# ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS TO BOUNDARY VALUE PROBLEMS ON TIME SCALES

JOHNNY HENDERSON, ALLAN PETERSON,
AND CHRISTOPHER C. TISDELL

*Received 13 August 2003 and in revised form 11 February 2004*

This work formulates existence, uniqueness, and uniqueness-implies-existence theorems for solutions to two-point vector boundary value problems on time scales. The methods used include maximum principles, a priori bounds on solutions, and the nonlinear alternative of Leray-Schauder.

## 1. Introduction

This paper considers the existence and uniqueness of solutions to the second-order vector dynamic equation

$$y^{\Delta\Delta}(t) = f(t, y(\sigma(t))) + P(t)y^{\Delta}(\sigma(t)), \quad t \in [a,b], \tag{1.1}$$

subject to any of the boundary conditions

$$y(a) = A, \qquad y(\sigma^2(b)) = B, \tag{1.2}$$

$$\alpha y(a) - \beta y^{\Delta}(a) = C, \qquad \gamma y(\sigma^2(b)) + \delta y^{\Delta}(\sigma(b)) = D, \tag{1.3}$$

$$\alpha y(a) - \beta y^{\Delta}(a) = C, \qquad y(\sigma^2(b)) = B, \tag{1.4}$$

$$y(a) = A, \qquad \gamma y(\sigma^2(b)) + \delta y^{\Delta}(\sigma(b)) = D, \tag{1.5}$$

where $f : [a,b] \times \mathbb{R}^d \to \mathbb{R}^d$; $P(t)$ is a $d \times d$ matrix; $A,B,C,D \in \mathbb{R}^d$; and $\alpha,\beta,\gamma,\delta \in \mathbb{R}$. The problems (1.1), (1.2); (1.1), (1.3); (1.1), (1.4); and (1.1), (1.5) are known as boundary value problems (BVPs) on "time scales."

To understand the notation used above and the idea of time scales, some preliminary definitions are useful.

*Definition 1.1.* A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$.

Since a time scale may or may not be connected, the concept of jump operators is useful.


Advances in Difference Equations 2004:2 (2004) 93–109
2000 Mathematics Subject Classification: 39A12
URL: http://dx.doi.org/10.1155/S1687183904308071

*Definition 1.2.* Define the forward (backward) jump operator $\sigma(t)$ at $t$ for $t < \sup \mathbb{T}$ (resp., $\rho(t)$ at $t$ for $t > \inf \mathbb{T}$) by

$$\sigma(t) = \inf\{\tau > t : \tau \in \mathbb{T}\}, \quad (\rho(t) = \sup\{\tau < t : \tau \in \mathbb{T}\}), \quad \forall t \in \mathbb{T}. \tag{1.6}$$

Also define $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$ if $\sup \mathbb{T} < \infty$, and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$ if $\inf \mathbb{T} > -\infty$. For simplicity and clarity denote $\sigma^2(t) = \sigma(\sigma(t))$ and $y^\sigma(t) = y(\sigma(t))$. Define the graininess function $\mu : \mathbb{T} \to \mathbb{R}$ by $\mu(t) = \sigma(t) - t$.

Throughout this work the assumption is made that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. Also assume throughout that $a < b$ are points in $\mathbb{T}$ with $[a,b] = \{t \in \mathbb{T} : a \leq t \leq b\}$.

The jump operators $\sigma$ and $\rho$ allow the classification of points in a time scale in the following way: if $\sigma(t) > t$, then call the point $t$ right-scattered; while if $\rho(t) < t$, then we call $t$ left-scattered. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then call the point $t$ right-dense; while if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then we call $t$ left-dense.

If $\mathbb{T}$ has a left-scattered maximum at $m$, then define $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^k = \mathbb{T}$.

*Definition 1.3.* Fix $t \in \mathbb{T}$ and let $y : \mathbb{T} \to \mathbb{R}^d$. Define $y^\Delta(t)$ to be the vector (if it exists) with the property that given $\epsilon > 0$ there is a neighbourhood $U$ of $t$ such that, for all $s \in U$ and each $i = 1, \ldots, d$,

$$\left| [y_i(\sigma(t)) - y_i(s)] - y_i^\Delta(t)[\sigma(t) - s] \right| \leq \epsilon \left| \sigma(t) - s \right|. \tag{1.7}$$

Call $y^\Delta(t)$ the (delta) derivative of $y(t)$ at $t$.

*Definition 1.4.* If $F^\Delta(t) = f(t)$, then define the integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a). \tag{1.8}$$

The following theorem is due to Hilger [12].

Theorem 1.5. *Assume that $f : \mathbb{T} \to \mathbb{R}^d$ and let $t \in \mathbb{T}^k$.*

(i) *If $f$ is differentiable at $t$, then $f$ is continuous at $t$.*

(ii) *If $f$ is continuous at $t$ and $t$ is right-scattered, then $f$ is differentiable at $t$ with*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}. \tag{1.9}$$

(iii) *If $f$ is differentiable and $t$ is right-dense, then*

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}. \tag{1.10}$$

(iv) *If $f$ is differentiable at $t$, then $f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t)$.*

*Definition 1.6.* Define $f \in C_{rd}(\mathbb{T}; \mathbb{R}^d)$ as right-dense continuous if, at all $t \in \mathbb{T}$,

(a) $f$ is continuous at every right-dense point $t \in \mathbb{T}$,

(b) $\lim_{s \to t^-} f(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$.

*Definition 1.7.* Define $S$ to be the set of all functions $y:\mathbb{T}\to\mathbb{R}^d$ such that

$$S=\{y:y\in C([a,\sigma^2(b)];\mathbb{R}^d),\ y^{\Delta\Delta}\in C_{rd}([a,b];\mathbb{R}^d)\}. \tag{1.11}$$

A solution to (1.1) is a function $y\in S$ which satisfies (1.1) for each $t\in[a,b]$.

In order to prove the existence of solutions to the BVPs (1.1), (1.2) through (1.1), (1.5), the following theorem will be used, which is referred to as the nonlinear alternative of Leray-Schauder.

Theorem 1.8. *Let $\Omega$ be an open, convex, and bounded subset of a Banach space $X$ with $0\in\Omega$ and let $T:\bar{\Omega}\to X$ be a compact operator. If $y\neq\lambda T(y)$ for all $y\in\partial\Omega$ and all $\lambda\in[0,1]$, then $y=T(y)$ for some $y\in\Omega$.*

*Proof.* This is a special case of Lloyd [15, Theorem 4.4.11]. □

Recently the study of dynamic equations on time scales has attracted much interest (see [1, 2, 3, 4, 6, 7, 8, 9, 10, 12, 13, 14]). This has been mainly due to its unification of the theory of differential and difference equations. The potential for applications is enormous—especially in those phenomena that manifest themselves partly in continuous time and partly in discrete time.

To the authors' knowledge, no papers have yet dealt with second-order systems of BVPs on time scales. The extension to systems is a natural one; for example, many occurrences in nature involve two or more populations coexisting in an environment, with the model being best described by a system of dynamic equations. (Beltrami [5, Section 5.6] discusses algae and copepod populations via second-order systems of BVPs.)

