, 0) = \mathcal{UK}(\beta)$ and $\mathcal{TD}(1, 1, \beta, \gamma, 0) = \mathcal{UQ}(\gamma)$, the classes of uniformly close-to-convex and quasiconvex functions introduced and investigated in [14].

(iii) $\mathcal{TD}(0, \alpha, 0, 0, 0) = \mathcal{Q}_\alpha$, the class of alpha quasiconvex functions, introduced and studied in [11].

(iv) $\mathcal{TD}(0, 0\, \beta, \gamma, \delta) = \mathcal{K}(\beta, \delta)$, the class of close-to-convex functions of order β type δ, [15].

(v) $\mathcal{TD}(0, 1, \beta, \gamma, \delta) = C^*(\gamma, \delta)$, the class of quasiconvex functions of order γ type δ, [16].

The conditions $k \geq 0$, $\alpha \geq 0$, $0 \leq \beta, \gamma, \delta < 1$ on the parameters are assumed throughout the entire paper unless otherwise mentioned.

Geometric Interpretation. A function $f \in \mathcal{A}$ is in the class $\mathcal{TD}(k, \alpha, \beta, \gamma, \delta)$ if and only if the functional $H(\alpha, \beta, \gamma, \delta; f, g)$ takes all the values in the conic domain Ω_k defined as follows:

$$\Omega_k = \left\{u + \iota v;\ u > k\sqrt{(u-1)^2 + v^2}\right\}. \tag{7}$$

Extremal functions for these conic regions are denoted by $p_k(z)$, which are analytic in E and map E onto Ω_k such that $p_k(0) = 1$ and $p_k'(0) > 1$. These functions are given as:

$$p_k(z) = \begin{cases} \dfrac{1+z}{1-z}, & k = 0, \\ 1 + \dfrac{2}{\pi^2}\left(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k = 1, \\ 1 + \dfrac{2}{1-k^2}\sinh^2\left[\left(\dfrac{2}{\pi}\arccos k\right)\operatorname{arctan} h\sqrt{z}\right], & 0 < k < 1, \\ 1 + \dfrac{1}{k^2-1}\sin\left(\dfrac{\pi}{2R(t)}\displaystyle\int_0^{u(z)/\sqrt{t}}\dfrac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right) + \dfrac{1}{k^2-1}, & k > 1, \end{cases} \tag{8}$$

where $u(z) = (z - \sqrt{t})/(1 - \sqrt{tz})$, $t \in (0, 1)$, $z \in E$, and z is chosen such that $k = \cosh(\pi R'(t)/4R(t))$, where $R(t)$ is Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$; see [2, 3].

2. Preliminaries Result

We require the following results which are essential in our investigations.

Lemma 2 (see [17, page 70]). *Let h be convex function in E and $q : E \mapsto C$ with $\operatorname{Re} q(z) > 0$, $z \in E$. If p is analytic in E with $p(0) = h(0)$, then*

$$p(z) + q(z)zp'(z) \prec h(z) \quad \text{implies} \quad p(z) \prec h(z). \tag{9}$$

Lemma 3 (see [17, page 195]). *Let h be convex function in E with $h(0) = 0$ and $A \geq 0$. Suppose that $j \geq 4/h'(0)$ and that $B(z)$, $C(z)$, and $D(z)$ are analytic in E and satisfy*

$$\operatorname{Re} B(z) \geq A + |C(z) - 1| - \operatorname{Re}(C(z) - 1) + jD(z), \tag{10}$$

for $z \in E$. If p is analytic in E with $p(z) = a_1z + a_2z^2 + a_3z^3 + \cdots$ and the following subordination relation holds:

$$Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z), \tag{11}$$

then

$$p(z) \prec h(z). \tag{12}$$

Lemma 4 (see [12]). *If $f(z) \prec h(z)$ and $g(z) \prec h(z)$, then for $\alpha \in [0, 1]$,*

$$(1-\alpha)f(z) + \alpha g(z) \prec h(z). \tag{13}$$

3. Main Results

First, we prove the following sufficiency criteria for the functions in the class $\mathcal{TD}(k, \alpha, \beta, \gamma, \delta)$.

