SMOOTH STRUCTURES ON SPHERE BUNDLES OVER SPHERES

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1. INTRODUCTION

Let E represent p-sphere bundle over a q-sphere with β ∈ π_{q-1}SO(p+1) the characteristic class of the corresponding p+1-disc bundle over the q-sphere. In [4] R. De Sapio gave a complete classification of the special case where β = 0. In [5] and [6] Kawakubo and Schultz respectively also gave a classification of E for this special case. This author in [7] gave a generalization of this special case to product of three ordinary spheres. In [1] a classification of E was given for p < q - 1 and where E has a cross-section and β ≠ 0. In [3] Schultz gave a classification of E for p ≥ q and E is without cross-section. We shall here remove the fact that E has a cross-section so that not every element of π_{q-1}SO(p+1) can be pulled back to the element π_{q-1}SO(p) in the homomorphism S_* : π_{q-1}SO(p) → π_{q-1}SO(p+1) induced by the inclusion s : SO(p) → SO(p+1). S^n denotes the unit n-sphere with the usual differential structure in the Euclidean
(n+1)-space $\mathbb{R}^{n+1} \times \mathbb{Z}^n$ denotes an homotopy n-sphere and $\Theta^n$ denotes the group of homotopy n-spheres. $H(p, k)$ denotes the subset of $\Theta^p$ which consists of those homotopy p-sphere $\Sigma^p$ such that $\Sigma^p \times S^k$ is diffeomorphic to $S^p \times S^k$. By [4, Lemma 4], $H(p, k)$ is a subgroup of $\Theta^p$ and it is not always zero and in fact in [7] we showed that if $k \geq p - 3$, $H(p, k) = \Theta^p$. We shall adopt the notation $E(\mathbb{Z}^q)$ to represent the total space of a p-sphere bundle over a homotopy q-sphere $\mathbb{Z}^q$. We will then prove the following:

**THEOREM.** If $M$ is a smooth, n-manifold homeomorphic to a p-sphere bundle over a q-sphere with total space $E$ where $n = p + q \geq 6$ and $p < q$ then there exists homotopy spheres $\mathbb{Z}^q$ and $\mathbb{Z}^n$ such that $M$ is diffeomorphic to $E(\mathbb{Z}^q) \# \mathbb{Z}^n$. We shall define a pairing

$$G : \pi_p SO(q) \times \pi_{q-1} SO(p+1) \to \Theta^{p+q}$$

and show that if $\beta \in \pi_{q-1} SO(p+1)$ is the characteristic class of a p-sphere bundle over an homotopy q-sphere $\mathbb{Z}^q$, then $G(\pi_p SO(q), \beta)$ equals the inertial group of $E(\mathbb{Z}^q)$. The above theorem together with the latter will give us the following.

**THEOREM.** Let $E$ be the total space of a p-sphere bundle over a q-sphere then the diffeomorphism classes of $(p+q)$-manifolds that are homeomorphic to $E$ are in one-to-one correspondence with the group

$$\frac{\Theta^q}{H(p, q)} \times \frac{\Theta^n}{\text{Image} \ G_B}, \text{ where } n = p + q \geq 6 \text{ and } p < q.$$

2. **CLASSIFICATION THEOREM**

In this section, we will prove the classification theorem for any manifold $M^n$ homeomorphic to $E$. We will apply the obstruction theory to smoothing of manifolds developed by Munkres in [8]. Since $p + q \geq 6$ and $2 \leq p < q$ then $E$ is simply-connected and the homology of $E$ has no 2-torsion, hence the "Hauptvermutung" of D. Sullivan [9] applies and this means that piecewise linear homeomorphism can be replaced by homeomorphism, we shall not distinguish the two.

**DEFINITION.** Let $M$ and $N$ be smooth closed n-manifolds and $L$ a closed subset of $M$ of dimension less than $n$. Let $f : M \to N$ be a homeomorphism such that for each simplex $\gamma$ of $L$, $f(\gamma)$ are contained in coordinate systems under which they are flat. $f$ is said to be a diffeomorphism modulo $L$ if $f|_{(M-L)}$ is a diffeomorphism and each simplex $\gamma$ of $L$ has a neighborhood $V$ such that $f$ is smooth on $V-L$ near $\gamma$. By [8, Theorem 2.8], if $M$ and $N$ are homeomorphic then there is a diffeomorphism modulo $(n-1)$-skeleton of $M$. If $f : M \to N$ is a diffeomorphism modulo $m$-skeleton $m < n$ then the obstruction to deforming
f to a diffeomorphism modulo (m-1)-skeleton g : M → N is an element λ(f) ∈ H_m(M, r^{n-m}) where r^{n-m} is a group of diffeomorphism of S^{n-m-1} modulo those that extend to diffeomorphisms of D^{n-m}. g is called the smoothing of f. If λ(f) = 0 then by [8, §4] smoothing g exist.

