Nonderogatory Unicyclic Digraphs

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A digraph is nonderogatory if its characteristic polynomial and minimal polynomial are equal. We find a characterization of nonderogatory unicyclic digraphs in terms of Hamiltonicity conditions. An immediate consequence of this characterization is that the complete product of difans and diwheels is nonderogatory.

A digraph (directed graph) \( \Gamma \) consists of a finite set of vertices together with a set of arcs, which are ordered pairs of vertices. The vertices and arcs are denoted \( V_\Gamma \) and \( E_\Gamma \), respectively. If \((u,v) \in E_\Gamma\), then \(u\) and \(v\) are adjacent and \((u,v)\) is an arc starting at vertex \(u\) and terminating at vertex \(v\). A walk in \(\Gamma\) from vertex \(u\) to vertex \(v\) is a finite vertex sequence

\[
u = u_0, u_1, u_2, \ldots, u_k = v
\]
(sometimes denoted by \( u_0 \rightarrow u_1 \rightarrow \cdots \rightarrow u_k \)), where \((u_{t-1}, u_t)\) is an arc of \( D \) for all \( 1 \leq t \leq k \). The number \( k \) is the length of the walk. If \( u = v \), we say that the walk is a closed walk. A path is a walk in which no vertex is repeated, and a closed path is a cycle.

Suppose that \( \{u_1, \ldots, u_n\} \) is the set of vertices of \( \Gamma \). The adjacency matrix of \( \Gamma \), denoted \( A_\Gamma \), is the square matrix of order \( n \) whose entry \( ij \) is the number of arcs starting at \( u_i \) and terminating at \( u_j \). The characteristic polynomial of the digraph \( \Gamma \), denoted by \( \Phi_\Gamma(x) \) (or simply \( \Phi_\Gamma \)), is the characteristic polynomial of the adjacency matrix \( A_\Gamma \), that is, \( \Phi_\Gamma(x) = |xI - A_\Gamma| \), where \( I \) is the identity matrix and \( |M| \) is the determinant of \( M \).

By the Cayley-Hamilton theorem, \( \Phi_\Gamma \) is an annihilating polynomial of \( A_\Gamma \), which means that \( \Phi_\Gamma(A_\Gamma) = 0 \). The monic polynomial of least degree which annihilates \( A_\Gamma \) is called the minimal polynomial of \( \Gamma \) and will be denoted by \( \mu_\Gamma(x) \) (or simply \( \mu_\Gamma \)).

**Definition 1.1.** Say that \( \Gamma \) is a nonderogatory digraph if \( \Phi_\Gamma(x) = \mu_\Gamma(x) \).

Recall that a linear directed graph \( L \) is a digraph in which every vertex has indegree and outdegree equal to 1; in other words, its components are cycles. The coefficient theorem for digraphs relates the coefficients of the characteristic polynomial of a digraph \( \Gamma \) with the set of linear directed subgraphs of \( \Gamma \).

**Theorem 1.2 (see [2, Theorem 1.2]).** Let \( \Gamma \) be a digraph with \( n \) vertices. Then the characteristic polynomial \( \Phi_\Gamma \) of \( \Gamma \) is

\[
\Phi_\Gamma = x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_n,
\]

where \( c_i = \sum_{L \in \mathcal{L}_i} (-1)^{p(L)} \). Here, \( \mathcal{L}_i \) is the set of linear directed subgraphs \( L \) of \( \Gamma \) with exactly \( i \) vertices, and \( p(L) \) is the number of components of \( L \).

A spanning path in a digraph \( \Gamma \) is called a Hamiltonian path. A path-Hamiltonian digraph is a digraph which contains a Hamiltonian path. Then we have a characterization of acyclic nonderogatory digraphs.

**Proposition 1.3.** Let \( \Gamma \) be a digraph with \( n \) vertices. The following are equivalent:

(i) \( \Gamma \) is an acyclic nonderogatory digraph;

(ii) \( \Gamma \) is path-Hamiltonian.

**Proof.** Recall that the entry \( uv \) of the power matrix \( A^{n-1} \) is precisely the number of walks in \( \Gamma \) of length \( n - 1 \) from \( u \) to \( v \) (see [2, Theorem 1.9]). By Theorem 1.2, if \( \Gamma \) is an acyclic digraph with \( n \) vertices, then its characteristic polynomial is simply \( \Phi_\Gamma = x^n \). Hence, \( \Gamma \) is nonderogatory if and only if \( A^{n-1} \neq 0 \). If \( \Gamma \) is acyclic, any walk is necessarily a path, and a path of length \( n - 1 \) is necessarily spanning and hence Hamiltonian. From this, it follows
that \( \Gamma \) is a nonderogatory acyclic digraph if and only if there exists a walk of length \( n - 1 \) if and only if \( \Gamma \) is path-Hamiltonian. \( \square \)

2. Nonderogatory unicyclic digraphs

Let \( \Gamma_p(u,v) \) denote the set of walks in \( \Gamma \) of length \( p \) from vertex \( u \) to vertex \( v \). By [2, Theorem 1.9], the entry \( uv \) of \( A^p \) is the number of walks from \( u \) to \( v \) of length \( p \).