Theorem 5. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{TD}(k, \alpha, \beta, \gamma, \delta)$, if*

$$\sum_{n=2}^{\infty} \Phi_n (k; \alpha, \beta, \gamma, \delta) < (1-\beta)(1-\gamma), \tag{14}$$

where

$$\begin{aligned} &\Phi_n (k; \alpha, \beta, \gamma, \delta) \\ &= (k+1)\left[(1-\alpha)(1-\gamma)n + \alpha(1-\beta)n^2\right]|a_n| \\ &\quad + \left[(k+2-\beta)(1-\gamma) + \alpha(k+1)(\gamma-\beta)n|b_n|\right]. \end{aligned} \tag{15}$$

Proof. Let us assume that relation (6) holds. Now, it is sufficient to show that

$$k|H(\alpha, \beta, \gamma, \delta; f, g) - 1| - \operatorname{Re}[H(\alpha, \beta, \gamma, \delta; f, g) - 1] < 1. \tag{16}$$

First, we consider

$$\begin{aligned} &|H(\alpha, \beta, \gamma, \delta; f, g) - 1| \\ &= \left| \frac{1-\alpha}{1-\beta}\left[\frac{f'(z)}{g'(z)} - \beta\right] + \frac{\alpha}{1-\gamma}\left[\frac{(zf'(z))'}{g'(z)} - \gamma\right] - 1 \right| \\ &= \left| \frac{1-\alpha}{1-\beta}\frac{f'(z)}{g'(z)} + \frac{\alpha}{1-\gamma}\frac{(zf'(z))'}{g'(z)} - \frac{(1-\alpha)\beta}{1-\beta} - \frac{\alpha\gamma}{1-\gamma} - 1 \right| \\ &= \left| \frac{(1-\alpha)(1-\gamma)f'(z)}{(1-\beta)(1-\gamma)g'(z)} + \frac{\alpha(1-\beta)(zf'(z))'}{(1-\beta)(1-\gamma)g'(z)} - \frac{[(1-\gamma)+\alpha(\gamma-\beta)]g'(z)}{(1-\beta)(1-\gamma)g'(z)} \right|. \end{aligned} \tag{17}$$

Using (1) and the series $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ in (17), we have

$$\begin{aligned} &|H(\alpha, \beta, \gamma, \delta; f, g) - 1| \\ &= \left| \frac{(1-\alpha)(1-\gamma)\left(1+\sum_{n=2}^{\infty} na_n z^{n-1}\right)}{(1-\beta)(1-\gamma)\left(1+\sum_{n=2}^{\infty} nb_n z^{n-1}\right)} + \frac{\alpha(1-\beta)\left(1+\sum_{n=2}^{\infty} n^2 a_n z^{n-1}\right)}{(1-\beta)(1-\gamma)\left(1+\sum_{n=2}^{\infty} nb_n z^{n-1}\right)} - \frac{[(1-\gamma)+\alpha(\gamma-\beta)]\left(1+\sum_{n=2}^{\infty} nb_n z^{n-1}\right)}{(1-\beta)(1-\gamma)\left(1+\sum_{n=2}^{\infty} nb_n z^{n-1}\right)} \right| \\ &= \left| \frac{(1-\alpha)(1-\gamma)\left(\sum_{n=2}^{\infty} na_n z^{n-1}\right)}{(1-\beta)(1-\gamma)\left(1+\sum_{n=2}^{\infty} nb_n z^{n-1}\right)} + \frac{\alpha(1-\beta)\left(\sum_{n=2}^{\infty} n^2 a_n z^{n-1}\right)}{(1-\beta)(1-\gamma)\left(1+\sum_{n=2}^{\infty} nb_n z^{n-1}\right)} - \frac{[(1-\gamma)+\alpha(\gamma-\beta)]\left(\sum_{n=2}^{\infty} nb_n z^{n-1}\right)}{(1-\beta)(1-\gamma)\left(1+\sum_{n=2}^{\infty} nb_n z^{n-1}\right)} \right| \\ &\le \left| \frac{(1-\alpha)(1-\gamma)\left(\sum_{n=2}^{\infty} n|a_n||z|^{n-1}\right)}{(1-\beta)(1-\gamma)\left(1-\sum_{n=2}^{\infty} n|b_n||z|^{n-1}\right)} + \frac{\alpha(1-\beta)\left(\sum_{n=2}^{\infty} n^2|a_n||z|^{n-1}\right)}{(1-\beta)(1-\gamma)\left(1-\sum_{n=2}^{\infty} n|b_n||z|^{n-1}\right)} + \frac{[(1-\gamma)+\alpha(\gamma-\beta)]\left(\sum_{n=2}^{\infty} n|b_n||z|^{n-1}\right)}{(1-\beta)(1-\gamma)\left(1-\sum_{n=2}^{\infty} n|b_n||z|^{n-1}\right)} \right| \\ &\le \left| \frac{(1-\alpha)(1-\gamma)\left(\sum_{n=2}^{\infty} n|a_n|\right)}{(1-\beta)(1-\gamma)\left(1-\sum_{n=2}^{\infty} n|b_n|\right)} + \frac{\alpha(1-\beta)\left(\sum_{n=2}^{\infty} n^2|a_n|\right)}{(1-\beta)(1-\gamma)\left(1-\sum_{n=2}^{\infty} n|b_n|\right)} + \frac{[(1-\gamma)+\alpha(\gamma-\beta)]\left(\sum_{n=2}^{\infty} n|b_n|\right)}{(1-\beta)(1-\gamma)\left(1-\sum_{n=2}^{\infty} n|b_n|\right)} \right|. \end{aligned} \tag{18}$$