**THEOREM 2.1.** If M is a smooth n-manifold homeomorphic to E where E denotes the total space of a p-sphere bundle over a q-sphere, 2 ≤ p < q and n = p + q then there exist homotopy spheres z^q and z^n such that M is diffeomorphic to E(z^q) # z^n where E(z^q) denotes the total space of a p-sphere bundle over the homotopy q-sphere z^q.

**PROOF.** E is the total space of a p-sphere bundle over a q-sphere with characteristic class [b] ∈ π_q ISO(p+1) then E = D^q × S^p ∪_{f_b} D^q × S^p where f_b : S^{q-1} × S^p → S^q × S^p is a diffeomorphism defined by f_b(x,y) = (x, b(x), y), (x, y) ∈ S^{q-1} × S^p.

\[
H_i(E) = \begin{cases} Z & \text{for } i = 0, p, q, p+q \\ 0 & \text{elsewhere} \end{cases}
\]

Since M^q is homeomorphic to E where n = p+q ≥ 6 2 ≤ p < q, then M^q is simply connected and since H_3(M, Z) has no 2-torsion, then "Hauptvermutung" of D. Sullivan [9] implies that there is a piecewise linear homeomorphism h : M^q → E which by [8, §5] is a diffeomorphism modulo (n-1)-skeleton. Since H_i(M, Z) = 0 for n-p+1 ≤ i ≤ n-1 then we can assume that h is a diffeomorphism modulo n-p = q skeleton. The obstruction to a diffeomorphism modulo q-1 skeleton is λ(h) ∈ H_q(M, r^p) = r^p. If [ϕ] = λ(h) ∈ r^p where ϕ : S^{p-1} × S^{p-1} is a diffeomorphism that represents λ(h) and let z^p denote the homotopy p-sphere where z^p = D^p_1 ∪_{id.} D^p_2. We define a map

\[
j : S^p → z^p
\]

such that

\[
j(x) = \begin{cases} x & \text{if } x ∈ D^p_1 \\ \phi^{-1}(\frac{x}{|x|}) & \text{if } x ∈ D^p_2. \end{cases}
\]

So j is an homeomorphism which is identity on D^p_1 and the radial extension of \(ϕ^{-1}\) on D^p_2 and so the first obstruction \(λ(j)\) to deforming j to a diffeomorphism is [\(ϕ^{-1}\)] = -λ(h).

We then define id × j : D^q × S^p → D^q × z^p where id is the identity, then id × j is a homeomorphism and it follows from [8, Def. 3.4] that the first obstruction \(λ(id × j)\) to
deforming \( \text{id} x_j \) to a diffeomorphism is also \(-\lambda(h)\). We can form a manifold \( E' \) by identifying two copies of \( D^q \times \Sigma^p \) along their common boundaries \( S^{q-1} \times \Sigma^p \) by the diffeomorphism \( f_b : S^{q-1} \times \Sigma^p \to S^{q-1} \times \Sigma^p \) where \( f_b(x,y) = (x,b(x),y) \) and 

\[ [b] \in \pi_{q-1} SO(p+1). \]

So \( E' = D^q \times \Sigma^p \cup_{f_b} D^q \times \Sigma^p \). We define a map 

\[ g : E = (D^q \times \Sigma^p)_1 \cup_{f_b} (D^q \times \Sigma^p)_2 \to (D^q \times \Sigma^p)_1 \cup_{f_b} (D^q \times \Sigma^p)_2 \]

on both \((D^q \times \Sigma^p)_1\), and \((D^q \times \Sigma^p)_2\), the map looks like 

\[ E = (D^q \times \Sigma^p)_1 \cup_{f_b} (D^q \times \Sigma^p)_2 = (D^q \times \Sigma^p)_1 \cup_{f_b} S^{q-1} \times \Sigma^p \cup_{\text{id}} (D^q \times \Sigma^p)_2 \]

\[ E' = (D^q \times \Sigma^p)_1 \cup_{f_b} (D^q \times \Sigma^p)_2 = (D^q \times \Sigma^p)_1 \cup_{f_b} S^{q-1} \times \Sigma^p \cup_{\text{id}} (D^q \times \Sigma^p)_2 \]

\( g \) is an homeomorphism and the first obstruction to a diffeomorphism is \( \lambda(\text{id} x_j) = -\lambda(h) \).