**Proposition 2.1.** Let \( \Gamma \) be a digraph with \( n \) vertices and unique cycle \( C \) of length \( r \geq 2 \). The following conditions are equivalent:

1. \( \Gamma \) is a nonderogatory digraph;
2. there exists \( u,v \in V_\Gamma \) such that \(|\Gamma_{n-1}(u,v)| \neq |\Gamma_{n-r-1}(u,v)|\).

**Proof.** From Theorem 1.2, the characteristic polynomial of \( \Gamma \) is

\[
\Phi_\Gamma = x^n - x^{n-r} = x^r(x' - 1). \tag{2.1}
\]

Since \( x' - 1 \) is a product of distinct linear factors, the minimal polynomial \( \mu_\Gamma \) has the form

\[
\mu_\Gamma = x^p(x' - 1), \tag{2.2}
\]

where \( 1 \leq p \leq n - r \). Hence, \( \Gamma \) is a nonderogatory digraph if and only if

\[
A^{n-r-1}(A^r - I) \neq 0 \tag{2.3}
\]

or equivalently,

\[
A^{n-1} \neq A^{n-r-1}. \tag{2.4}
\]

\( \square \)

In the next results, we give further insight into condition 2 of Proposition 2.1.

Assume that \( \{x_1, \ldots, x_r\} \) are the vertices of \( C \), and \( (x_i, x_{i+1}) \), for \( i = 1, \ldots, r - 1 \), together with \( (x_r, x_1) \) are the arcs of \( C \). For each \( 1 \leq j \leq r \), let \( C(x_j) \) denote the closed walk \( x_j, x_{j+1}, \ldots, x_{j-1} \). For \( u, v \in V_\Gamma \), define

\[
\Gamma^p_{\pi}(u,v) = \{ \pi \in \Gamma_p(u,v) : \pi \text{ contains } C(x_j) \text{ for some } x_j (1 \leq j \leq r) \}, \\
\Gamma^*_{\pi}(u,v) = \{ \pi \in \Gamma_p(u,v) : \pi \text{ contains } x_j \text{ for some } x_j (1 \leq j \leq r) \}. \tag{2.5}
\]

**Lemma 2.2.** Let \( \Gamma \) be a digraph with \( n \) vertices and unique cycle \( C \) of length \( r \geq 2 \). Then \(|\Gamma^*_{n-1}(u,v)| = |\Gamma^*_{n-r-1}(u,v)|\) for every \( u, v \in V_\Gamma \).

**Proof.** Note that, \( \pi = u \cdots x_j \cdots v \in \Gamma^*_{n-r-1}(u,v) \) if and only if \( \overline{\pi} = u \cdots C(x_j) \cdots v \in \Gamma^*_{n-1}(u,v) \). Consequently, the function \( \Psi : \Gamma^*_{n-r-1}(u,v) \to \Gamma^*_{n-1}(u,v) \) defined as \( \Psi(\pi) = \overline{\pi} \) is bijective and the result follows. \( \square \)

Note that, in particular, \( \Gamma^*_{n-r-1}(u,v) = \emptyset \) if and only if \( \Gamma^*_{n-1}(u,v) = \emptyset \) and in this case,

\[
|\Gamma^*_{n-1}(u,v)| = |\Gamma^*_{n-r-1}(u,v)| = 0. \tag{2.6}
\]
Let \( \Gamma - C \) be the digraph obtained from \( \Gamma \) by deleting the vertices of \( C \) and the arcs incident to them.

**Lemma 2.3.** Let \( \Gamma \) be a digraph with \( n \) vertices and unique cycle \( C \) of length \( r \geq 2 \). Then

1. \( \Gamma_{n-1}(u,v) \setminus \Gamma_{n-1}^*(u,v) \) is the (possibly empty) set of Hamiltonian paths of \( \Gamma \) from \( u \) to \( v \);
2. \( \Gamma_{n-r-1}(u,v) \setminus \Gamma_{n-r-1}^*(u,v) \) is the (possibly empty) set of Hamiltonian paths of \( \Gamma - C \) from \( u \) to \( v \).