This paper deals with two specific types of second-order equations. Sections 2, 3, 4, 5, and 6 treat the nonlinear equation

$$y^{\Delta\Delta}(t)=f(t,y^{\sigma}(t)),\quad t\in[a,b], \tag{1.12}$$

and Section 7 treats the linear equation

$$y^{\Delta\Delta}(t)=P(t)y^{\Delta}(\sigma(t))+Q(t)y^{\sigma}(t)+h(t),\quad t\in[a,b], \tag{1.13}$$

where $P$ and $Q$ are $d\times d$ matrices functions and $h$ is a $d\times 1$ vector function.

In particular, the paper is organized as follows.

In Section 2, the necessary a priori bounds on solutions to the BVPs (1.12), (1.2) through (1.12), (1.5) are formulated via some simple lemmas involving inequalities on $f$ and on the boundary conditions.

In Section 3, the a priori bounds from Section 2 are used in conjunction with the nonlinear alternative to prove the existence of solutions to the BVPs (1.12), (1.2) through (1.12), (1.5).

In Section 4, the inequalities on $f$ from Section 3 are slightly strengthened and some extra qualitative information about solutions is obtained. Solutions are shown to be nonincreasing or nondecreasing in norm.

In Section 5, BVPs on infinite intervals are investigated and some existence theorems are presented. The proofs rely on the existence of solutions on finite intervals and so use the theorems of Section 3. A standard diagonalization argument is also employed.

In Sections 6 and 7, some simple maximum principles are used to prove the uniqueness of solutions to (1.12), (1.2) and (1.13), (1.2). A simple uniqueness-implies-existence theorem is also presented for (1.13), (1.2).

The theory of time scales dates back to Hilger [12]. The monographs [6, 14] also provide an excellent introduction. Of particular motivation for the research in this paper were the works [1, 2, 3, 4, 8, 9, 10].

## 2. A priori bounds on solutions

In order to apply Theorem 1.8, a priori bounds on solutions to the BVPs are needed. In this section conditions on $f$ and on the boundary conditions are formulated, under which these bounds are guaranteed.

The following maximum principle will be very useful throughout the rest of the paper and can be found in [10].

Lemma 2.1. *If a function $r:\mathbb{T}\to\mathbb{R}$ has a local maximum at a point $c\in[a,\sigma^2(b)]$, then $r^{\Delta\Delta}(\rho(c))\le 0$ provided that $c$ is not simultaneously left-dense and right-scattered and that $r^{\Delta\Delta}(\rho(c))$ exists.*

Let $y$ be a solution to (1.1). In what follows, the maximum principle of Lemma 2.1 will be applied to the "Lyapunov-type" function $r(t)=\|y(t)\|^2$, and then used to show that $r$ is bounded on $[a,\sigma^2(b)]$ (and therefore solutions $y$ are bounded on $[a,\sigma^2(b)]$). In order to guarantee that $r^{\Delta\Delta}(\rho(c))$ exists, both $y^{\Delta\Delta}(t)$ and $[y(\sigma(t))]^{\Delta}$ must exist, since $r(t)=\langle y(t),y(t)\rangle$ is the inner product of two functions. As remarked in [6], the product of two functions is not necessarily differentiable even if each of the functions is twice differentiable. Therefore, for the rest of the paper, assume that $\sigma(t)$ is such that for those solutions $y\in S$, $[y(\sigma(t))]^{\Delta}$ exists.

Lemma 2.2. *Let $R>0$ be a constant such that*

$$\langle u,f(t,u)\rangle>0,\quad \forall t\in[a,b],\ \|u\|\ge R. \tag{2.1}$$

*If $y$ is a solution to (1.12) and $\|y(t)\|$ does not achieve its maximum value at $t=a$ or $t=\sigma^2(b)$, then $\|y(t)\|<R$ for $t\in[a,\sigma^2(b)]$.*

*Proof.* Assume that the conclusion of the lemma is false. Therefore $r(t):=\|y(t)\|^2-R^2$ must have a nonnegative maximum in $[a,\sigma^2(b)]$. By hypothesis, this maximum must occur in $(a,\sigma^2(b))$. Choose $c\in(a,\sigma^2(b))$ such that

$$r(c)=\max\{r(t);\ t\in[a,\sigma^2(b)]\}\ge 0, \tag{2.2}$$

$$r(t)<r(c),\quad \text{for } c<t<\sigma^2(b). \tag{2.3}$$

First, we show that the point $c$ cannot be simultaneously left-dense and right-scattered. Assume the contrary by letting $\rho(c)=c<\sigma(c)$. If $r^{\Delta}(c)\ge 0$, then $r(\sigma(c))\ge r(c)$, and this contradicts (2.3). If $r^{\Delta}(c)<0$, then $\lim_{t\to c^-}r^{\Delta}(t)=r^{\Delta}(c)<0$. Therefore there exists a $\delta>0$

such that $r^{\Delta}(t) < 0$ on $(c-\delta, c]$. Hence $r(t)$ is strictly decreasing on $(c-\delta, c]$ and this contradicts the way $c$ was chosen.

Therefore the point $c$ cannot be simultaneously left-dense and right-scattered.

By Lemma 2.1 we must have

$$r^{\Delta\Delta}(\rho(c)) \leq 0. \tag{2.4}$$

So using the product rule (see [6]) we have

$$\begin{aligned} r^{\Delta\Delta}(\rho(c)) &= 2\langle y^{\sigma}(\rho(c)), f(\rho(c), y^{\sigma}(\rho(c)))\rangle + ||y^{\Delta}(\rho(c))||^2 + ||y^{\sigma\Delta}(\rho(c))||^2 \\ &\geq 2\langle y^{\sigma}(\rho(c)), f(\rho(c), y^{\sigma}(\rho(c)))\rangle > 0, \quad \text{by (2.1)}, \end{aligned} \tag{2.5}$$

which contradicts (2.4). Therefore $\|y(t)\| < R$ for $t \in [a, \sigma^2(b)]$. (Notice at $c$ that $\|y^{\sigma}(\rho(c))\| = \|y(c)\| \geq R$, since $c$ is not simultaneously left-dense and right-scattered.) This concludes the proof. □

The following lemma provides a priori bounds on solutions to the Dirichlet BVP (1.12), (1.2).

Lemma 2.3. *If $f$ and $R$ satisfy the conditions of Lemma 2.2 with $\|A\|, \|B\| < R$, then every solution $y$ to the Dirichlet BVP (1.12), (1.2) satisfies $\|y(t)\| < R$ for $t \in [a, \sigma^2(b)]$.*

*Proof.* This result follows immediately from Lemma 2.2. □

The following lemma provides a priori bounds on solutions to the Sturm-Liouville BVP (1.12), (1.3).

