CONTINUITY FOR SOME MULTILINEAR OPERATORS
OF INTEGRAL OPERATORS ON
TRIEBEL-LIZORKIN SPACES

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The continuity for some multilinear operators related to certain fractional singular integral operators on Triebel-Lizorkin spaces is obtained. The operators include Calderon-Zygmund singular integral operator and fractional integral operator.

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1. Introduction. Let $T$ be a Calderon-Zygmund singular integral operator; a well-known result of Coifman et al. (see [6]) states that the commutator $[b,T] = T(bf) - bTf$ (where $b \in BMO$) is bounded on $L^p(R^n)$ for $1 < p < \infty$; Chanillo (see [1]) proves a similar result when $T$ is replaced by the fractional integral operator; in [10, 11], these results on the Triebel-Lizorkin spaces and the case $b \in \text{Lip} \beta$ (where $\text{Lip} \beta$ is the homogeneous Lipschitz space) are obtained. The main purpose of this paper is to discuss the continuity for some multilinear operators related to certain fractional singular integral operators on the Triebel-Lizorkin spaces. In fact, we will establish the continuity on the Triebel-Lizorkin spaces for the multilinear operators only under certain conditions on the size of the operators. As to the applications, the continuity for the multilinear operators related to the Calderon-Zygmund singular integral operator and fractional integral operator on the Triebel-Lizorkin spaces is obtained.

2. Notations and results. Throughout this paper, $Q$ will denote a cube of $R^n$ with side parallel to the axes, and for a cube $Q$, let $f_Q = |Q|^{-1} \int_Q f(x)dx$ and $f^\#(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f(y) - f_Q|dy$. For $1 \leq r < \infty$ and $0 \leq \delta < n$, let

$$M_{\delta, r}(f)(x) = \sup_{x \in Q} \left( \frac{1}{|Q|^{1-r\delta/n}} \int_Q |f(y)|^r dy \right)^{1/r};$$

(2.1)

we denote $M_{\delta, r}(f) = M_r(f)$ if $\delta = 0$, which is the Hardy-Littlewood maximal function when $r = 1$. For $\beta > 0$ and $p > 1$, let $\dot{F}^p_{\beta, \infty}$ be the homogeneous Triebel-Lizorkin space; the Lipschitz space $\dot{\Lambda}_\beta$ is the space of functions $f$ such that

$$\|f\|_{\dot{\Lambda}_\beta} = \sup_{x, h \in R^n \atop h \neq 0} \frac{\left| \Delta_h^{[\beta]+1} f(x) \right|}{|h|^\beta} < \infty,$$

(2.2)

where $\Delta_h^k$ denotes the $k$th difference operator (see [11]).
We are going to consider the fractional singular integral operator as follows.

**Definition 2.1.** Let \( T : S \to S' \) be a linear operator. \( T \) is called a fractional singular integral operator if there exists a locally integrable function \( K(x, y) \) on \( \mathbb{R}^n \times \mathbb{R}^n \) such that

\[
T(f)(x) = \int K(x, y)f(y)dy
\]  

(2.3)

for every bounded and compactly supported function \( f \). Let \( m \) be a positive integer and \( A \) a function on \( \mathbb{R}^n \). Denote that \( R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha A(y)(x - y)\alpha \). (2.4)

The multilinear operator related to fractional singular integral operator \( T \) is defined by

\[
T_A(f)(x) = \int \frac{R_{m+1}(A; x, y)}{|x - y|^m} K(x, y)f(y)dy.
\]  

(2.5)

Note that when \( m = 0 \), \( T_A \) is just the commutator of \( T \) and \( A \) while when \( m > 0 \), it is nontrivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2, 3, 4, 5, 7, 8, 14]). The main purpose of this paper is to study the continuity for the multilinear operator on Triebel-Lizorkin spaces. We will prove the following theorem in Section 3.