**Proof.** (1) Clearly, a Hamiltonian path of \( \Gamma \) from \( u \) to \( v \) is a path of length \( n - 1 \) which does not contain \( C \). Conversely, assume that \( \pi \in \Gamma_{n-1}(u,v) \setminus \Gamma_{n-1}^*(u,v) \). Then \( \pi \) is a walk of length \( n - 1 \) from \( u \) to \( v \) that does not contain the cycle \( C \). Since \( C \) is the unique cycle of \( \Gamma \), then clearly \( \pi \) is a spanning path of \( \Gamma \) from \( u \) to \( v \).

(2) First, note that \( \Gamma - C \) is a digraph with \( n - r \) vertices. It is clear that every Hamiltonian path in \( \Gamma - C \) belongs to \( \Gamma_{n-r-1}(u,v) \setminus \Gamma_{n-r-1}^*(u,v) \). Conversely, if \( \sigma \in \Gamma_{n-r-1}(u,v) \setminus \Gamma_{n-r-1}^*(u,v) \), then \( \sigma \) is a walk in \( \Gamma - C \) of length \( n - r - 1 \). Since \( \Gamma - C \) is acyclic, it follows that \( \sigma \) is a Hamiltonian path in \( \Gamma - C \). \( \square \)

For every \( u, v \in V_{\Gamma} \), we define

\[
\begin{align*}
\hat{h}^*(u,v) &= \left| \Gamma_{n-1}(u,v) \setminus \Gamma_{n-1}^*(u,v) \right|, \\
\hat{h}^*(u,v) &= \left| \Gamma_{n-r-1}(u,v) \setminus \Gamma_{n-r-1}^*(u,v) \right|.
\end{align*}
\] (2.7)

By Lemma 2.3, \( \hat{h}^*(u,v) \) (resp., \( \hat{h}^*(u,v) \)) is the number of Hamiltonian paths in \( \Gamma \) (resp., \( \Gamma - C \)) from vertex \( u \) to vertex \( v \).

If \( \Gamma - C \) is path-Hamiltonian, then its structure is very simple, as we can see in the next result.

**Proposition 2.4.** Let \( \Omega \) be an acyclic path-Hamiltonian digraph with \( n \) vertices. Then \( \Omega \) is obtained from \( P_n = u_1 \rightarrow u_2 \rightarrow \cdots \rightarrow u_n \) by adding some edges of the form \( u_j \rightarrow u_k \), where \( k > j \).

**Proof.** Let \( \pi : u_1 \rightarrow u_2 \rightarrow \cdots \rightarrow u_n \) be a spanning path of \( \Omega \) and assume that \( u_j \rightarrow u_k \) belongs to \( \Omega \), where \( k < j \). Then

\[
u_k \rightarrow \cdots \rightarrow u_j \rightarrow u_k \] (2.8)

is a cycle in \( \Omega \), which is a contradiction. \( \square \)

In particular, an acyclic path-Hamiltonian digraph \( \Omega \) has a unique Hamiltonian path: it starts in the unique vertex of indegree 0 and it ends in the unique vertex of outdegree 0 of \( \Omega \).

Now, we can characterize nonderogatory unicyclic digraphs in terms of Hamiltonicity conditions. We distinguish the following two cases:

(a) \( \Gamma - C \) is path-Hamiltonian, and so \( \Gamma - C \) has the form given in Proposition 2.4;
(b) \( \Gamma - C \) is not path-Hamiltonian.
Theorem 2.5. Let $\Gamma$ be a digraph with $n$ vertices and unique cycle $C$ of length $r \geq 2$.

(1) If $\Gamma - C$ is not path-Hamiltonian, then

$$\Gamma \text{ is nonderogatory } \iff \Gamma \text{ is path-Hamiltonian.} \quad (2.9)$$

(2) If $\Gamma - C$ is path-Hamiltonian with unique Hamiltonian path from $u$ to $v$, then

$$\Gamma \text{ is nonderogatory } \iff h^*(u,v) \neq 1. \quad (2.10)$$

Proof. By Proposition 2.1, $\Gamma$ is nonderogatory if and only if there exists $u,v \in V_\Gamma$ such that $|\Gamma_{n-1}(u,v)| \neq |\Gamma_{n-r-1}(u,v)|$. Note that $\Gamma_{n-1}(u,v)$ and $\Gamma_{n-r-1}(u,v)$ can be expressed as disjoint unions

$$\Gamma_{n-1}(u,v) = \Gamma_{n-1}^\circ(u,v) \cup \Gamma_{n-1}^*(u,v),$$

$$\Gamma_{n-r-1}(u,v) = \Gamma_{n-r-1}^*(u,v) \cup \Gamma_{n-r-1}^*(u,v). \quad (2.11)$$