It follows that the obstructions to smoothing the composition \( g \cdot h : M \to E' \) is 

\( \lambda(g \cdot h) = \lambda(g) + \lambda(h) = -\lambda(h) + \lambda(h) = 0 \). It follows that \( g \cdot h : M \to E' \) is a diffeomorphism modulo \((q-1)\)-skeleton. However in [7, Remark 1] we showed that \( D^q \times \Sigma^p \) is diffeomorphic to \( D^q \times \Sigma^p \) if \( p < q + 2 \) and so by our hypothesis \( p < q \) then it follows that \( D^q \times \Sigma^p \) is diffeomorphic to \( D^q \times \Sigma^p \). This implies that \( E \) and \( E' \) are diffeomorphic hence \( g' : M \to E \) is a diffeomorphism modulo \((q-1)\)-skeleton. Since \( H_i(M,\mathbb{Z}) = 0 \) for \( p + 1 < i < q-1 \), there is no more obstruction to deforming \( g' \) to a diffeomorphism until we get to \((p-1)\) skeleton. We can then assume that \( g' \) is a diffeomorphism modulo \( p \)-skeleton. The first obstruction to deforming \( g' \) to a diffeomorphism modulo \((p-1)\)-skeleton is \( \lambda(g';) \in H_p(M,\pi^q) = \pi^p \). Let \( [\phi] = \lambda(g') \in \pi^q \) where \( \phi : S^{q-1} \to S^{q-1} \) is a diffeomorphism which represents \( \lambda(g') \in \pi^q \). We define \( (\phi \cdot \text{id}) : S^{q-1} \times \Sigma^p \to S^{q-1} \times \Sigma^p \)

where \( (\phi \cdot \text{id})(x,y) = (\phi(x),y) \) and if \( b = [b] \in \pi_{q-1} SO(p+1) \) we also define \( f_b : S^{q-1} \times \Sigma^p \to S^{q-1} \times \Sigma^p \) where \( f_b(x,y) = (x,b(x),y) \). We then have two orientation preserving diffeomorphisms of \( S^{q-1} \times \Sigma^p \) unto itself which we can compose to get \( (\phi \cdot \text{id}) \cdot f_b : S^{q-1} \times \Sigma^p \to S^{q-1} \times \Sigma^p \) where \( (\phi \cdot \text{id}) \cdot f_b(x,y) = (\phi(x),b(x),y) \). We then construct a manifold by attaching two copies of \( D^q \times \Sigma^p \) along their common boundary \( S^{q-1} \times \Sigma^p \) using the diffeomorphism \( (\phi \cdot \text{id}) \cdot f_b \) to have \( D^q_1 \times \Sigma^p \cup_{f_b} D^q_2 \times \Sigma^p \). Notice that this manifold is a \( p \)-sphere bundle over a homotopy \( q \)-sphere \( S^q = D^q_1 \cup_0 D^q_2 \) whose characteristic map is
$\beta = [b] \in \pi_{q-1}SO(p+1)$. We define a map

$$h : D^q \times S^p \bigcup_{f_b} D_2^q \times S^p = D_1^q \times S^p \bigcup_{(\phi \times \text{id}) \cdot f_b} D_2^q \times S^p$$

by

$$h(x,y) = \begin{cases} (x,y) & \text{if } (x,y) \in D_1^q \times S^p \\ (x \cdot \phi^{-1}(1/|x|), y) & \text{if } (x,y) \in D_2^q \times S^p \end{cases}$$

Hence $h$ is identity on $D_1^q \times S^p$ and a radial extension of $\phi^{-1}$ on $D_2^q$. It then follows that $h$ is an homeomorphism with the first obstruction to a diffeomorphism being $[\phi^{-1}] = -\lambda(g')$. Then by [8, 3.8] the first obstruction to deforming the composition $g' \circ h = g : M \times D_1^q \times S^p \bigcup_{(\phi \times \text{id}) \cdot f_b} D_2^q \times S^p$ into a diffeomorphism is $\lambda(g) = \lambda(g' \circ h) = \lambda(g') + \lambda(h) = -\lambda(h) + \lambda(h) = 0$ and hence $g$ is a diffeomorphism modulo $(p-1)$-skeleton. Since $H_i(M, \mathbb{Z}) = 0$ for $0 < i < p$ then we can assume that $g$ is a diffeomorphism modulo one point. Since $D_1^q \times S^p \bigcup_{(\phi \times \text{id}) \cdot f_b} D_2^q \times S^p$ is a $p$-sphere bundle over a homotopy $q$-sphere $\Sigma^q$ with characteristic map $[b] \in \pi_{q-1}SO(p+1)$, we shall denote it by $E(\Sigma^q)$.