**Theorem 2.2.** Let \( 0 < \beta < 1 \) and \( D^\alpha A \in \dot{\Lambda}_\beta \) for \( |\alpha| = m \). Suppose \( T \) is the fractional singular integral operator such that \( T \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) for \( 0 \leq \delta < n, 1 < p < n/\delta \), and \( 1/p - 1/q = \delta/n \). If \( T \) satisfies the size condition

\[
|T_A(f)(x) - T_A(f)(x_0)| \leq C \sum_{|\alpha| = m} ||D^\alpha A||_{\dot{\Lambda}_\beta} |Q|^\beta/n M_{\delta, 1}(f)(x)
\]  

(2.6)

for any cube \( Q = Q(x_0, l) \) with \( \text{supp} \, f \subset (2Q)^c, x \in Q \), and some \( 0 \leq \delta < n \), then

(a) \( T_A \) maps \( L^p(\mathbb{R}^n) \) continuously into \( L^q(\mathbb{R}^n) \) for \( 0 \leq \delta < n, 1 < p < n/\delta \), and \( 1/p - 1/q = \delta/n \);

(b) \( T_A \) maps \( L^p(\mathbb{R}^n) \) continuously into \( L^q(\mathbb{R}^n) \) for \( 0 \leq \delta < n - \beta, 1 < p < n/(\delta + \beta) \), and \( 1/p - 1/q = (\delta + \beta)/n \).

From the theorem, we get the following corollary.

**Corollary 2.3.** Fix \( \varepsilon > 0, 0 < \beta < \min(1, \varepsilon), \delta \geq 0, \) and \( D^\alpha A \in \dot{\Lambda}_\beta \) for \( |\alpha| = m \). Let \( K \) be a locally integrable function on \( \mathbb{R}^n \times \mathbb{R}^n \) satisfying

\[
|K(x, y)| \leq C|x - y|^{-n + \delta},
\]

\[
|K(y, x) - K(z, x)| \leq C|y - z|^\varepsilon |x - z|^{-n - \varepsilon + \delta}
\]  

(2.7)
if $2|y - z| \leq |x - z|$. Denote that (2.3) holds and denote the multilinear operator of $T$ by (2.5) for every bounded and compactly supported function $f$ and $x \in (\text{supp} f)^c$. Suppose $T$ is bounded from $L^p(R^n)$ to $L^q(R^n)$ for $0 \leq \delta < n$, $1 < p < n/\delta$, and $1/q = 1/p - \delta/n$. Then

(a) $T_A$ maps $L^p(R^n)$ continuously into $L^{p,\infty}(R^n)$ for $0 \leq \delta < n$, $1 < p < n/\delta$, and $1/p - 1/q = \delta/n$;

(b) $T_A$ maps $L^p(R^n)$ continuously into $L^q(R^n)$ for $0 \leq \delta < n - \beta$, $1 < p < n/(\delta + \beta)$, and $1/p - 1/q = (\delta + \beta)/n$.

3. Proof of Theorem 2.2. To prove the theorem, we need the following lemmas.

**Lemma 3.1** [11]. For $0 < \beta < 1$ and $1 < p < \infty$,

$$\|f\|_{L^p} \approx \left\| \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q| \, dx \right\|_{L_p} \quad (3.1)$$

**Lemma 3.2** [11]. For $0 < \beta < 1$ and $1 < p < \infty$,

$$\|f\|_{L^p} \approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q| \, dx \quad (3.2)$$

**Lemma 3.3** [1]. Suppose that $1 < r < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then $\|M_{\delta,r}(f)\|_{L^q} \leq C\|f\|_{L^p}$.

**Lemma 3.4** [5]. Let $A$ be a function on $R^n$ and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then

$$|R_m(A;x,y)| \leq C|x - y|^m \sum_{|\alpha| = m} \left( \frac{1}{|Q(x,y)|} \int_{Q(x,y)} |D^\alpha A(z)|^q \, dz \right)^{1/q}, \quad (3.3)$$

where $Q(x,y)$ is the cube centered at $x$ and having side length $5\sqrt{n}|x - y|$.