Lemma 2.4. *If $f$ and $R$ satisfy the conditions of Lemma 2.2 with $\alpha, \beta, \gamma, \delta > 0$, then every solution $y$ to the Sturm-Liouville BVP (1.12), (1.3) satisfies*

$$||y(t)|| < \max\left\{\frac{\|C\|}{\alpha}, \frac{\|D\|}{\gamma}, R\right\} + 1, \quad \text{for } t \in [a, \sigma^2(b)]. \tag{2.6}$$

*Proof.* Let $M = \max\{\|C\|/\alpha, \|D\|/\gamma, R\}$ and assume that $r(t) := \|y(t)\|^2 - (M+1)^2$ has a nonnegative maximum at $t = a$. Then

$$\begin{aligned} r^{\Delta}(a) &= \langle y(a) + y^{\sigma}(a), y^{\Delta}(a)\rangle \\ &= \langle 2y(a) + \mu(a)y^{\Delta}(a), y^{\Delta}(a)\rangle \\ &= 2\langle y(a), y^{\Delta}(a)\rangle + \mu(a)||y^{\Delta}(a)||^2 \leq 0. \end{aligned} \tag{2.7}$$

It follows that

$$2\langle y(a), y^{\Delta}(a)\rangle \leq -\mu(a)||y^{\Delta}(a)||^2 \leq 0 \tag{2.8}$$

and therefore

$$\langle y(a), y^{\Delta}(a)\rangle \leq 0. \tag{2.9}$$

Hence

$$0 \geq \langle y(a), \beta y^{\Delta}(a)\rangle = \langle y(a), \alpha y(a) - C\rangle = \alpha\|y(a)\|^2\left(1 - \frac{\langle y(a), C\rangle}{\alpha\|y(a)\|^2}\right), \tag{2.10}$$

and therefore $(1 - \langle y(a), C\rangle/\alpha\|y(a)\|^2) \leq 0$. Hence we have

$$1 \leq \frac{\langle y(a), C\rangle}{\alpha\|y(a)\|^2} \leq \frac{|\langle y(a), C\rangle|}{\alpha\|y(a)\|^2} \leq \frac{\|y(a)\|\|C\|}{\alpha\|y(a)\|^2} = \frac{\|C\|}{\alpha\|y(a)\|}. \tag{2.11}$$

Thus, rearranging (2.11) we obtain $\|y(a)\| \leq \|C\|/\alpha \leq M$. If a nonnegative maximum occurs at $t = \sigma^2(b)$, then

$$\begin{aligned} r^{\Delta}(\sigma(b)) &= \langle y(\sigma(b)) + y^{\sigma}(\sigma(b)), y^{\Delta}(\sigma(b))\rangle \\ &= \langle 2y^{\sigma}(\sigma(b)) - \mu(\sigma(b))y^{\Delta}(\sigma(b)), y^{\Delta}(\sigma(b))\rangle \\ &= 2\langle y(\sigma^2(b)), y^{\Delta}(\sigma(b))\rangle - \mu(\sigma(b))\|y^{\Delta}(\sigma(b))\|^2 \geq 0. \end{aligned} \tag{2.12}$$

It follows that

$$2\langle y(\sigma^2(b)), y^{\Delta}(\sigma(b))\rangle \geq \mu(\sigma(b))\|y^{\Delta}(\sigma(b))\|^2 \geq 0 \tag{2.13}$$

and therefore

$$\langle y(\sigma^2(b)), y^{\Delta}(\sigma(b))\rangle \geq 0. \tag{2.14}$$

Hence

$$\begin{aligned} 0 &\leq \langle y(\sigma^2(b)), \delta y^{\Delta}(\sigma(b))\rangle \\ &= \langle y(\sigma^2(b)), D - \gamma y(\sigma^2(b))\rangle \\ &= \gamma\|y(\sigma^2(b))\|^2\left(\frac{\langle y(\sigma^2(b)), D\rangle}{\gamma\|y(\sigma^2(b))\|^2} - 1\right), \end{aligned} \tag{2.15}$$

and therefore $(\langle y(\sigma^2(b)), D\rangle/\gamma\|y(\sigma^2(b))\|^2 - 1) \geq 0$. Hence we have

$$1 \leq \frac{\langle y(\sigma^2(b)), D\rangle}{\gamma\|y(\sigma^2(b))\|^2} \leq \frac{|\langle y(\sigma^2(b)), D\rangle|}{\gamma\|y(\sigma^2(b))\|^2} \leq \frac{\|y(\sigma^2(b))\|\|D\|}{\gamma\|y(\sigma^2(b))\|^2} = \frac{\|D\|}{\gamma\|y(\sigma^2(b))\|}. \tag{2.16}$$

Thus, rearranging (2.16) we obtain $\|y(\sigma^2(b))\| \leq \|D\|/\gamma \leq M$. If a maximum occurs in $(a, \sigma^2(b))$, then $\|y(t)\| < R$, $t \in [a, \sigma^2(b)]$ by Lemma 2.2. This concludes the proof. □

The question now arises on whether the conditions $\alpha,\beta,\gamma,\delta > 0$ can be removed from Lemma 2.4. By "piecing together" parts of Lemmas 2.3 and 2.4, results for the BVPs (1.12), (1.4) and (1.12), (1.5) are now presented.

Lemma 2.5. *Let $f$ and $R$ satisfy the conditions of Lemma 2.2. If $\alpha,\beta > 0$ and $\|B\| < R$, then every solution $y$ to the BVP (1.12), (1.4) satisfies*

$$\|y(t)\| < \max\left\{\frac{\|C\|}{\alpha},R\right\} + 1, \quad \text{for } t \in [a,\sigma^2(b)]. \tag{2.17}$$

Lemma 2.6. *Let $f$ and $R$ satisfy the conditions of Lemma 2.2. If $\gamma,\delta > 0$ and $\|A\| < R$, then every solution $y$ to the BVP (1.12), (1.5) satisfies*

$$\|y(t)\| < \max\left\{\frac{\|D\|}{\gamma},R\right\} + 1, \quad \text{for } t \in [a,\sigma^2(b)]. \tag{2.18}$$

*Proofs.* The proofs follow lines similar to those of Lemmas 2.3 and 2.4 and so are omitted. □

## 3. Existence of solutions

In this section, some existence results are presented for the BVPs (1.12), (1.2) through (1.12), (1.5). The proofs rely on the a priori bounds on solutions of Section 2 and the nonlinear alternative.

The following theorem gives the existence of solutions to the Dirichlet BVP on time scales.

