ON SOME PROPERTIES OF BANACH OPERATORS. II

A. B. THAHEEM and A. R. KHAN

Received 30 September 2003

Using the notion of a Banach operator, we have obtained a decompositional property of a Hilbert space, and the equality of two invertible bounded linear multiplicative operators on a normed algebra with identity.

2000 Mathematics Subject Classification: 46C05, 47A10, 47A50, 47H10.

1. Introduction. This paper is a continuation of our earlier work [7] on Banach operators. We recall that if \( X \) is a normed space and \( \alpha : X \to X \) is a mapping, then following [4], \( \alpha \) is said to be a \textit{Banach operator} if there exists a constant \( k \) such that \( 0 \leq k < 1 \) and \( \| \alpha^2(x) - \alpha(x) \| \leq k \| \alpha(x) - x \| \) for all \( x \in X \). Banach operators are generalizations of contraction maps and play an important role in the fixed point theory; their consideration goes back to Cheney and Goldstein [2] in the study of proximity maps on convex sets (see [4] and the references therein).

In [7], we established some decompositional properties of a normed space using Banach operators. We showed that if \( \alpha \) is a linear Banach operator on a normed space \( X \), then \( N(\alpha - 1) = N((\alpha - 1)^2) \), \( N(\alpha - 1) \cap R(\alpha - 1) = \{0\} \) and in case \( X \) is finite dimensional, we get the decomposition \( X = N(\alpha - 1) \oplus R(\alpha - 1) \), where \( N(\alpha - 1) \) and \( R(\alpha - 1) \) denote the null space and the range space of \( (\alpha - 1) \), respectively, and \( 1 \) denotes the identity operator on \( X \). In [7, Proposition 2.3], we proved a decompositional property of a general bounded linear operator on a Hilbert space, namely, if \( \alpha \) is a bounded linear operator on a Hilbert space \( H \) such that \( \alpha \) and \( \alpha^* \) have common fixed points, then \( N(\alpha - 1) + R(\alpha - 1) \) is dense in \( H \).

In this paper, also we prove some properties of Banach operators on a Hilbert space. We show (Proposition 2.1) that if \( \alpha \) is a bounded linear Banach operator on a Hilbert space \( H \) such that the sets of fixed points of \( \alpha \) and \( \alpha^* \) are the same, then \( H \) admits a decomposition \( H = N(\alpha - 1) \oplus M \), where \( M = R(\alpha - 1) \), \( R(\alpha - 1) \) denotes the closure of \( R(\alpha - 1) \). It follows as a corollary of Proposition 2.1 that \( \alpha \) commutes with both orthogonal projections onto \( N(\alpha - 1) \) and onto \( M \).

As in [7], we also study the operator equation \( \alpha + c\alpha^{-1} = \beta + c\beta^{-1} \) for a pair of invertible bounded linear multiplicative Banach operators \( \alpha \) and \( \beta \) on a normed algebra \( X \) with identity, where \( c \) is an appropriate real or complex number. We prove the following result (Proposition 2.3): assume that \( \alpha(x) + c\alpha^{-1}(x) = \beta(x) + c\beta^{-1}(x) \) for all \( x \in X \), where \( c \) is a real or complex number such that \( |c| \geq 1, \|\alpha\|^2 \leq |c|/2, \|\beta\|^2 \leq |c|/2 \). If \( \beta \) is inner, then \( \alpha = \beta \). We briefly recall that this operator equation has been extensively studied for automorphisms on von Neumann algebras. We refer to [1, 5, 6] for more details about this operator equation.
2. The results

**Proposition 2.1.** Let $\alpha$ be a bounded linear Banach operator on a Hilbert space $H$ such that the sets of fixed points of $\alpha$ and $\alpha^*$ are the same. Then the following hold:

(i) $N(\alpha - 1) \perp R(\alpha - 1)$,

(ii) $H = N(\alpha - 1) \oplus M$, where $M = R(\alpha - 1)$.

**Proof.** To prove (i), let $x \in N(\alpha - 1)$ and $y \in R(\alpha - 1)$. Then $\alpha(x) = x$ and $y = \alpha(z) - z$ for some $z \in H$. Therefore, $\alpha^*(x) = x$ and hence

$$\langle x, y \rangle = \langle x, \alpha(z) - z \rangle = \langle x, \alpha(z) \rangle - \langle x, z \rangle = \langle \alpha^*(x), z \rangle - \langle x, z \rangle = \langle x, z \rangle - \langle x, z \rangle = 0.$$  

(2.1)

Thus $N(\alpha - 1) \perp R(\alpha - 1)$.