**Proof of Theorem 2.2.** Fix a cube $Q = Q(x_0,l)$ and $\hat{x} \in Q$. Let $\hat{Q} = 5\sqrt{n}Q$ and $\hat{\hat{Q}} = A(x) - \sum_{|\alpha| = m} (1/\alpha!)(D^\alpha A)_{\hat{x}} x^\alpha$. Then $R_m(A;x,y) = R_m(\hat{A};x,y)$ and $D^\alpha \hat{A} = D^\alpha A - (D^\alpha A)_{\hat{x}}$ for $|\alpha| = m$. We write, for $f_1 = f\chi_{\hat{Q}}$ and $f_2 = f\chi_{R^n \setminus \hat{Q}}$,

$$T_A(f)(x) = \int_{\hat{Q}} \frac{R_m(A;x,y)}{|x - y|^m} K(x,y) f(y) \, dy \quad (3.4)$$
\[
\begin{align*}
|T_A(f)(x) - T_A(f_2)(x_0)| & \leq T\left( R_m(\tilde{A}; x, \cdot) f_1 \right)(x) + \sum_{|\alpha|=m} \frac{1}{\alpha!} |T\left( \frac{(x - \cdot)^\alpha}{|x - \cdot|^m} \tilde{A} f_1 \right)(x)| \\
& \quad + |T_A(f_2)(x) - T_A(f_2)(x_0)| := I(x) + II(x) + III(x).
\end{align*}
\]

Thus,

\[
\begin{align*}
\frac{1}{|Q|^{1+\beta/n}} \int_Q |T_A(f)(x) - T_A(f_2)(x_0)| dx & \leq \frac{1}{|Q|^{1+\beta/n}} \int_Q |I(x)| dx + \frac{1}{|Q|^{1+\beta/n}} \int_Q |II(x)| dx + \frac{1}{|Q|^{1+\beta/n}} \int_Q |III(x)| dx \\
& := I + II + III.
\end{align*}
\]

Now, we estimate I, II, and III, respectively. First, for \(x \in Q\) and \(y \in \tilde{Q}\), using Lemmas 3.4 and 3.2, we get

\[
R_m(\tilde{A}; x, y) \leq C|x - y|^m \sum_{|\alpha|=m} \sup_{x \in \tilde{Q}} |D^\alpha A(x) - (D^\alpha A)_{\tilde{Q}}| \\
\leq C|x - y|^m |Q|^{\beta/n} \sum_{|\alpha|=m} ||D^\alpha A||_{\lambda^\beta}.
\]

Thus, taking \(r, s\) such that \(1 \leq r < p\) and \(1/s = 1/r - \delta/n\), by \((L^r, L^s)\) boundedness of \(T\) and Hölder’s inequality, we obtain

\[
I \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\lambda^\beta} \frac{1}{|Q|} \int_Q |T(f_1)(x)| dx \\
\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\lambda^\beta} ||T(f_1)||_{L^r} |Q|^{-1/s} \\
\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\lambda^\beta} ||f_1||_{L^r} |Q|^{-1/s} \\
\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\lambda^\beta} \left( \frac{1}{|Q|^{1-r\delta/n}} \int_{\tilde{Q}} |f(y)|^r dy \right)^{1/r} \\
\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\lambda^\beta} M_{\delta, r}(f)(\tilde{x}).
\]

Second, using the inequality (see [11])

\[
||(D^\alpha A - (D^\alpha A)_{\tilde{Q}}) f\chi_{\tilde{Q}}||_{L^r} \leq C|Q|^{1/s+\beta/n} ||D^\alpha A||_{\lambda^\beta} M_{\delta, r}(f)(x),
\]

\[
\text{(3.5)}
\]

\[
\text{(3.6)}
\]

\[
\text{(3.7)}
\]

\[
\text{(3.8)}
\]

\[
\text{(3.9)}
\]
we gain

\[
II \leq \frac{C}{|Q|^{1+\beta/n}} \sum_{|\alpha|=m} \left\| T\left((D^\alpha A - (D^\alpha A)_{\tilde{Q}}) f_{\chi_{Q}}\right) \right\|_{L^s}|Q|^{1-1/s} \\
\leq C|Q|^{-\beta/n-1/s} \sum_{|\alpha|=m} \left\| (D^\alpha A - (D^\alpha A)_{\tilde{Q}}) f_{\chi_{Q}} \right\|_{L^r} \\
\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\tilde{\lambda}_\beta M_{\delta,r}(f)}(\tilde{x}).
\] (3.10)