Theorem 3.1. *Let $R > 0$ be a constant. Suppose that $f(t,u)$ is continuous on $[a,b]\times\mathbb{R}^d$ and satisfies (2.1). If $\|A\|,\|B\| < R$, then the Dirichlet BVP (1.12), (1.2) has at least one solution $y \in S$ satisfying $\|y(t)\| < R$ on $[a,\sigma^2(b)]$.*

*Proof.* The BVP (1.12), (1.2) is equivalent (see [6, Corollary 4.76]) to the integral equation

$$y(t) = \int_a^{\sigma(b)} G(t,s) f(s,y^\sigma(s))\Delta s + \phi(t), \quad t \in [a,\sigma^2(b)], \tag{3.1}$$

where

$$G(t,s) = \begin{cases} -\dfrac{(t-a)(\sigma^2(b)-\sigma(s))}{(\sigma^2(b)-a)}, & \text{for } t \le s, \\ -\dfrac{(\sigma(s)-a)(\sigma^2(b)-t)}{(\sigma^2(b)-a)}, & \text{for } \sigma(s) \le t, \end{cases} \tag{3.2}$$

$$\phi(t) = \frac{A\sigma^2(b) - Ba + (B-A)t}{\sigma^2(b)-a}.$$

Thus, we want to prove that there exists at least one $y$ satisfying (3.1). Define an operator $T : C([a,\sigma^2(b)];\mathbb{R}^d) \to C([a,\sigma^2(b)];\mathbb{R}^d)$ by

$$(Ty)(t) = \int_a^{\sigma(b)} G(t,s)f(s,y^\sigma(s))\Delta s + \phi(t). \tag{3.3}$$

If we can prove that there exists a $y$ such that $T(y) = y$, then there exists a solution to (3.1). To show that $T$ has a fixed point, consider the equation

$$y = \lambda T(y), \quad \text{for } \lambda \in [0,1]. \tag{3.4}$$

Define an open, bounded subset of the Banach space $S$ by $\Omega = \{y \in S : \|y\| < R\}$, where here $\|\cdot\|$ is the sup norm. Note that (3.4) is equivalent to the BVP

$$\begin{gathered} y^{\Delta\Delta}(t) = \lambda f(t,y(\sigma(t))), \quad t \in [a,b], \\ y(a) = \lambda A, \qquad y(\sigma^2(b)) = \lambda B. \end{gathered} \tag{3.5}$$

Now show that all solutions to (3.5) must satisfy $y \in \Omega$, and consequently $y \notin \partial\Omega$ for all $\lambda \in [0,1]$. Obviously $y \in \Omega$ for $\lambda = 0$. So consider (3.5) for $\lambda \in (0,1]$. Note that, by (2.1),

$$\langle u, \lambda f(t,u)\rangle = \lambda\langle u, f(t,u)\rangle > 0, \quad \forall t \in [a,b],\ \|u\| \geq R. \tag{3.6}$$

Also $\|\lambda A\|, \|\lambda B\| \leq \|A\|, \|B\| < R$. Therefore Lemma 2.3 is applicable to solutions of (3.5). Hence all solutions $y$ to (3.5) must satisfy $\|y(t)\| < R$ for $t \in [a,\sigma^2(b)]$. Hence $y \notin \partial\Omega$.

Since $f$ is continuous, $T$ is continuous and it can be shown that $T$ is a compact operator by the Arzela-Ascoli theorem. Therefore, Theorem 1.8 is applicable to $T$ and $T$ must have a fixed point. Hence the BVP has a solution. This concludes the proof. □

The following theorem gives the existence of solutions to the Sturm-Liouville BVP on time scales.

Theorem 3.2. *Let $R > 0$ be a constant. Suppose that $f$ is continuous on $[a,b] \times \mathbb{R}^d$ and satisfies inequality (2.1). If $\alpha,\beta,\gamma,\delta > 0$, then the Sturm-Liouville BVP (1.12), (1.3) has at least one solution $y \in S$ satisfying (2.6).*

*Proof.* The BVP (1.12), (1.3) is equivalent to the integral equation

$$y(t) = \int_a^{\sigma(b)} G(t,s)f(s,y^\sigma(s))\Delta s + \phi(t), \quad t \in [a,\sigma^2(b)], \tag{3.7}$$

where

$$\begin{gathered} G(t,s) = \begin{cases} -\dfrac{[\beta+(t-a)\alpha][\delta+(\sigma^2(b)-\sigma(s))\gamma]}{p}, & \text{for } t \leq s, \\ -\dfrac{[\beta+(\sigma(s)-a)\alpha][\delta+(\sigma^2(b)-t)\gamma]}{p}, & \text{for } \sigma(s) \leq t, \end{cases} \\ p = \alpha\gamma(\sigma^2(b)-a) + \alpha\delta + \beta\gamma, \\ \phi(t) = \frac{[(\gamma\sigma^2(b)+\delta)C + (\beta-\alpha a)D + (D\alpha - C\gamma)t]}{p}. \end{gathered} \tag{3.8}$$

Thus, we want to prove that there exists at least one $y$ satisfying (3.7). Define an operator $T : C([a,\sigma^2(b)];\mathbb{R}^d) \to C([a,\sigma^2(b)];\mathbb{R}^d)$ by

$$(Ty)(t) = \int_a^{\sigma(b)} G(t,s)f(s,y^\sigma(s))\Delta s + \phi(t). \tag{3.9}$$

If we can prove that there exists a $y$ such that $T(y) = y$, then there exists a solution to (3.7). To show that $T$ has a fixed point, consider the equation

$$y = \lambda T(y), \quad \text{for } \lambda \in [0,1]. \tag{3.10}$$

Define an open, bounded subset of the Banach space $S$ by

$$\Omega = \left\{ y \in S : \|y\| < \max\left\{ \frac{\|C\|}{\alpha}, \frac{\|D\|}{\gamma}, R \right\} + 1 \right\}. \tag{3.11}$$

Note that (3.10) is equivalent to the BVP

$$\begin{gathered} y^{\Delta\Delta} = \lambda f(t,y^\sigma), \quad t \in [a,b], \\ \alpha y(a) - \beta y^\Delta(a) = \lambda C, \qquad \gamma y(\sigma^2(b)) + \delta y^\Delta(\sigma(b)) = \lambda D. \end{gathered} \tag{3.12}$$

Now show that all solutions to (3.12) must satisfy $y \in \Omega$, and consequently $y \notin \partial\Omega$ for all $\lambda \in [0,1]$. Obviously $y \in \Omega$ for $\lambda = 0$. So consider (3.12) for $\lambda \in (0,1]$. Note that, by (2.1), (3.6) holds. Since $\lambda\alpha, \lambda\beta, \lambda\gamma, \lambda\delta > 0$, we get that Lemma 2.4 is applicable to solutions of (3.12), hence

$$||y(t)|| \le \max\left\{ \frac{\|\lambda C\|}{\alpha}, \frac{\|\lambda D\|}{\gamma}, R \right\} \le \max\left\{ \frac{\|C\|}{\alpha}, \frac{\|D\|}{\gamma}, R \right\} \tag{3.13}$$

for $t \in [a,\sigma^2(b)]$. Hence all solutions $y$ to (3.12) must satisfy

$$||y(t)|| < \max\left\{ \frac{\|C\|}{\alpha}, \frac{\|D\|}{\gamma}, R \right\} + 1 \tag{3.14}$$

for $t \in [a,\sigma^2(b)]$ and therefore $y \notin \partial\Omega$.

Since $f$ is continuous, $T$ is continuous and it can be shown that $T$ is a compact operator by the Arzela-Ascoli theorem. Therefore, Theorem 1.8 is applicable to $T$ and $T$ must have a fixed point. Hence the BVP has a solution. This concludes the proof. □

The following result gives the existence of solutions to the BVP (1.12), (1.4), and we will use this in Section 4 when dealing with BVPs on infinite intervals.

