COMPUTATION OF RELATIVE INTEGRAL BASES FOR ALGEBRAIC NUMBER FIELDS

MAHMOOD HAGHIGHI
Department of Computer Science
Bradley University
Peoria, IL 61625

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ABSTRACT. At first we are given conditions for existence of relative integral bases for extension \((K;k) = n\). Then we will construct relative integral bases for extensions \(O_{K_6}(\sqrt[6]{3})/O_{k_2}(\sqrt[3]{3}), O_{K_6}(\sqrt[6]{3})/O_{k_3}(\sqrt[3]{3}), O_{K_6}(\sqrt[6]{3})/Z\).

KEY WORDS AND PHRASES. Integral Bases and Principal Ideal.

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1. EXISTENCE OF A RELATIVE INTEGRAL BASES.

The following criterion has been shown in [1] for existence of a Relative Integral Bases, for any finite extension \(K/k\).

**THEOREM 1.1.** Let \((K;k) = n\), and let \(h_k\) be an odd integer then \(O_K\) has a "relative integral bases" over \(O_k\) if \(d_{K/k}\) is a principal ideal. See also [2].

**COROLLARY 1.2.** If \(O_K\) is P.I.D., then \(h_k = 1\) and \(d_{K/k}\) is P.I. Therefore for every finite extension of \(k\) where \(O_k\) is P.I.D., a relative integral bases exists.

Let \(K_1 = O_{k_1}, K_2 = O(\sqrt[3]{3}), K_3 = O(\sqrt[3]{3}), K_6 = O(\sqrt[3]{3})\). Since \(h_k_1 = h_k_2 = h_k_3 = 1\), so \(O_k_1, O_k_2, O_k_3\) are P.I.D. and then by corollary 1.2, relative integral bases for extensions \(K_6/k_1, K_6/k_2, K_6/k_3\) exists.

Now, we will compute the relative discriminant for the extensions. Let \((K;k) = n\) and for some \(\theta \in K, O_K = O_k(\theta)\) and \(\theta\) satisfies an equation \(F(\theta) = 0\) of degree \(n\).

**THEOREM 1.2.** If \(F(\theta) = \prod(\theta - \theta(t))\), where \(\theta, \theta(1), \theta(2), \ldots, \theta(n)\) are conjugates [3].

Since extensions \(K_2/K_1, K_3/K_1\) have discriminants divisible by 3 [3], by theorem in [3] discriminants \(K_6/k_2, K_6/k_3, K_6/k_1\) are also divisible by 3 and 3 is completely ramified in \(k_1, k_2, k_3\).

For extension \(K_6/k_2\), \(\theta = \sqrt[6]{3}\) we therefore have:

\[D_{K_6/k_2} = (\theta - \theta(1))(\theta - \theta(2)) = (\sqrt[6]{3} - \rho \sqrt[6]{3}) \cdot (\sqrt[6]{3} - \rho^2 \cdot \sqrt[6]{3})\],

\[D_{K_6/k_2} = (-3)^{4/3}\] for \(\rho = -\frac{1 + \sqrt[3]{3}}{2}\). By the definition in [4],

\[d_{K_6/k_2} = N_{K_6/k_2}(D_{K_6/k_2}) = (-3)^4.\]
For extension \( K_6/k_3 \), \( \theta = 6\sqrt{-3} \), \( D_{K_6/k_3} = (\theta - \theta(1)) = (-3)^{1/6} \), then \( d_{K_6/k_3} = (-3)^{1/2} \).

By theorem in [4], \( D_{K_6/k_1} = D_{K_6/k_2} = D_{K_2/k_1} = (3)^{4/3} \cdot (-3)^{1/2} = (3)^{11/6} \), then \( d_{K_6/k_1} = (-3)^{11} \).

Now we will construct relative integral bases for the extensions. See also [5] for associated work.

For \( K_3/k_1 \), \( \Omega_{K_3} = (1, 3\sqrt{-3}, \sqrt{-3}) \cdot Z, [3] \).

For \( K_2/k_1 \), \( \Omega_{K_2} = (1, \frac{1 + \sqrt{-3}}{2}) \cdot Z, [3] \).

2. RELATIVE INTEGRAL BASES FOR \( O_6(6\sqrt{-3})/O_2(\sqrt{-3}) \).

Let \( O_6 = (1, \alpha, \beta)O_2 \) for \( \alpha, \beta \in O_6 \). By theorem in [6], \( \text{disc} (1, \alpha, \beta) = d_{K_6/k_2} \),

\[
\begin{array}{c|c|c}
\alpha & \beta & d_{K_6/k_2} \\
1 & \alpha & \rho^2 \beta \\
1 & \beta & \rho^2 \alpha
\end{array}
\]

Now \( \alpha^2 \beta^2 (3\alpha^2 - 3\beta^2) = (-3)^6 \) and from here \( \alpha \cdot \beta = \sqrt{-3} \).

We may take \( \alpha = 6\sqrt{-3} \) and \( \beta = 6\sqrt{(-3)^2} \), because they satisfy an \( \alpha \cdot \beta = \sqrt{-3} \) and they are in \( O_6 \).

Since \( N_{6/3}(\alpha) = 3\sqrt{-3} \) and \( N_{6/3}(\beta) = 3\sqrt{(-3)^2} \) are in \( O_3 \), we have:

\( O_6 = (1, 6\sqrt{-3}, 6\sqrt{(-3)^2})O_2 \).