Now,

$$\begin{aligned} &k|H(\alpha, \beta, \gamma, \delta; f, g) - 1| - \operatorname{Re}[H(\alpha, \beta, \gamma, \delta; f, g) - 1] \\ &\le (k+1)|H(\alpha, \beta, \gamma, \delta; f, g) - 1| \\ &\le (k+1)\left[\frac{(1-\alpha)(1-\gamma)\left(\sum_{n=2}^{\infty} n|a_n|\right)}{(1-\beta)(1-\gamma)\left(1-\sum_{n=2}^{\infty} n|b_n|\right)} + \frac{\alpha(1-\beta)\left(\sum_{n=2}^{\infty} n^2|a_n|\right)}{(1-\beta)(1-\gamma)\left(1-\sum_{n=2}^{\infty} n|b_n|\right)} + \frac{[(1-\gamma)+\alpha(\gamma-\beta)]\left(\sum_{n=2}^{\infty} n|b_n|\right)}{(1-\beta)(1-\gamma)\left(1-\sum_{n=2}^{\infty} n|b_n|\right)} \right]. \end{aligned} \tag{19}$$

The last inequality is bounded above by 1, if

$$(k+1)\left[(1-\alpha)(1-\gamma)\left(\sum_{n=2}^{\infty} n|a_n|\right)+\alpha(1-\beta)\left(\sum_{n=2}^{\infty} n^2|a_n|\right)\right.$$
$$\left.+\left[(1-\gamma)+\alpha(\gamma-\beta)\right]\left(\sum_{n=2}^{\infty} n|b_n|\right)\right]$$
$$\leq(1-\beta)(1-\gamma)\left(1-\sum_{n=2}^{\infty} n|b_n|\right). \tag{20}$$

Hence,

$$\sum_{n=2}^{\infty}\Phi_n(k;\alpha,\beta,\gamma,\delta)\leq(1-\beta)(1-\gamma), \tag{21}$$

where $\Phi_n(k;\alpha,\beta,\gamma,\delta)$ is given by (15). This completes the proof. □

When we take $\alpha=0$, $k=1$, and $g(z)=z$ in the above theorem, we obtain the following sufficient condition for the functions to be in the class $\mathscr{UK}(\beta)$ which is proved in [14].

Corollary 6 (see [14]). *A function $f\in A$ is said to be in the class $\mathscr{UK}(\beta)$ if*

$$\sum_{n=2}^{\infty} n|a_n|\leq\frac{(1-\beta)}{2}. \tag{22}$$

Corollary 7 (see [14]). *A function $f\in A$ is said to be in the class $\mathscr{UQ}(\gamma)$ if*

$$\sum_{n=2}^{\infty} n^2|a_n|\leq\frac{(1-\gamma)}{2}. \tag{23}$$

The above corollary is obtained when we take $\alpha=1$, $k=1$, and $g(z)=z$ in Theorem 5.