To prove (ii), it is enough to show that $N(\alpha - 1) \perp M$. By (i) and the continuity of $\alpha$, $N(\alpha - 1) \perp M$. So, $N(\alpha - 1) \subseteq M^\perp$. Conversely, assume that $z \in M^\perp$. Then $\langle z, y \rangle = 0$ for all $y \in M$; in particular, $\langle z, (\alpha - 1)x \rangle = 0$ for all $x \in H$ because $R(\alpha - 1) \subseteq M$. Thus $\langle z, \alpha(x) \rangle = \langle z, x \rangle$ for all $x \in H$. So, $\langle \alpha^*(z), x \rangle = \langle z, x \rangle$ for all $x \in H$. This shows that $\langle \alpha^*(z) - z, x \rangle = 0$ for all $x \in H$. Therefore, $\alpha^*(z) - z = 0$ or $\alpha^*(z) = z$, that is, $z$ is a fixed point of $\alpha^*$ and hence by assumption, $\alpha(z) = z$, that is, $z \in N(\alpha - 1)$. So, $M^\perp \subseteq N(\alpha - 1)$. Thus $N(\alpha - 1) = M^\perp$ and hence $H = N(\alpha - 1) \oplus M$.  

**Corollary 2.2.** Let $\alpha$ be a bounded linear Banach operator on a Hilbert space $H$ such that the sets of fixed points of $\alpha$ and $\alpha^*$ are the same. Then $\alpha$ commutes with both orthogonal projections, onto $N(\alpha - 1)$ and onto $M$.

**Proof.** Since $R(\alpha - 1)$ is $\alpha$-invariant, so is $M$. Also, $M^\perp = N(\alpha - 1)$ is $\alpha$-invariant. Thus $M$ reduces $\alpha$ and hence $\alpha$ commutes with both orthogonal projections, onto $N(\alpha - 1)$ and onto $M$ [3].

It easily follows that the orthogonal projection $P$ onto $N(\alpha - 1)$ is the largest orthogonal projection such that $\alpha P = P$.

We conclude this paper with a result about an operator equation similar to the one considered in [7].

**Proposition 2.3.** Let $\alpha, \beta$ be invertible bounded linear multiplicative Banach operators on a normed algebra $X$ with identity such that $\alpha(x) + c \alpha^{-1}(x) = \beta(x) + c \beta^{-1}(x)$ for all $x \in X$, where $c$ is a real or complex number with $|c| \geq 1$, $\|\alpha\|^2 \leq |c|/2$, $\|\beta\|^2 \leq |c|/2$. If $\beta$ is inner, then $\alpha = \beta$.

**Proof.** It follows from [7, Proposition 3.2] that $\alpha$ and $\beta$ commute. Therefore,

$$(\alpha \beta - c)(\beta^{-1} - \alpha^{-1})(x) = \alpha(x) - \alpha \beta \alpha^{-1}(x) - c \beta^{-1}(x) + c \alpha^{-1}(x)$$

$$= \alpha(x) - \beta \alpha(\alpha^{-1}(x)) - c \beta^{-1}(x) + c \alpha^{-1}(x)$$

$$= \alpha(x) - \beta(x) - c \beta^{-1}(x) + c \alpha^{-1}(x)$$

$$= (\alpha(x) + c \alpha^{-1}(x)) - (\beta(x) + c \beta^{-1}(x)) = 0.$$  

(2.2)
Put \((\beta^{-1} - \alpha^{-1})(x) = y\). Then we obtain \((\alpha \beta - c)(y) = 0\), that is, \(\alpha \beta(y) = cy\). Therefore, by assumption, we get \(|c||y|| = ||cy|| = ||\alpha \beta(y)|| \leq ||\alpha|| ||\beta|| ||y|| \leq (|c|/2)||y||\), that is, \(|c||y|| \leq (|c|/2)||y||\). This implies that \(||y|| = 0\) and hence \((\beta^{-1} - \alpha^{-1})(x) = 0\) for all \(x \in X\), that is, \(\beta^{-1}(x) = \alpha^{-1}(x)\) for all \(x \in X\). Since \(\alpha\) is onto, therefore replacing \(x\) by \(\alpha(x)\), we get \(\beta^{-1}(\alpha(x)) = x\) or \(\alpha(x) = \beta(x)\) for all \(x \in X\).

\[\Box\]

**ACKNOWLEDGMENTS.** The authors are grateful to King Fahd University of Petroleum and Minerals and Saudi Basic Industries Corporation (SABIC) for supporting Fast Track Research Project no. FT-2002/01. We also thank the referees for useful suggestions. The second author is on leave from Bahauddin Zakariya University, Multan, Pakistan, and is indebted to the university’s authorities for granting leave.

**REFERENCES**


A. B. Thaheem: Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

**E-mail address:** athaheem@kfupm.edu.sa

A. R. Khan: Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

**E-mail address:** arahim@kfupm.edu.sa
Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Deadline</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>December 1, 2008</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>March 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>June 1, 2009</td>
</tr>
</tbody>
</table>

**Guest Editors**

José Roberto Castilho Piqueira, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

Elbert E. Neher Macau, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil; elbert@lac.inpe.br

Celso Grebogi, Center for Applied Dynamics Research, King’s College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk

Hindawi Publishing Corporation
http://www.hindawi.com