THEOREM 3.3. *Let $R > 0$ be a constant. Suppose that $f$ is continuous on $[a,b] \times \mathbb{R}^d$ and satisfies (2.1). If $\alpha, \beta > 0$ and $\|B\| < R$, then the BVP (1.12), (1.4) has at least one solution $y \in S$ satisfying $\|y(t)\| < \max\{\|C\|/\alpha, R\} + 1$ for $t \in [a,\sigma^2(b)]$.*

*Proof.* The BVP (1.12), (1.4) is equivalent to the integral equation

$$y(t) = \int_a^{\sigma(b)} G(t,s)f(s,y^\sigma(s))\Delta s + \phi(t), \quad t \in [a,\sigma^2(b)], \tag{3.15}$$

where

$$G(t,s)=\begin{cases}-\dfrac{[\beta+(t-a)\alpha][(\sigma^2(b)-\sigma(s))]}{p}, & \text{for } t\le s,\\ -\dfrac{[\beta+(\sigma(s)-a)\alpha][(\sigma^2(b)-t)]}{p}, & \text{for } \sigma(s)\le t,\end{cases} \tag{3.16}$$

$$p=\alpha(\sigma^2(b)-a)+\beta, \qquad \phi(t)=\frac{[\sigma^2(b)C+(\beta-\alpha a)B+(B\alpha-C)t]}{p}.$$

Thus, we want to prove that there exists at least one $y$ satisfying (3.15). Define an operator $T: C([a,\sigma^2(b)];\mathbb{R}^d)\to C([a,\sigma^2(b)];\mathbb{R}^d)$ by

$$(Ty)(t)=\int_a^{\sigma(b)} G(t,s)f(s,y^\sigma(s))\Delta s+\phi(t). \tag{3.17}$$

If we can prove that there exists a $y$ such that $T(y)=y$, then there exists a solution to (3.7). To show that $T$ has a fixed point, consider the equation

$$y=\lambda T(y), \quad \text{for } \lambda\in[0,1]. \tag{3.18}$$

Define an open, bounded subset of the Banach space $S$ by

$$\Omega=\left\{y\in S:\|y\|<\max\left\{\frac{\|C\|}{\alpha},R\right\}+1\right\}. \tag{3.19}$$

Note that (3.18) is equivalent to the BVP

$$\begin{gathered} y^{\Delta\Delta}=\lambda f(t,y^\sigma), \quad t\in[a,b],\\ \alpha y(a)-\beta y^\Delta(a)=\lambda C, \qquad y(\sigma^2(b))=\lambda B. \end{gathered} \tag{3.20}$$

Now show that all solutions to (3.20) must satisfy $y\in\Omega$ and consequently $y\notin\partial\Omega$ for all $\lambda\in[0,1]$. Obviously $y\in\Omega$ for $\lambda=0$. So consider (3.20) for $\lambda\in(0,1]$. Note that, by (2.1), (3.6) holds. Since $\lambda\alpha,\lambda\beta>0$ and $\|\lambda B\|\le\|B\|<R$, we see that Lemma 2.5 is applicable to solutions of (3.20), and hence

$$\|y\|\le\max\left\{\frac{\|\lambda C\|}{\alpha},R\right\}\le\max\left\{\frac{\|C\|}{\alpha},R\right\}. \tag{3.21}$$

Therefore, all solutions $y$ to (3.20) must satisfy $\|y\|<\max\{\|C\|/\alpha,R\}+1$ and $y\notin\partial\Omega$.

Since $f$ is continuous, $T$ is continuous and it can be shown that $T$ is a compact operator by the Arzela-Ascoli theorem. Therefore Theorem 1.8 is applicable to $T$ and $T$ must have a fixed point. Hence the BVP has a solution. This concludes the proof. □

Similarly, the following result holds.

Theorem 3.4. *Let $R>0$ be a constant. Suppose that $f$ is continuous on $[a,b]\times\mathbb{R}^d$ and satisfies (2.1). If $\|A\|<R$ and $\gamma,\delta>0$, then the BVP (1.1), (1.5) has at least one solution $y\in S$ satisfying*

$$||y(t)||<\max\left\{\frac{\|D\|}{\gamma},R\right\}+1, \quad \textit{for } t\in[a,\sigma^2(b)]. \tag{3.22}$$

*Proof.* The proof is similar to that of Theorem 3.3 and so is omitted. □

*Remark 3.5.* Theorems 3.1, 3.2, 3.3, and 3.4 establish bounds on *all* solutions to the respective BVPs (1.12), (1.2) through (1.12), (1.5). If there is no concern about bounding all of the solutions to the BVPs, then inequality (2.1) may be weakened to

$$\langle u, f(t,u)\rangle > 0, \quad \forall t \in [a,b],\ \|u\| = R, \tag{3.23}$$

and existence results will still hold, as the following theorems demonstrate.

THEOREM 3.6. *Let the conditions of Theorem 3.1 hold with (2.1) replaced by (3.23). Then the Dirichlet BVP (1.12), (1.2) has at least one solution $y \in S$ satisfying $\|y(t)\| < R$ on $[a,\sigma^2(b)]$ (and there may exist further solutions satisfying $\|y(t_0)\| \geq R$ for some $t_0 \in [a,\sigma^2(b)]$).*

*Proof.* Consider the modified dynamic equation

$$y^{\Delta\Delta} = m(t,y^\sigma), \quad t \in [a,b], \tag{3.24}$$

subject to the boundary conditions (1.2), where

$$m(t,y^\sigma) = \begin{cases} Rf((t,Ry^\sigma/||y^\sigma||)/||y^\sigma||), & \text{for } ||y^\sigma|| \geq R, \\ f(t,y^\sigma), & \text{for } ||y^\sigma|| \leq R. \end{cases} \tag{3.25}$$

Similar to the proof of Theorem 3.1, define an operator $T : C([a,\sigma^2(b)];\mathbb{R}^d) \to C([a,\sigma^2(b)];\mathbb{R}^d)$ by

$$(Ty)(t) = \int_a^{\sigma(b)} G(t,s)m(s,y^\sigma(s))\Delta s + \phi(t), \tag{3.26}$$

where $G$ and $\phi$ are given in the proof of Theorem 3.1. To show that $T$ has a fixed point, consider the equation

$$y = \lambda T(y), \quad \text{for } \lambda \in [0,1]. \tag{3.27}$$

Define an open, bounded subset of the Banach space $S$ by $\Omega = \{y \in S : \|y\| < ME + N + 1\}$, where here $\|\cdot\|$ is the sup norm, $E$ is the bound on $m$ and

$$M = \max_{t\in[a,\sigma^2(b)]} \int_a^{\sigma(b)} |G(t,s)|\Delta s, \qquad N = \max_{t\in[a,\sigma^2(b)]} |\phi(t)|. \tag{3.28}$$