Theorem 8. *Let $f\in\mathscr{TD}(k,\alpha,\beta,\gamma,\delta)$ and $\alpha\geq 4(\beta/(1+3\beta))$. Then, $f\in\mathscr{KD}(k,0,\delta)$.*

Proof. Let

$$\frac{f'(z)}{g'(z)}=p(z), \tag{24}$$

where $p(z)$ is analytic and $p(0)=1$. Now differentiating (24), we have

$$\frac{(zf'(z))'}{g'(z)}=zp'(z)+p(z)\psi(z), \tag{25}$$

where $\psi(z)=(zg'(z))'/g'(z)$. Using (24) and (25) in relation (6), we obtain

$$\begin{aligned}H&(\alpha,\beta,\gamma,\delta;f(z))\\&=\frac{1-\alpha}{1-\beta}\left[p(z)-\beta\right]+\frac{\alpha}{1-\gamma}\left[zp'(z)+p(z)\psi(z)-\gamma\right]\\&=\frac{\alpha}{1-\gamma}zp'(z)+\left[\frac{(1-\alpha)(1-\gamma)+\alpha(1-\beta)\psi(z)}{(1-\beta)(1-\gamma)}\right]p(z)\\&\quad-\frac{\beta(1-\alpha)(1-\gamma)+\alpha\gamma(1-\beta)}{(1-\beta)(1-\gamma)}\\&=B(z)zp'(z)+C(z)p(z)+D(z),\end{aligned} \tag{26}$$

where

$$B(z)=\frac{\alpha}{1-\gamma},\qquad C(z)=\frac{(1-\alpha)(1-\gamma)+\alpha(1-\beta)\psi(z)}{(1-\beta)(1-\gamma)},$$
$$D(z)=-\frac{\beta(1-\alpha)(1-\gamma)+\alpha\gamma(1-\beta)}{(1-\beta)(1-\gamma)}. \tag{27}$$

Now, since $f\in\mathscr{TD}(k,\alpha,\beta,\gamma,\delta)$, we have

$$B(z)zp'(z)+C(z)p(z)+D(z)\prec p_k(z). \tag{28}$$

Replacing $p(z)$ by $p_*(z)=p(z)-1$ and $p_k(z)$ by $p_k^*(z)=p_k(z)-1$, the above subordination is equivalent to

$$B(z)zp_*'(z)+C(z)p_*(z)+D_*(z)\prec p_k^*(z), \tag{29}$$

where $D_*(z)=C(z)+D(z)-1$. Using Lemma 3 with $A=0$, we obtain

$$p_*(z)\prec p_k^*(z). \tag{30}$$

This implies that

$$\frac{f'(z)}{g'(z)}=p(z)\prec p_k(z). \tag{31}$$

Hence, $f\in\mathscr{KD}(k,0,\delta)$. This completes the proof. □

Corollary 9. *Let $f\in\mathscr{TD}(k,\alpha,0,0,0)=\mathscr{Q}_\alpha$. Then, $f\in\mathscr{KD}(0,0,0)=\mathscr{K}$. That is, $\mathscr{Q}_\alpha\subset\mathscr{K}$, $\alpha\geq 0$.*

The above result is well-known inclusion proved in [11].

For $f\in A$, consider the following integral operator defined by

$$F(z)=I_m[f]=\frac{m+1}{z^m}\int_0^z t^{m-1}f(t)\,dt,\quad m=1,2,3\ldots. \tag{32}$$

This operator was given by Bernardi [18] in 1969. In particular, the operator I_1 was considered by Libera [19]. Now let us prove the following.

Theorem 10. *Let* $f \in \mathcal{TD}(k,\alpha,\beta,\gamma,\delta)$*. Then,* $I_m[f] \in \mathcal{KD}(k,0,\delta)$.

Proof. Let the function g be such that (6) is satisfied. It can easily be seen that according to [4], the function $G = I_m[f] \in \mathcal{CD}(k,\delta)$, and from (32), we deduce

$$(1+m) f'(z) = (1+m) F'(z) + zF''(z), \tag{33}$$

$$(1+m) g'(z) = (1+m) G'(z) + zG''(z). \tag{34}$$

If we let $p(z) = F'(z)/G'(z)$ and $q(z) = 1/(m+1+(zG''(z)/G'))$, then simple computations yield us

$$\begin{aligned}\frac{f'(z)}{g'(z)} &= \frac{(1+m) F'(z) + zF''(z)}{(1+m) G'(z) + zG''(z)} \\ &= \frac{zp(z) g'(z) + zp(z) g''(z) + zg'(z) p'(z)}{(1+m) G'(z) + zG''(z)} \\ &= p(z) + q(z) zp'(z).\end{aligned} \tag{35}$$