Since $g$ is a diffeomorphism modulo one point then it is known that there is an homotopy $n$-sphere $\Sigma^n$ such that $M$ is diffeomorphic to $E(\Sigma^q) \# \Sigma^n$. Hence the proof.

3. INERTIAL GROUPS

Since by Theorem 2.1, every manifold homeomorphic to $E$ is diffeomorphic to $E(\Sigma^q) \# \Sigma^n$ for some homotopy spheres $\Sigma^q, \Sigma^n$, classification of such manifolds reduces to classification of manifolds of the form $E(\Sigma^q) \# \Sigma^n$. To complete this classification, we then need to investigate what happens when we vary the homotopy spheres and in particular we need to investigate the Inertial group of $E(\Sigma^q)$. We will investigate these in this section.

**Lemma 3.1.** Let $\Sigma^q_1$ and $\Sigma^q_2$ be homotopy $q$-spheres such that $\Sigma^q_i = D_1^q \bigcup_{\phi^i} D_2^q$, $i = 1, 2$ then $E(\Sigma^q_1)$ is diffeomorphic to $E(\Sigma^q_2)$ if and only if $\Sigma^q_1 \pm \Sigma^q_2 \in H(q,p)$.

**Proof.** Suppose $E(\Sigma^q_1)$ is diffeomorphic to $E(\Sigma^q_2)$. This means that $D^q_1 \times S^p \bigcup_{(\phi^1 \times \text{id}) \cdot f_b} D^q_2 \times S^p$ is diffeomorphic to $D^q_1 \times S^p \bigcup_{(\phi^2 \times \text{id}) \cdot f_b} D^q_2 \times S^p$ where $\phi^i : S^q-1 \times S^p \times S^q-1 \times S^p$ is the diffeomorphism defined by $\phi^i(x,y) = (\phi^i(x), y)$ and $f_b : S^q-1 \times S^p \times S^q-1 \times S^p$ is defined by $f_b(x,y) = (x, b(x), y)$ where $[b] = b \in \pi_{q-1}SO(p+1)$ is the characteristic map of the bundle. The manifold $E(\Sigma^q_2)$ can be regarded as the boundary of the $(p+1)$-disc bundle over $\Sigma_2$ which is denoted by
\[ D_q \times D^{p+1} \cup_{(2 \times 1 \times 1)} D_2 \times D^{p+1} = D(\Sigma_2^q). \]

So if \( E(\Sigma_1^q) \) is diffeomorphic to \( E(\Sigma_2^q) \) then since \( \Sigma_1^q \) can be embedded in \( E(\Sigma_1^q) \) it follows that \( \Sigma_1^q \) embeds in \( E(\Sigma_2^q) \). But \( \Sigma_2^q \) naturally embeds in \( E(\Sigma_2^q) \) and so we have \( \Sigma_1^q \) and \( \Sigma_2^q \) sitting in \( E(\Sigma_2^q) \), if we translate \( \Sigma_1^q \) away from \( \Sigma_2^q \) we can run a tube between them to obtain an embedding \( \Sigma_1^q \# (\Sigma_2^q) \rightarrow E(\Sigma_2^q) \) so that the embedding is homotopically trivial and so by the engulfing result of [10, chapter 7] it means that \( \Sigma_1^q \# (\Sigma_2^q) \) can be embedded in the interior of a \((p+q+1)\)-disc in \( E(\Sigma_2^q) \) and by [11, 3.5] the embedding is isotopic to a nuclear embedding into the interior of \( S^q \times D^{p+1} \). However the embedding \( \Sigma_1^q \# (\Sigma_2^q) \rightarrow S^q \times D^{p+1} \) is a homotopy equivalence, it then follows by Smale's theorem [12, Theorem 4.1] that \( \Sigma_1^q \# (\Sigma_2^q) \times D^{p+1} \) is diffeomorphic to \( S^q \times D^{p+1} \) and so it follows that \( \Sigma_1^q \# (\Sigma_2^q) \times S^p \) is diffeomorphic to \( S^q \times S^p \). Since \( S^q \times S^p \) embeds in \( R^{p+q+1} \) with trivial normal bundle then it follows that \( \Sigma_1^q \# (\Sigma_2^q) \) embeds in \( R^{p+q+1} \) with trivial normal bundle. This shows that each \( \Sigma_i^q \) for \( i = 1, 2 \) embeds in \( R^{p+q+1} \) with trivial normal bundle and by [11, §3.5] the embedding is isotopic to an embedding of \( \Sigma_i^q \) into the interior of \( S^q \times D^{p+1} \). However for \( i = 1, 2 \) the embedding \( \Sigma_i^q \rightarrow S^q \times D^{p+1} \) is a homotopy equivalence hence it follows from [12, Theorem 4.1] that \( \Sigma_i^q \times D^{p+1} \) is diffeomorphic to \( S^q \times D^{p+1} \) which implies that \( \Sigma_1^q \times D^{p+1} \) is diffeomorphic to \( \Sigma_2^q \times D^{p+1} \). Now since \( \Sigma_i^q = D_1 \cup D_2 \) where \( \phi_i : S^{q-1} \rightarrow S^{q-1} \) represents \( \Sigma_i^q \in r^q \) \( i = 1, 2 \), then we can write