It is easy to see that $\|\lambda T(y)\| \leq \lambda(ME+N) < ME+N+1$ for all $\lambda \in [0,1]$ and that Theorem 1.8 is applicable. Therefore, the BVP (3.24), (1.2) has a solution $y \in \Omega$. To show that this is a solution of the BVP (1.12), (1.2), see that, for $\|y\| \geq R$,

$$\langle y^\sigma, m(t,y^\sigma)\rangle = \langle p, f(t,p)\rangle > 0, \tag{3.29}$$

for $\|y\| \geq R = \|p\|$ by (3.23) and $p = Ry^\sigma/\|y^\sigma\|$. Therefore, all solutions to (3.24), (1.2) satisfy $\|y\| < R$ and are solutions to the BVP (1.12), (1.2). This concludes the proof. □

Theorem 3.7. *Let the conditions of Theorem 3.2 hold with (2.1) replaced by (3.23) and* $\max\{\|C\|/\alpha, \|D\|/\beta\} < R$. *Then the Sturm-Liouville BVP (1.12), (1.3) has at least one solution* $y \in S$ *satisfying* $\|y(t)\| < R$ *on* $[a, \sigma^2(b)]$ *(and there may exist further solutions satisfying* $\|y(t_0)\| \geq R$ *for some* $t_0 \in [a, \sigma^2(b)]$*).*

*Proof.* Consider the modified BVP (3.24), (1.3). Similar to the proof of Theorem 3.2, define an operator $T : C([a, \sigma^2(b)]; \mathbb{R}^d) \to C([a, \sigma^2(b)]; \mathbb{R}^d)$ by (3.26), where $G$ and $\phi$ are given in the proof of Theorem 3.2. To show that $T$ has a fixed point, consider equation (3.27). Define an open, bounded subset of the Banach space $S$ by $\Omega = \{y \in S : \|y\| < ME + N + 1\}$, where here $\|\cdot\|$ is the sup norm, $E$ is the bound on $m$, and (3.28) holds. It is easy to see that $\|\lambda T(y)\| \leq \lambda(ME + N) < ME + N + 1$ for all $\lambda \in [0, 1]$ and see that Theorem 1.8 is applicable. Therefore, the BVP (3.24), (1.3) has a solution $y \in \Omega$. To show that this is a solution of the BVP (1.12), (1.3), see that, for $\|y\| \geq R$, (3.29) holds, for $\|y\| \geq R = \|p\|$ by (3.23) and $p = Ry^\sigma/\|y^\sigma\|$. Therefore, all solutions to (3.24), (1.3) satisfy $\|y\| < R$ and are solutions to the BVP (1.12), (1.3). This concludes the proof. □

Theorem 3.8. *Let the conditions of Theorem 3.3 hold with (2.1) replaced by (3.23) and* $\|C\|/\alpha < R$. *Then the BVP (1.12), (1.4) has at least one solution* $y \in S$ *satisfying* $\|y(t)\| < R$ *on* $[a, \sigma^2(b)]$ *(and there may exist further solutions satisfying* $\|y(t_0)\| \geq R$ *for some* $t_0 \in [a, \sigma^2(b)]$*).*

Theorem 3.9. *Let the conditions of Theorem 3.4 hold with (2.1) replaced by (3.23) and* $\|D\|/\beta < R$. *Then the BVP (1.12), (1.5) has at least one solution* $y \in S$ *satisfying* $\|y(t)\| < R$ *on* $[a, \sigma^2(b)]$ *(and there may exist further solutions satisfying* $\|y(t_0)\| \geq R$ *for some* $t_0 \in [a, \sigma^2(b)]$*).*

*Proofs.* The proofs follow the modification technique of Theorems 3.6 and 3.7 and so are omitted for brevity. □

## 4. On nonincreasing solutions

Some results about the qualitative nature of solutions for the BVPs

$$y^{\Delta\Delta} = f(t, y^\sigma), \quad t \in [a, b], \tag{4.1}$$

$$y(a) = A, \qquad y(\sigma^2(b)) = 0, \tag{4.2}$$

$$y(a) = 0, \qquad y(\sigma^2(b)) = B, \tag{4.3}$$

are now proved. In particular, by strengthening inequality (2.1), the solutions furnished by Theorem 3.1 may be shown to be nondecreasing or nonincreasing in norm.

Corollary 4.1. *Let the conditions of Theorem 3.1 hold for the BVP (4.1), (4.2) with (2.1) strengthened to*

$$\langle u, f(t, u)\rangle > 0, \quad \forall t \in [a, b] \text{ and all } u \neq 0. \tag{4.4}$$

*Then the solutions to (4.1), (4.2) guaranteed by Theorem 3.1 satisfy that* $\|y(t)\|$ *is nonincreasing on* $[a, \sigma^2(b)]$.

*Proof.* Note that (4.4) implies that $r(t) := \|y(t)\|^2$ cannot have a nonnegative maximum in $(a,\sigma^2(b))$ for any solution $y$, and therefore $r$ must have a maximum at either $t = a$ or $t = \sigma^2(b)$ with $\max r(t) = \max\{r(a), r(\sigma^2(b))\} = A^2$. □

Corollary 4.2. *Let the conditions of Theorem 3.1 hold for the BVP (4.1), (4.3) with (2.1) strengthened to (4.4). Then the solutions $y$ to (4.1), (4.3) guaranteed by Theorem 3.1 satisfy $\|y(t)\|$ is nondecreasing on $[a,\sigma^2(b)]$.*

*Proof.* The proof is similar to that of Corollary 4.1. □

## 5. BVPs on infinite intervals

This section formulates the existence theorems for solutions to the following BVPs on infinite intervals:

$$y^{\Delta\Delta} = f(t, y^\sigma), \quad t \in [a,\infty), \tag{5.1}$$

$$y(a) = A, \quad y(t) \text{ is bounded for } t \in [a,\infty), \tag{5.2}$$

$$\alpha y(a) - \beta y^\Delta(a) = C, \quad y(t) \text{ is bounded for } t \in [a,\infty). \tag{5.3}$$

In particular, Theorems 3.1 and 3.3 will be useful.

Let $[a,\infty) = \cup_{k=1}^{\infty}[a,t_k]$. Throughout this section assume that there exists $t_n \in \mathbb{T}$ and $n \in \mathbb{N}$ such that

$$a < t_1 < t_2 < \cdots < t_n < \cdots \quad \text{with } t_n \uparrow \infty \text{ as } n \longrightarrow \infty. \tag{5.4}$$

Theorem 5.1. *Suppose that $f$ is continuous on $[a,\infty) \times \mathbb{R}^d$ and satisfies*

$$\langle u, f(t,u)\rangle > 0, \quad \forall t \in [a,\infty),\ \|u\| \geq R, \tag{5.5}$$

*where $R > 0$. Then for each $\|A\| < R$, the BVP (5.1), (5.2) has at least one solution $y \in C([a,\infty);\mathbb{R}^d)$ with $\|y(t)\| < R$ on $[a,\infty)$.*

*Proof.* Fix $n \in \mathbb{N}$ and consider the BVP

$$\begin{aligned} y^{\Delta\Delta} &= f(t, y^\sigma), \quad t \in [a,t_n], \\ y(a) &= A, \qquad y(\sigma^2(t_n)) = 0. \end{aligned} \tag{5.6}$$