Let

$$\frac{f'(z)}{g'(z)} = p(z) + zp'(z) q(z) = h(z), \tag{36}$$

where $h(z)$ is analytic and $p(0) = 1$. From (36), we have

$$\frac{(zf'(z))'}{g'(z)} = \psi(z) h(z) + zh'(z), \tag{37}$$

where $\psi(z) = (zg'(z))'/g'(z)$. Using (36) and (37) in (6), we have

$$\begin{aligned}&H(\alpha,\beta,\gamma,\delta; f(z)) \\ &\quad= \frac{1-\alpha}{1-\beta}[h(z)-\beta] + \frac{\alpha}{1-\gamma}[zh'(z) + h(z)\psi(z) - \gamma] \\ &\quad= \frac{\alpha}{1-\gamma} zh'(z) \\ &\qquad+ \left[\frac{(1-\alpha)(1-\gamma) + \alpha(1-\beta)\psi(z)}{(1-\beta)(1-\gamma)}\right] h(z) \\ &\qquad- \frac{\beta(1-\alpha)(1-\gamma) + \alpha\gamma(1-\beta)}{(1-\beta)(1-\gamma)} \\ &\quad= B(z) zh'(z) + C(z) h(z) + D(z),\end{aligned} \tag{38}$$

where

$$\begin{aligned}B(z) &= \frac{\alpha}{1-\gamma}, \qquad C(z) = \frac{(1-\alpha)(1-\gamma) + \alpha(1-\beta)\psi(z)}{(1-\beta)(1-\gamma)}, \\ D(z) &= -\frac{\beta(1-\alpha)(1-\gamma) + \alpha\gamma(1-\beta)}{(1-\beta)(1-\gamma)}.\end{aligned} \tag{39}$$

Now proceeding in the similar manner as in the proof of Theorem 5 and using Lemma 3 with $A = 0$, we obtain

$$\frac{f'(z)}{g'(z)} = h(z) \prec p_k(z). \tag{40}$$

From (36), it implies that

$$p(z) + zp'(z) q(z) \prec p_k(z). \tag{41}$$

By employing Lemma 2, we immediately obtain the desired result. □

Theorem 11. *For* $\alpha > \alpha_1 \geq 0$,

$$\mathcal{TD}(k,\alpha,\beta,\gamma,\delta) \subseteq \mathcal{TD}(k,\alpha_1,\beta,\gamma,\delta). \tag{42}$$

Proof. Let $f \in \mathcal{TD}(k,\alpha,\beta,\gamma,\delta)$. Then, consider

$$\begin{aligned}H(\alpha_1,\beta,\gamma,\delta; f, g) = \frac{1-\alpha_1}{1-\beta}\left[\frac{f'(z)}{g'(z)} - \beta\right] \\ + \frac{\alpha_1}{1-\gamma}\left[\frac{(zf'(z))'}{g'(z)} - \gamma\right].\end{aligned} \tag{43}$$

After some simple computations, we have

$$\begin{aligned}&H(\alpha_1,\beta,\gamma,\delta; f, g) \\ &\quad= \frac{1}{1-\beta}\left(1-\frac{\alpha_1}{\alpha}\right)\left[\frac{f'(z)}{g'(z)} - \beta\right] \\ &\qquad+ \frac{\alpha_1}{\alpha}\left[\frac{1-\alpha}{1-\beta}\left(\frac{f'(z)}{g'(z)} - \beta\right) + \frac{\alpha}{1-\gamma}\left(\frac{(zf'(z))'}{g'(z)} - \gamma\right)\right] \\ &\quad= \left(1-\frac{\alpha_1}{\alpha}\right) H(0,\beta,\gamma,\delta; f, g) + \frac{\alpha_1}{\alpha} H(\alpha,\beta,\gamma,\delta; f, g).\end{aligned} \tag{44}$$

Now, since $f \in \mathcal{TD}(k,\alpha,\beta,\gamma,\delta)$, we have $H(\alpha,\beta,\gamma,\delta; f, g) \prec p_k(z)$. Also, Theorem 8 gives us that $H(0,\beta,\gamma,\delta; f, g) \prec p_k(z)$. The use of Lemma 4 leads us to the required relation; that is, $H(\alpha_1,\beta,\gamma,\delta; f, g) \prec p_k(z)$. This completes the proof. □

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