\[
\Sigma_i^q \times D^{p+1} = D_1 \times D^{p+1} \cup_{\phi_i \times 1} D_2 \times D^{p+1}
\]

along \( S^{q-1} \times D^{p+1} \) by the diffeomorphism \( \phi_i \times 1 : S^{q-1} \times D^{p+1} \rightarrow S^{q-1} \times D^{p+1} \) defined by \( (\phi_i \times 1)(x,y) = (\phi_i(x),y) \) where \( (x,y) \in S^{q-1} \times D^{p+1} \). So \( \Sigma_1^q \times D^{p+1} \) is diffeomorphic to \( \Sigma_2^q \times D^{p+1} \) implies \( D_1 \times D^{p+1} \cup_{\phi_i \times 1} D_2 \times D^{p+1} \) is diffeomorphic to \( D_1 \times D^{p+1} \cup_{\phi_i \times 1} D_2 \times D^{p+1} \).

Now consider the manifold \( D(S^q) = D_1 \times D^{p+1} \cup_{f_b} D_2 \times D^{p+1} \) which is a \((p+1)\)-disc bundle over a \( q \)-sphere with characteristic map \([b] \in \pi_{q-1} SO(p+1)\). We then form the quotient space

\[
D(S^q) \cup \Sigma_1^q \times D^{p+1} = (D_1 \times D^{p+1} \cup_{f_b} D_2 \times D^{p+1}) \cup (D_1 \times D^{p+1} \cup_{\phi_i \times 1} D_2 \times D^{p+1})
\]

by identifying \( D_1 \times D^{p+1} \subset D(S^q) \) and \( D_2 \times D^{p+1} \subset \Sigma_1^q \times D^{p+1} \) by the relation \( (x,y) = (x,y)(x \in D_1^q, y \in D^{p+1}) \). The manifold \( D(S^q) \cup \Sigma_2^q \times D^{p+1} \) is similarly constructed. Since \( \Sigma_1^q \times D^{p+1} \) is diffeomorphic to \( \Sigma_2^q \times D^{p+1} \). Let \( d : \Sigma_1^q \times D^{p+1} \rightarrow \Sigma_2^q \times D^{p+1} \) be the
Diffeomorphism and since any diffeomorphism fixes a disc, we can assume that \( d \) is identity on the disc \( D^{p+q+1} = D_1^q \times D^{p+1} \), then we can define a diffeomorphism.

\[
g : D(S^q) \cup \Sigma_1^q \times D^{p+1} \to D(S^q) \cup \Sigma_2^q \times D^{p+1}
\]

where

\[
g(x) = \begin{cases} 
  d(x) & \text{for } x \in \Sigma_1^q \times D^{p+1} \\
  x & \text{for } x \in D(S^q).
\end{cases}
\]

This means that \( g = d \) on \( \Sigma_1^q \times D^{p+1} \) and identity on \( D(S^q) \). \( g \) is well defined because \( d \) is identity on the disc connecting \( D(S^q) \) and \( \Sigma_1^q \times D^{p+1} \) and \( g \) is a diffeomorphism. The manifold \( D(S^q) \cup \Sigma_1^q \times D^{p+1} \) can be clearly seen as follows. Let \( (\phi_i \times \text{id}) \cdot f_b : S^{q-1} \times D^{p+1} \to S^{q-1} \times D^{p+1} \) be the diffeomorphism defined by \( ((\phi_i \times \text{id}) \cdot f_b)(x,y) = (\phi_i (x), b(x) \cdot y) \), \( (x,y) \in S^{q-1} \times D^{p+1} \) then attaching two manifolds \( D^q_+ \times D^{p+1} \) and \( D^q_- \times D^{p+1} \) by the diffeomorphism \( (\phi_i \times \text{id}) \cdot f_b \) we have \( D^q_+ \times D^{p+1} \cup D^q_- \times D^{p+1} \) we get a \((p+1)\)-disc bundle over the homotopy q-sphere \( \Sigma_1^q = D_1^q \cup D_2^q \) \( i = 1, 2 \). However, from the way \( (\phi_i \times \text{id}) \cdot f_b \) is constructed it is easily seen that \( D(S^q) \cup \Sigma_1^q \times D^{p+1} = D^q_+ \times D^{p+1} \cup D^q_- \times D^{p+1} = D(S^q) \) hence \( g \) is the diffeomorphism of \( D(\Sigma_1^q) \) onto \( D(\Sigma_2^q) \) then it follows that \( a(D(\Sigma_1^q)) = E(\Sigma_1^q) \) is diffeomorphic to \( a(D(\Sigma_2^q)) = E(\Sigma_2^q) \).