It is clear from Theorem 3.1 that (5.6) has a solution $y_n \in C([a,\sigma^2(t_n)];\mathbb{R}^d)$ with $\|y_n(t)\| < R$ for $t \in [a,t_n]$. (Note also that $y_n^{\Delta\Delta} \in C_{rd}[a,\sigma^2(t_n)];\mathbb{R}^d$.) This argument can be used for each $n \in \mathbb{N}$. The theorem then follows from Ascoli's selection theorem (see [11]) applied to a sequence of intervals $[a,t_n]$ as $n \to \infty$. □

Theorem 5.2. *Suppose that $f$ is continuous on $[a,\infty) \times \mathbb{R}^d$ and satisfies (5.5), where $R > 0$. If $\alpha,\beta > 0$, then the Sturm-Liouville BVP (5.1), (5.3) has at least one solution $y \in C([a,\infty);\mathbb{R}^d)$ satisfying*

$$||y(t)|| < \max\left\{\frac{\|C\|}{\alpha}, R\right\} + 1, \quad \textit{for } t \in [a,\infty). \tag{5.7}$$

*Proof.* Fix $n \in \mathbb{N}$ and consider the BVP

$$\begin{gathered} y^{\Delta\Delta} = f(t, y^{\sigma}), \quad t \in [a, t_n], \\ \alpha y(a) - \beta y^{\Delta}(a) = C, \quad y(\sigma^2(t_n)) = 0. \end{gathered} \tag{5.8}$$

It is clear from Theorem 3.3 that (5.8) has a solution $y_n \in C([a, \sigma^2(t_n)]; \mathbb{R}^d)$ with $\|y_n\| < M + 1$ for $t \in [a, t_n]$. (Note also that $y_n^{\Delta\Delta} \in C_{rd}([a, \sigma^2(t_n)]; \mathbb{R}^d)$.) This argument can be used for each $n \in \mathbb{N}$. The theorem then follows from Ascoli's selection theorem applied to a sequence of intervals $[a, t_n]$ as $n \to \infty$. □

## 6. On uniqueness of solutions

This section provides some results which guarantee the uniqueness of solutions to the Dirichlet BVP (1.12), (1.2).

THEOREM 6.1. *If $f$ satisfies*

$$\langle u - v, f(t, u) - f(t, v) \rangle > 0, \quad \forall t \in [a, b],\ u \neq v, \tag{6.1}$$

*then (1.12) has, at most, one solution satisfying (1.2).*

*Proof.* Assume that $y$ and $z$ are solutions to the Dirichlet BVP (1.12), (1.2). Then $y - z$ satisfies the BVP

$$\begin{gathered} y^{\Delta\Delta}(t) - z^{\Delta\Delta}(t) = f(t, y(\sigma(t))) - f(t, z(\sigma(t))), \quad t \in [a, b], \\ y(a) - z(a) = 0, \qquad y(\sigma^2(b)) - z(\sigma^2(b)) = 0. \end{gathered} \tag{6.2}$$

Consider $r(t) := \|y(t) - z(t)\|^2$, $t \in [a, \sigma^2(b)]$. Now $r$ must have a positive maximum at some point $c \in [a, \sigma^2(b)]$. From the boundary conditions, $c \in (a, \sigma^2(b))$. Choosing $c$ in the same fashion as in the proof of Lemma 2.2, it can be shown via the same reasoning that $c$ cannot be simultaneously left-dense and right-scattered. Therefore, by Lemma 2.1 we must have (2.4). So using the product rule we have

$$r^{\Delta\Delta}(\rho(c)) \geq 2\langle y^{\sigma}(\rho(c)) - z^{\sigma}(\rho(c)), f(\rho(c), y^{\sigma}(\rho(c))) - f(\rho(c), z^{\sigma}(\rho(c))) \rangle > 0, \tag{6.3}$$

which contradicts (2.4). (Notice at $c$ that $\|y^{\sigma}(\rho(c)) - z^{\sigma}(\rho(c))\| = \|y(c) - z(c)\|$, since $c$ is not simultaneously left-dense and right-scattered.) Therefore $r(t) = \|y(t) - z(t)\|^2 = 0$ for $t \in [a, \sigma^2(b)]$, and solutions of the BVP (1.12), (1.2) must be unique. This concludes the proof. □

## 7. Uniqueness implies existence

In this section, a uniqueness-implies-existence result is formulated for the BVP (1.13), (1.2). Since the nonlinear alternative is not required, the continuity requirements of the matrices $P(t)$ and $Q(t)$ may be relaxed to $P, Q \in C_{rd}$.

The following is a vector analogue of a result of Bohner and Peterson [6].

Theorem 7.1. *Let $P, Q \in C_{rd}$ and suppose that the BVP*

$$\begin{aligned} y^{\Delta\Delta}(t) &= P(t)y^{\Delta}(\sigma(t)) + Q(t)y(\sigma(t)), \quad t \in [a,b], \\ y(a) &= 0, \qquad y(\sigma^2(b)) = 0, \end{aligned} \tag{7.1}$$

*has only the zero solution. Then the BVP (1.13), (1.2) has a unique solution for each $h \in C_{rd}([a,b];\mathbb{R}^d)$.*

*Proof.* The proof is omitted as it follows lines similar to that of Bohner and Peterson [6] with only minor modifying changes. □

Theorem 7.2. *Let $P(t)$ and $Q(t)$ be $d \times d$ matrices satisfying*

$$\langle (2Q(t) - P(t)P^T(t))u, u\rangle > 0, \tag{7.2}$$

*for $t \in [a,b]$, $u \neq 0$. Then (1.13) has a unique solution satisfying the boundary conditions (1.2).*

*Proof.* Since (1.13) is linear, the difference of two solutions to the BVP (1.13), (1.2) is also a solution of the BVP

$$\begin{aligned} y^{\Delta\Delta}(t) &= Q(t)y^{\sigma} + P(t)y^{\Delta}(\sigma(t)), \quad t \in [a,b], \\ y(a) &= 0, \qquad y(\sigma^2(b)) = 0, \end{aligned} \tag{7.3}$$

and it needs to be shown that the only solution to (7.3) is $y = 0$.

Assume the contrary, let $y$ be a nontrivial solution to (7.3) and put $r(t) = \|y(t)\|^2$. Now $r$ must have a positive maximum at some point $c \in [a, \sigma^2(b)]$. From the boundary conditions, $c \in (a, \sigma^2(b))$. Choosing $c$ in the same fashion as in the proof of Lemma 2.2 it can be shown via the same reasoning that $c$ cannot be simultaneously left-dense and right-scattered. Therefore by Lemma 2.1, (2.4) holds. So using the product rule we have

$$r^{\Delta\Delta}(\rho(c)) = 2\langle y^{\sigma}(\rho(c)), f(\rho(c), y^{\sigma}(\rho(c)))\rangle + ||y^{\Delta}(\rho(c))||^2 + ||y^{\sigma\Delta}(\rho(c))||^2. \tag{7.4}$$