For III, using the size condition of \( T \), we have

\[
III \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\tilde{\lambda}_\beta M_{\delta,1}(f)}(\tilde{x}).
\] (3.11)

We now put these estimates together, and taking the supremum over all \( Q \) such that \( \tilde{x} \in Q \), and using Lemmas 3.1 and 3.3, we obtain

\[
\|T_A(f)\|_{F_{q,\infty}} \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\tilde{\lambda}_\beta} \|f\|_{L^p}.
\] (3.12)

This completes the proof of (a).

For (b), by the same argument as in the proof of (a), we have

\[
\frac{1}{|Q|} \int_Q |T_A(f)(y) - T_A(f_2)(x_0)| \, dy \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\tilde{\lambda}_\beta} (M_{\delta+\beta,r}(f) + M_{\delta+\beta,1}(f)).
\] (3.13)

Thus, we get the sharp estimate of \( T_A \) as follows:

\[
(T_A(f))^\# \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\tilde{\lambda}_\beta} (M_{\delta+\beta,r}(f) + M_{\delta+\beta,1}(f)).
\] (3.14)

Now, using Lemma 3.3, we gain

\[
\|T_A(f)\|_{L^q} \leq C \| (T_A(f))^\# \|_{L^q} \\
\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\tilde{\lambda}_\beta} (\|M_{\delta+\beta,r}(f)\|_{L^q} + \|M_{\delta+\beta,1}(f)\|_{L^q}) \leq C \|f\|_{L^p}.
\] (3.15)

This completes the proof of (b) and the theorem.

**Proof of Corollary 2.3.** It suffices to verify that \( T \) satisfies the size condition in Theorem 2.2.
Suppose \( \text{supp} f \subset \hat{Q}^c \) and \( x \in Q = Q(x_0, l) \). We write

\[
T_A(f)(x) - T_A(f)(x_0) = \int_{\mathbb{R}^n} \left[ \frac{K(x, y)}{|x - y|^m} - \frac{K(x_0, y)}{|x_0 - y|^m} \right] R_m(\hat{A}; x, y) f(y) dy
\]

\[+ \int_{\mathbb{R}^n} \frac{K(x_0, y)f(y)}{|x_0 - y|^m} \left[ R_m(\hat{A}; x, y) - R_m(\hat{A}; x_0, y) \right] dy\]

\[= \sum_{|\alpha| = m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \left( \frac{K(x, y)(x - y)^\alpha}{|x - y|^m} - \frac{K(x_0, y)(x_0 - y)^\alpha}{|x_0 - y|^m} \right) D^\alpha \hat{A}(y) f(y) dy\]

\[:= I_1 + I_2 + I_3.\]

By Lemma 3.4 and the following inequality, for \( b \in \hat{\Lambda}_\beta \):

\[
|b(x) - b_Q| \leq \frac{1}{|Q|} \int_Q \|b\|_{\hat{\Lambda}_\beta} |x - y|^\beta dy \leq \|b\|_{\hat{\Lambda}_\beta} (|x - x_0| + 1)^\beta,
\]

we get

\[
|R_m(\hat{A}; x, y)| \leq \sum_{|\alpha| = m} \|D^\alpha A\|_{\hat{\Lambda}_\beta} (|x - y| + 1)^{m + \beta}.
\]