Hence the theorem is proved.

REMARK 1. This theorem implies that \( E(\Sigma_1^q) \) is diffeomorphic to \( E(\Sigma_2^q) \) if and only if \( \Sigma_1^q \) and \( \Sigma_2^q \) are equivalent in the quotient group \( \mathfrak{e}^q / \mathfrak{h}(q,p) \).

To complete this classification, we need to determine the inertial group of \( E(\Sigma^q) \). The inertial group \( \mathfrak{I}(M) \) of an oriented closed smooth \( n \)-dimensional manifold \( M \) is defined to be the subgroup of \( \mathfrak{e}^n \) consisting of those homotopy \( n \)-spheres \( \Sigma^n \) such that \( M \) is diffeomorphic to \( M \).

Let \( E_B \) represent the total space of a \( p \)-sphere bundle over a real \( q \)-sphere with characteristic class \( \beta \in \pi_{q-1}SO(p+1) \). In \([13]\) we defined a map \( G_B : \pi_p SO(q) \to \mathfrak{e}^{p+q} \) and showed that the image of this map equals the inertial group of \( E_B \) where \( p < q \) and \( E_B \) has no cross-section. We shall similarly define a map \( G_{\phi, \beta} : \pi_p SO(q) \to \mathfrak{e}^{p+q} \) and show that the image of this map equals the inertial group of \( E(\Sigma^q) \) where \( E(\Sigma^q) \) is the total space of \( p \)-sphere bundle over a homotopy sphere \( \Sigma^q = D_1^q \cup D_2^q \). Let \( \alpha \in \pi_p SO(q) \) we define

\[
G_{\phi, \beta}(\alpha) = S^{q-1} \times D^{p+1} \bigcup_{\phi^{-1}(\phi \times \text{id}) \cdot f_b} D^q \times S^p \text{ where } [\alpha] = \alpha \text{ and } [\beta] = \beta \in \pi_{q-1}SO(p+1) \text{ and}
\]
fa-1(φ×id)⋅fb : S^{q-1} × S^p → S^{q-1} × S^p is a diffeomorphism defined by
fa-1(φ×id)⋅fb(x,y) = (a^{-1}(b(x)⋅y) ∗ (x),b(x)⋅y). One can easily show that G_{φ, B} is well-defined and that
its image is an homotopy (p+q)-sphere as similarly shown in [13].

**Lemma 3.2.** Let E(ζ^q) denote the total space of a p-sphere bundle over an homotopy
q-sphere ζ^q = D^q_1 = D^q_1 ∪ D^q_2 with characteristic class B ∈ π_{q-1}SO(p+1) then
G_{φ, B^p}(SO(q)) = I(E(ζ^q)).

**Proof.** If ζ^{p+q} ∈ I(E(ζ^q)) then this means there is a diffeomorphism
d : E(ζ^q) # ζ^{p+q} → E(ζ^q), that is,

d : (D^{q+1}_1 × S^p ∪ D^{q+1}_2 × S^p) # ζ^{p+q} → D^q_1 × S^p ∪ D^q_2 × S^p

since p < q then π_p(E(ζ^q)) is infinitely cyclic and d(α×ζ^q) represents a generator and
so is homotopic to the inclusion 0 × S^p → E(ζ^q). By Haefliger's theorem [14], d|0 × S^p
and the inclusion 0 × S^p → E(ζ^q) are isotopic and by isotopy extension theorem and
tubular neighborhood theorem, d is isotopic to a map which we shall again denote by d
such that d(D^q × S^p = D^q) × S^p where d(x,y) = (a(y)×x,y) for [a] ∈ π_pSO(q) and (x,y) ∈ D^q × S^p. We now remove D^q × S^p from E(ζ^q) # ζ^{p+q} = (D^q × S^p ∪ D^q × S^p) # ζ^{p+q}
by surgery away from the connected sum and replace it with S^{q-1} × D^{p+1}. After this opera-
tion on the summand E(ζ^q) of the connected sum, we have the manifold S^{q-1} × D^{p+1}