Using the identity $\langle Ab, c\rangle = \langle b, A^T c\rangle$, it can be verified that

$$\begin{aligned} 2\langle P(\rho(c))y^{\Delta}(\sigma(\rho(c))) &+ Q(\rho(c))y^{\sigma}(\rho(c)), y^{\sigma}(\rho(c))\rangle + ||y^{\Delta}(\sigma(\rho(c)))||^2 \\ &= ||y^{\Delta}(\sigma(\rho(c))) + P^T(\rho(c))y^{\sigma}(\rho(c))||^2 \\ &\quad + \langle (2Q(\rho(c)) - P(\rho(c))P^T(\rho(c)))y^{\sigma}(\rho(c)), y^{\sigma}(\rho(c))\rangle, \end{aligned} \tag{7.5}$$

and therefore

$$r^{\Delta\Delta}(\rho(c)) \geq \langle (2Q(\rho(c)) - P(\rho(c))P^T(\rho(c)))y^{\sigma}(\rho(c)), y^{\sigma}(\rho(c))\rangle. \tag{7.6}$$

Hence (7.2) implies $r^{\Delta\Delta}(\rho(c)) > 0$, which contradicts (2.4). It follows that $r(t) = 0$ on $[a, \sigma^2(b)]$. That is, the only solution to (7.3) is $y = 0$. The existence of solutions to (1.13), (1.2) now follows from Theorem 7.1 and this completes the proof. □

The paper is now concluded with an example of a nonlinear vector BVP on a number of different time scales. First, we give $y^{\Delta\Delta}$ for three simple examples.

*Example 7.3.* Let $\mathbb{T} = \mathbb{Z}$. Here $\sigma(t) = t + 1$ and

$$y^{\Delta\Delta}(t) = y(t+2) - 2y(t+1) + y(t). \tag{7.7}$$

*Example 7.4.* Let $h > 0$ and $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$. Here $\sigma(t) = t + h$ and

$$y^{\Delta\Delta}(t) = \frac{y(t+2h) - 2y(t+h) + y(t)}{h^2}. \tag{7.8}$$

*Example 7.5.* Let $q > 1$ and $\mathbb{T} = q^{\mathbb{N}_0}$. Here $\sigma(t) = qt$ and

$$y^{\Delta\Delta}(t) = \frac{y(q^2 t) - (q+1)y(qt) + qy(t)}{q(q-1)^2 t^2}. \tag{7.9}$$

Dynamic equations on the time scale $\mathbb{T} = q^{\mathbb{N}_0}$ are called $q$-difference equations and there are many important applications of $q$-difference equations.

*Example 7.6.* Consider the BVP

$$\begin{pmatrix} y_1^{\Delta\Delta} \\ y_2^{\Delta\Delta} \end{pmatrix} = \begin{pmatrix} t^2 \left(y_1^\sigma\right)^3 - y_2^\sigma \\ y_1^\sigma + y_2^\sigma \exp\left(y_1^\sigma y_2^\sigma\right) \end{pmatrix}, \quad t \in [a,b],\ a > 0, \tag{7.10}$$

subject to the boundary conditions

$$\left(y_1(a), y_2(a)\right) = (1,1), \qquad \left(y_1(\sigma^2(b)), y_2(\sigma^2(b))\right) = (0,0). \tag{7.11}$$

Note that the conditions of Theorem 3.3 are satisfied, and thus the BVP (7.10), (7.11) has a solution $y = (y_1, y_2)$ on each of the above time scales, $\mathbb{T} = \mathbb{Z}$, $h\mathbb{Z}$ and $q^{\mathbb{N}_0}$, satisfying $\|y(t)\| < 2$. Note also that the inequality (4.4) holds, and therefore $\|y(t)\|$ is nonincreasing on $[a, \sigma^2(b)]$.

*Remark 7.7.* The question on how to ensure that $[y(\sigma(t))]^{\Delta}$ exists is naturally raised. This "smoothness" requirement will be satisfied if, for example, $\sigma$ is differentiable or when the points in $\mathbb{T}$ are isolated (left-scattered and right-scattered). However, if these cases are excluded, then the method of upper and lower solutions for systems of equations can be developed with the a priori bounds on solutions being obtained in a component-wise fashion. These arguments are essentially minor extensions of the ideas in [4].

**Acknowledgment**

The third author gratefully acknowledges the financial support of the University of New South Wales (UNSW) and The Australian Research Council's Discovery Projects (DP0450752).

## References


[1] R. P. Agarwal, M. Bohner, and D. O'Regan, *Time scale boundary value problems on infinite intervals*, J. Comput. Appl. Math. **141** (2002), no. 1-2, 27–34.

[2] R. P. Agarwal, M. Bohner, D. O'Regan, and A. Peterson, *Dynamic equations on time scales: a survey*, J. Comput. Appl. Math. **141** (2002), no. 1-2, 1–26.

[3] R. P. Agarwal and D. O'Regan, *Nonlinear boundary value problems on time scales*, Nonlinear Anal. **44** (2001), no. 4, 527–535.

[4] E. Akin, *Boundary value problems for a differential equation on a measure chain*, Panamer. Math. J. **10** (2000), no. 3, 17–30.

[5] E. Beltrami, *Mathematics for Dynamic Modeling*, Academic Press, Massachusetts, 1987.

[6] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales*, Birkhäuser Boston, Massachusetts, 2001.

[7] M. Bohner and A. Peterson (eds.), *Advances in Dynamic Equations on Time Scales*, Birkhäuser Boston, Massachusetts, 2003.

[8] L. Erbe and A. Peterson, *Green's functions and comparison theorems for differential equations on measure chains*, Dynam. Contin. Discrete Impuls. Systems **6** (1999), no. 1, 121–137.

[9] L. Erbe, A. Peterson, and R. Mathsen, *Existence, multiplicity, and nonexistence of positive solutions to a differential equation on a measure chain*, J. Comput. Appl. Math. **113** (2000), no. 1-2, 365–380.

[10] T. Gnana Bhaskar, *Comparison theorem for a nonlinear boundary value problem on time scales*, J. Comput. Appl. Math. **141** (2002), no. 1-2, 117–122.

[11] P. Hartman, *Ordinary Differential Equations*, John Wiley & Sons, New York, 1964.

[12] S. Hilger, *Analysis on measure chains—a unified approach to continuous and discrete calculus*, Results Math. **18** (1990), no. 1-2, 18–56.

[13] B. Kaymakçalan, *Existence and comparison results for dynamic systems on a time scale*, J. Math. Anal. Appl. **172** (1993), no. 1, 243–255.

[14] V. Lakshmikantham, S. Sivasundaram, and B. Kaymakcalan, *Dynamic Systems on Measure Chains*, Mathematics and Its Applications, vol. 370, Kluwer Academic Publishers, Dordrecht, 1996.

[15] N. G. Lloyd, *Degree Theory*, Cambridge University Press, Cambridge, 1978.



Johnny Henderson: Department of Mathematics, Baylor University, Waco, TX 76798-7328, USA
*E-mail address*: johnny_henderson@baylor.edu

Allan Peterson: Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588-0323, USA
*E-mail address*: apeterso@math.unl.edu

Christopher C. Tisdell: School of Mathematics, The University of New South Wales, Sydney, NSW 2052, Australia
*E-mail address*: cct@maths.unsw.edu.au