Note that by \( |x - y| \sim |x_0 - y| \) for \( x \in Q \) and \( y \in \mathbb{R}^n \setminus \hat{Q} \), we obtain, by the condition of \( K \),

\[
|I_1| \leq C \int_{\mathbb{R}^n \setminus \hat{Q}} \left( \frac{|x - x_0|}{|x_0 - y|^{m+1-\delta}} + \frac{|x - x_0|^\varepsilon}{|x_0 - y|^{m+\varepsilon-\delta}} \right) |R_m(\hat{A}; x, y)| \|f(y)\| dy
\]

\[\leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{\hat{\Lambda}_\beta} \sum_{k=0}^{\infty} \frac{1}{\binom{2k+1}{k} Q^{2k} Q^{1-\delta/n}} \left( \frac{|x - x_0|}{|x_0 - y|^{n+1-\delta-\beta}} + \frac{|x - x_0|^\varepsilon}{|x_0 - y|^{n+\varepsilon-\delta-\beta}} \right) |f(y)| dy\]

\[\leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{\hat{\Lambda}_\beta} |Q|^{\beta/n} \sum_{k=1}^{\infty} \left( 2^{k(\beta-1)} + 2^{k(\beta-\varepsilon)} \right) \frac{1}{2^k Q^{1-\delta/n}} \int_{2kQ} |f(y)| dy\]

\[\leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{\hat{\Lambda}_\beta} |Q|^{\beta/n} M_{\delta, 1}(f)(x).
\]

For \( I_2 \), by the formula (see [3])

\[
R_m(\hat{A}; x,y) - R_m(\hat{A}; x_0, y) = \sum_{|\eta| < m} \frac{1}{\eta!} R_{m-|\eta|} (D^\eta \hat{A}; x, x_0)(x - y)^\eta
\]

(3.20)
and Lemma 3.4, we get
\[
|I_2| \leq C \int_{R^n \setminus Q} \frac{|R_m(\hat{A}; x, y) - R_m(\hat{A}; x_0, y)|}{|x_0 - y|^{m+n-\delta}} |f(y)| \, dy
\]
\[
\leq C \sum_{|\alpha|=m} |||D^\alpha A|||_{\hat{\beta}} \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^kQ} \frac{|x - x_0|^{\beta+1}}{|x_0 - y|^{n+1-\delta}} |f(y)| \, dy
\]
\[
\leq C \sum_{|\alpha|=m} |||D^\alpha A|||_{\hat{\beta}} |Q|^{\beta/n} \sum_{k=1}^{\infty} 2^{-k} \frac{1}{|2^kQ|^{1-\delta/n}} \int_{2^kQ} |f(y)| \, dy
\]
\[
\leq C \sum_{|\alpha|=m} |||D^\alpha A|||_{\hat{\beta}} |Q|^{\beta/n} M_{\delta,1}(f)(x).
\]

For \(I_3\), similar to the estimates of \(I_1\), we obtain
\[
|I_3| \leq C \int_{R^n \setminus Q} \left( \frac{|x - x_0|}{|x_0 - y|^{n+1-\delta}} + \frac{|x - x_0|^\epsilon}{|x_0 - y|^{n+\epsilon-\delta}} \right) |||D^\alpha A|||_{\hat{\beta}} |Q|^{\beta/n} \sum_{k=0}^{\infty} (2^{k(\beta-1)} + 2^{k(\beta-\epsilon)}) \frac{1}{|2^kQ|^{1-\delta/n}} \int_{2^kQ} |f(y)| \, dy
\]
\[
\leq C \sum_{|\alpha|=m} |||D^\alpha A|||_{\hat{\beta}} |Q|^{\beta/n} \sum_{k=1}^{\infty} 2^{-k} \frac{1}{|2^kQ|^{1-\delta/n}} \int_{2^kQ} |f(y)| \, dy
\]
\[
\leq C \sum_{|\alpha|=m} |||D^\alpha A|||_{\hat{\beta}} |Q|^{\beta/n} M_{\delta,1}(f)(x).
\]

Thus (2.6) holds. This completes the proof of the corollary.