U (φ×id)⋅fb

D^q × S^p. Since the diffeomorphism (φ×id)⋅fb : S^{q-1} × S^p → S^{q-1} × S^p extend to
the diffeomorphism of S^{q-1} × D^{p+1} onto itself then S^{q-1} × D^{p+1} U D^q × S^p is
diffeomorphic to S^{q-1} × D^{p+1} U D^q × S^p, the diffeomorphism g is defined thus

\[
\begin{array}{c}
\text{S}^{q-1} \times D^{p+1} \quad \text{id} \\
\downarrow (φ×id)⋅fb \\
\text{S}^{q-1} \times D^{p+1} \quad \text{id}
\end{array}
\]

where

\[
g(x,y) = \begin{cases} 
(x,y) & \text{if } (x,y) \in D^q × S^p \\
((φ×id)⋅fb)(x,y) & \text{if } (x,y) \in S^{q-1} × D^{p+1}.
\end{cases}
\]

However, by [7, Lemma 2.1.2], S^{q-1} × D^{p+1} U D^q × S^p is diffeomorphic to the standard
(p+q)-sphere S^{p+q}, hence after this surgery E(ζ^q) is reduced to S^{p+q} and so E(ζ^q) # ζ^{p+q}
is reduced to S^{p+q} × ζ^{p+q} = ζ^{p+q}. 

We perform the corresponding modification (under d) on $E(\mathcal{E}^q)$ to remove the p-sphere $0 \times S^p$ with product structure $d(DQ \times S^p)$ in $E(\mathcal{E}^q)$. From this modification we obtain a manifold $S^{q-1} \times D^{p+1} \cup D^q \times S^p$ where $\psi = (d^{-1}|S^{q-1} \times S^p) \cdot (q \times \text{id}) \cdot f_b$ and this is diffeomorphic to $\Sigma^{p+q}$ because of the way we performed the surgery using $d$. However, this manifold $S^{q-1} \times D^{p+1} \cup D^q \times S^p = G_{\phi \cdot \beta} (a)$ by the definition of $G_{\phi \cdot \beta}$, thus there exists an element $a \in \pi_1 \text{SO}(q)$ (namely) $d|D^q \times S^p$ which gives $a \in \pi_1 \text{SO}(q)$ such that $\Sigma^{p+q} = G_{\phi \cdot \beta} (a)$ and so $\Sigma^{p+q} \in G_{\phi \cdot \beta} (\pi_1 \text{SO}(q))$, hence $I(E(\mathcal{E}^q)) \subseteq G_{\phi \cdot \beta} (\pi_1 \text{SO}(q))$. Conversely suppose $\Sigma^{p+q} \in G_{\phi \cdot \beta} (\pi_1 \text{SO}(q))$ then for some $a \in \pi_1 \text{SO}(q)$, $\Sigma^{p+q} = S^{q-1} \times D^{p+1} \cup (a-1)^{-1} \cdot (q \times \text{id}) \cdot f_b$ $D^q \times S^p$ where $\phi$ is a diffeomorphism of $S^{q-1}$ onto itself representing $\Sigma^q = D^q \cup \Delta^q$ and $f_a$ and $f_b$ are as defined earlier. Notice that $G_{\phi \cdot \beta} (a)$ is thus the obstruction to the construction of a diffeomorphism $S^{p+q} \to \Sigma^{p+q}$. To construct a diffeomorphism from $S^{p+q} \to \Sigma^{p+q}$, we map $S^{q-1} \times D^{p+1} \subseteq S^{p+q}$ to itself using $(q \times \text{id}) \cdot f_b$ to have $S^{p+q} = S^{q-1} \times D^{p+1} \cup D^q \times S^p$ $(\phi \times \text{id}) \cdot f_b \upharpoonright \Sigma^{p+q} = S^{q-1} \times D^{p+1} \cup (a-1)^{-1} \cdot (q \times \text{id}) \cdot f_b$ $D^q \times S^p$ and try to extend it to $D^q \times S^p$. On the boundary $S^{q-1} \times S^p$ of $D^q \times S^p$, the map is $f_{b-1} \cdot (\phi^{-1} \times \text{id}) \cdot f_a \cdot (q \times \text{id}) \cdot f_b$. So this means that $\Sigma^{p+q} = G_{\phi \cdot \beta} (a)$ is the obstruction to extending the diffeomorphism $f_{b-1} \cdot (\phi^{-1} \times \text{id}) \cdot f_a \cdot (q \times \text{id}) \cdot f_b : S^{q-1} \times S^p \to S^{q-1} \times S^p$ to a diffeomorphism of $D^q \times S^p$ onto itself. We can then define a map $E(\mathcal{E}^P) \to E(\mathcal{E}^Q)$ using the diffeomorphism $f_a : D^q \times S^p \to D^q \times S^p$ where $f_a(x,y) = (a(y), x, y)$ $(x, y) \in D^q \times S^p$ we then have $E(\mathcal{E}^P) = D^q \times S^p \cup (q \times \text{id}) \cdot f_b$ $D^q \times S^p$ $f_a \upharpoonright E(\mathcal{E}^Q) = D^q \times S^p \cup (q \times \text{id}) \cdot f_b$ $D^q \times S^p$ On the boundary $S^{q-1} \times S^p$ of $D^q \times S^p$, this map is $f_{b-1} \cdot (\phi^{-1} \times \text{id}) \cdot f_a \cdot (q \times \text{id}) \cdot f_b$ and the obstruction to extending this to a diffeomorphism of $E(\mathcal{E}^Q)$ onto itself is the
obstruction to extending the map \( f_{-1}(\phi^{-1}\times \text{id}) \cdot f_{\phi}(\times \text{id}) \cdot f_{\phi} \) to the diffeomorphism of \( D^2 \times S^p \) onto itself which is \( \mathbb{Z}^{p+q} \). It then follows that \( E(\mathbb{Z}^q) \oplus E(\mathbb{Z}^q) \neq \mathbb{Z}^{p+q} \) is a diffeomorphism and so \( \mathbb{Z}^{p+q} \in I(E(\mathbb{Z}^q)) \) hence