It follows from Lemma 2.2 that

$$\Gamma \text{ is nonderogatory } \iff h^*(u,v) \neq h^*(u,v) \quad \text{for some } u,v \in V_\Gamma. \quad (2.12)$$

(1) Assume that $\Gamma - C$ is not path-Hamiltonian. Then $h^*(u,v) = 0$ for every $u,v \in V_\Gamma$. It follows from (2.12) and Lemma 2.3 that

$$\Gamma \text{ is nonderogatory } \iff h^*(u,v) \neq 0 \quad \text{for some } u,v \in V_\Gamma \iff \Gamma \text{ is path-Hamiltonian.} \quad (2.13)$$

(2) Assume that $\Gamma - C$ is path-Hamiltonian with unique Hamiltonian path from vertex $u$ to vertex $v$. Then $h^*(u,v) = 1$ and $h^*(w,z) = 0$ for every other pair of vertices $w$ and $z$ in $V_\Gamma$. It follows from (2.12) that if $h^*(u,v) \neq 1$, then $\Gamma$ is nonderogatory. Now, assume that $h^*(u,v) = 1$. We will show that $h^*(w,z) = 0$ for every other pair of vertices $w$ and $z$ in $V_\Gamma$, which implies that $\Gamma$ is derogatory. Let $\pi$ be a Hamiltonian path in $\Gamma$ from $u$ to $v$ and suppose that there is a Hamiltonian path $\sigma$ in $\Gamma$ from $w$ to $z$. If $w \neq u$, then there exists an arc $w' \rightarrow u$ in $\Gamma$. Since $\pi$ induces a path from $u$ to $w'$, then a cycle in $\Gamma$ is formed different from $C$. This is a contradiction, so $w$ must equal $u$. Similarly, $z = v$. □

Example 2.6. In this example we turn our attention to a well-known operation between digraphs, the so-called coalescence of digraphs, considered in [8].

Let $F_r \cdot W_q$ be the coalescence of $F_r$ and $W_q$ with respect to the hub of both digraphs (see Figure 2.1).

$F_r \cdot W_q$ is unicyclic with unique cycle $C = t_{r+1}, t_{r+2}, \ldots, t_{r+q-2}, t_{r+q-1}, t_{r+1}$. Note that $F_r \cdot W_q - C = F_r$, which is path-Hamiltonian with unique Hamiltonian path $t_r \rightarrow t_1 \rightarrow \cdots \rightarrow t_{r-1}$. Moreover, $h^*(t_r, t_{r-1}) = 0$. Hence, by Theorem 2.5, $F_r \cdot W_q$ is nonderogatory.

Example 2.7. Consider the digraph $D$ shown in Figure 2.2.

$D$ is unicyclic with unique cycle $C = u_5, u_2, u_3, u_4, u_5$ of length 4. Observe that neither $D - C$ nor $D$ is path-Hamiltonian. Consequently, by Theorem 2.5, $D$ is derogatory.
The results of Gan [4] follow immediately from Theorem 2.5, as we can see in the next example. But first recall that the complete product $D_1 \otimes D_2$ of digraphs $D_1$ and $D_2$ is the digraph obtained by joining every vertex of $D_1$ to every vertex of $D_2$ with an arc directed from the vertex of $D_1$ to the vertex of $D_2$.

**Example 2.8.** (1) Consider the complete product $F_q \otimes W_r$ (see Figure 2.3).

Note that $F_q \otimes W_r$ is unicyclic with unique cycle $C = v_1, v_2, \ldots, v_{r-1}, v_r$; $(F_q \otimes W_r) - C$ is path-Hamiltonian with unique Hamiltonian path $v_{r+q} \rightarrow v_{r+1} \rightarrow v_{r+2} \rightarrow \cdots \rightarrow v_{r+q-1} \rightarrow v_r$ and $h^*(v_{r+q}, v_r) = 0$. Thus, by Theorem 2.5, $F_q \otimes W_r$ is non-derogatory.

(2) Consider the complete product $W_q \otimes F_r$ (see Figure 2.4).

Note that $W_q \otimes F_r$ is unicyclic with unique cycle $C = v_{r+1}, v_{r+2}, \ldots, v_{r+q-1}, v_{r+1}$; $(W_q \otimes F_r) - C$ is path-Hamiltonian with unique Hamiltonian path $v_{r+q} \rightarrow v_r \rightarrow v_1 \rightarrow \cdots \rightarrow v_{r-2} \rightarrow v_{r-1}$ and $h^*(v_{r+q}, v_{r-1}) > 1$. Thus, by Theorem 2.5, $W_q \otimes F_r$ is nonderogatory.
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References


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