3. RELATIVE INTEGRAL BASES FOR \( O_6(6\sqrt{-3})/O_3(3\sqrt{-3}) \).

Let \( O_6 = (1, \alpha)O_3 \) for \( \alpha \in O_6 \). Again by theorem [6]

\[
\text{disc}(1, \alpha) = \begin{vmatrix} 1 & \alpha \\ 1 & -\alpha \end{vmatrix} = 4\alpha^2 = d_{K_6/k_3} = 3\sqrt{-3},
\]

Note \( \alpha = \frac{6\sqrt{-3}}{2} \not\in O_6 \), because \( N_{6/3}(\alpha) = \frac{6\sqrt{-3}}{2} \cdot \frac{-\sqrt{-3}}{2} = \frac{-3\sqrt{-3}}{4} \epsilon O_3 \). Hence, \( (1, \alpha) \) is not a relative integral bases.

We define \( \alpha = \frac{\beta + 3\sqrt{-3}}{2} \) for \( \beta \epsilon O_3 \) such that \( N_{6/3}(\alpha) \) is divisible by \( 2.2 = 4 \) and \( \alpha \epsilon O_6 \). If we take \( \beta = 3\sqrt{(-3)^2} \epsilon O_3 \), it satisfies the conditions, this is because

\[
\frac{\beta + 3\sqrt{-3}}{2} \cdot \frac{\beta - 6\sqrt{-3}}{2} = \frac{3\sqrt{(-3)^2} - 6\sqrt{(-3)^2}}{4} = 3\sqrt{-3} \epsilon O_3, \text{ by theorem [6]},
\]

Also, \( \text{disc}(1, \alpha) = d_{K_6/k_3} \), so that:

\( O_6 = \left(1, \frac{3\sqrt{(-3)^2} + 6\sqrt{-3}}{2}\right) \cdot O_3 \).

4. RELATIVE INTEGRAL BASES FOR \( O_6(6\sqrt{-3})/Z \).

Since \( K_6 = Q(6\sqrt{-3}) \), at first we start by:

\( O_6 = (1, \theta, \theta^2, \theta^3, \theta^4, \theta^5) \cdot Z \)

Let \( \theta = 6\sqrt{-3} \epsilon O_6 \). Since \( \text{disc}(1, \theta, \theta^2, \theta^3, \theta^4, \theta^5) = 2^2 \cdot 2^2 \cdot 2^2 \cdot d_{K_6/k_1} \), we can apply
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We will build a new bases \( a_1^* = \{a_1: 0 \leq 1 \leq 5\} \). By the theorem [3] we check which \( a_1 \) is going to be changed. \( a_0^* = a_0/2 = 1/2 \notin 0_6 \). Thus there is no change for the first bases element \( a_0^* = 1 \).

\[
a_1^* = \frac{g_1 a_0 + a_1}{2} = \frac{g_1 a_0 + g_1}{2} \quad \text{for} \quad 0 \leq g_1 \leq 1.
\]

For any value of \( g_1 \), \( a_1^* \) is not in \( 0_6 \).

This is because

\[
N_{6/3}(a_1^*) = \frac{1}{2} \cdot 4 \cdot 2 \cdot 3 = \frac{1}{2} \cdot 3 \cdot 2 \notin 0_3 \quad \text{and also since} \quad N_{6/3}(\theta/2) \notin 0_3, \quad \text{so there is no change for} \quad a_1.
\]

\[
a_2^* = \frac{g_1 a_0 + g_2 a_1 + a_2}{2} \quad \text{for} \quad 0 \leq g_1 \leq 1.
\]

For any value of \( g_1, a_2^* \notin 0_6 \), then there will be no change for \( a_2 \).

\[
a_3^* = \frac{g_1 a_0 + g_2 a_1 + g_3 a_2 + a_3}{2} \quad \text{for} \quad 0 \leq g_1 \leq 1.
\]

In this case for \( g_1 = g_2 = g_3 = 1 \),

\[
a_3^* = 6 \sqrt{-3} e_{06}. \quad \text{This is because:}
\]

\[
a_3^* = \frac{1}{2} \cdot 4 \cdot 2 \cdot 3 = \frac{1}{2} \cdot 3 \cdot 2 \notin 0_3, \quad \text{and for other values of} \quad g_1, a_3^* \notin 0_6.
\]

\[
a_4^* = \frac{g_1 a_0 + g_2 a_1 + g_3 a_2 + g_4 a_3 + a_4}{2} \quad \text{for} \quad 0 \leq g_1 \leq 1.
\]

In this case for \( g_2 = g_4 = 1 \),

\[
a_4^* = \frac{6 \sqrt{-3} + 6 \sqrt{-3} e_{06}. \quad \text{This is because}}
\]

\[
N_{6/3}(a_4^*) = \frac{6 \sqrt{-3} + 6 \sqrt{-3} e_{06}. \quad \text{This is because}}
\]

\[
a_5^* = \frac{g_1 a_0 + g_2 a_1 + g_3 a_2 + g_4 a_3 + g_5 a_4 + a_5}{2}, \quad \text{for} \quad g_2 = g_5 = 1,
\]

\[
a_5^* = \frac{6 \sqrt{-3} + 6 \sqrt{-3} e_{06}. \quad \text{This is because}}
\]

\[
\text{disc}(a_0, a_1, a_2, a_3, a_4, a_5) = \frac{2^2 \cdot 2^2 \cdot 2^2}{2^2 \cdot 2^2 \cdot 2^2} \cdot \delta_{k6/k1}, \quad \text{and each} \quad a_1, a_4^* \quad \text{are in} \quad 0_6, \quad \text{then}
\]

\[
0_6 = \left\{1, 6 \sqrt{-3}, 6 \sqrt{-3}, 6 \sqrt{-3}, 6 \sqrt{-3}, 6 \sqrt{-3} \right\} \cdot 2.
\]
REFERENCES


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