\[
E(E(\mathbb{Z}^q)) = G_{\phi} \cdot \beta^p \cdot SO(q)
\]

**REMARK 2.** We note that if \( p = 2, 4, 5, 6 \pmod 8 \) and \( p < q-1 \) then \( \pi_p \cdot SO(q) = 0 \) and so the image of \( G \) is trivial and hence in this particular case, the inertial group of \( E(\mathbb{Z}^q) \) is trivial and this coincides with the result of [4, Proposition 1].

**REMARK 3.** By [15], inertial group \( I(M) \) of a smooth manifold \( M \) is a diffeotopy invariant of \( M \). So if \( 2p > q+1 \) then we can deduce that the inertial group \( I(E(\mathbb{Z}^q)) \) of a \( p \)-sphere bundle over an homotopy \( q \)-sphere \( \mathbb{Z}^q \) is equal to the inertial group \( I(E_B) \) of a \( p \)-sphere bundle over the standard \( q \)-sphere, where \( \beta \in \pi_{q-1} \cdot SO(p+1) \) classifies the associated disc bundle. Let \( D(\mathbb{Z}^q) \) be the associated \((p+1)\)-disc bundle over the homotopy \( q \)-sphere where \( E(\mathbb{Z}^q) \) is the boundary of \( D(\mathbb{Z}^q) \). \( \mathbb{Z}^q \) has the homotopy type of \( D(\mathbb{Z}^q) \) and \( \mathbb{Z}^q \) has the homotopy type of \( S^q \), it follows that \( S^q \) has the homotopy type of \( D(\mathbb{Z}^q) \). Since \( 2p > q+1 \) then it follows that \( 2(p+q+1) > 3q + 3 \) and since \( p + q > 5 \) and \( p = 3 \) then \( D(\mathbb{Z}^q) \) and \( E(\mathbb{Z}^q) \) are simply connected and from [12: Theorem 4.4], it follows that \( D(\mathbb{Z}^q) \) is diffeomorphic to a \((p+1)\)-disc bundle \( D(S^q) \) over the \( q \)-sphere \( S^q \) hence the boundary \( \partial D(\mathbb{Z}^q) = E(\mathbb{Z}^q) \) of \( D(\mathbb{Z}^q) \) is diffeomorphic to the boundary \( \partial D(S^q) = E_B \) of \( D(S^q) \). It then follows by [15] that \( I(E(\mathbb{Z}^q)) = I(E_B) \). This means that the inertial group of \( S_B \) in [13] coincides with Lemma 3.2.

Combination of Lemmas 3.1 and 3.2 give the following.

**THEOREM 3.3.** Let \( E \) be the total space of a \( p \)-sphere bundle over a \( q \)-sphere with characteristic map \( \beta \in \pi_{q-1} \cdot SO(p+1) \) then the diffeomorphism classes of \( p+q \)-manifolds that are homeomorphic to \( E \) are in one-to-one correspondence with the group

\[
\mathbb{Z}^q \times \frac{\mathbb{Z}^n}{\mathbb{Z}^{n+p}} \times \text{Image } G_B
\]

where \( p+q = n \geq 6 \) and \( p < q \).

**REFERENCES**


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