4. Applications. In this section, we will apply Theorem 2.2 and Corollary 2.3 to some particular operators such as the Calderon-Zygmund singular integral operator and fractional integral operator.

APPLICATION 1 (Calderon-Zygmund singular integral operator). Let \(T\) be the Calderon-Zygmund operator defined by (2.3) (see [9, 12, 13]); the multilinear operator related to \(T\) is defined by
\[
TAf(x) = \int_{R^n} \frac{R_m(A; x, y)}{|x - y|^m} K(x, y) f(y) \, dy.
\]

Then it is easy to see that \(T\) satisfies the conditions in Corollary 2.3 with \(\delta = 0\); thus \(T_A\) is bounded from \(L^p(R^n)\) to \(L^{p, \infty}(R^n)\) for \(D^\alpha A \in \hat{\lambda}_\beta, |\alpha| = m, 0 < \beta < 1, 1 < p < \infty\) and from \(L^p(R^n)\) to \(L^q(R^n)\) for \(D^\alpha A \in \hat{\lambda}_\beta, |\alpha| = m, 0 < \beta < 1, 1 < p < n/\beta, \) and \(1/p - 1/q = \beta/n\).

APPLICATION 2 (fractional integral operator with rough kernel). For \(0 \leq \delta < n\), let \(T_\delta\) be the fractional integral operator with rough kernel defined by (see [7, 8, 14])
\[
T_\delta f(x) = \int_{R^n} \frac{\Omega(x - y)}{|x - y|^{n-\delta}} f(y) \, dy;
\]

(4.2)
the multilinear operator related to $T_\delta$ is defined by

$$T_\delta^A f(x) = \int_{\mathbb{R}^n} \frac{R_{m+1}(A; x, y)}{|x-y|^{m+n-\delta}} \Omega(x-y) f(y) dy,$$

(4.3)

where $\Omega$ is homogeneous of degree zero on $\mathbb{R}^n$, $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$, and $\Omega \in \text{Lip}_p(S^{n-1})$ for $0 < y \leq 1$, that is, there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \leq M|x - y|^y$. Then $T_\delta$ satisfies the conditions in Corollary 2.3. In fact, for supp $f \subset (2Q)^c$ and $x \in Q = Q(x_0, l)$, by the condition of $\Omega$, we have (see [14])

$$\left| \frac{\Omega(x-y)}{|x-y|^{n-\delta}} - \frac{\Omega(x_0-y)}{|x_0-y|^{n-\delta}} \right| \leq C \left( \frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} + \frac{|x-x_0|^y}{|x_0-y|^{n+y-\delta}} \right);$$

(4.4)

thus, similar to the proof of Corollary 2.3, we get

$$|T_\delta^A(f)(x) - T_\delta^A(f)(x_0)|$$

$$\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\lambda, \beta} \sum_{k=0}^{\infty} 2k+1\tilde{Q} \int_{2k\tilde{Q}} |f(y)| dy$$

$$\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\lambda, \beta} |Q|^{\beta/n} \sum_{k=1}^{\infty} (2k^{(\beta-1)} + 2^{k(\beta-1)}) \frac{1}{|2k\tilde{Q}|^{1-\beta/n}} \int_{2k\tilde{Q}} |f(y)| dy$$

$$\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\lambda, \beta} |Q|^{\beta/n} M_{\delta, 1}(f)(x).$$

(4.5)

Therefore, $T_\delta^A$ is bounded from $L^p(\mathbb{R}^n)$ to $F^\beta,\infty_q(\mathbb{R}^n)$ for $D^\alpha A \in \lambda, \beta$, $|\alpha| = m$, $0 < \beta < y$, $1 < p < n/\beta$ and from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $D^\alpha A \in \lambda, \beta$, $|\alpha| = m$, $0 < \beta < 1$, $1 < p < n/(\delta + \beta)$, and $1/p - 1/q = (\delta + \beta)/n.$

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This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

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