ON A NEW GENERALIZATION OF ALZER’S INEQUALITY

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Abstract. Let \( \{a_n\}_{n=1}^{\infty} \) be an increasing sequence of positive real numbers. Under certain conditions of this sequence we use the mathematical induction and the Cauchy mean-value theorem to prove the following inequality:

\[
\frac{a_n}{a_{n+m}} \leq \left( \frac{1/n \sum_{i=1}^{n} a_i^r}{(1/(n+m)) \sum_{i=1}^{n+m} a_i^r} \right)^{1/r},
\]

where \( n \) and \( m \) are natural numbers and \( r \) is a positive number. The lower bound is best possible. This inequality generalizes the Alzer’s inequality (1993) in a new direction. It is shown that the above inequality holds for a large class of positive, increasing and logarithmically concave sequences.

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1. Introduction. Several authors including Alzer [1], Sandor [8], and Ume [10] proved the following inequality:

\[
\frac{n}{n+1} < \left( \frac{1/n \sum_{i=1}^{n} i^r}{(1/(n+1)) \sum_{i=1}^{n+1} i^r} \right)^{1/r},
\]

where \( r > 0 \) and \( n \in \mathbb{N} \). The proof of this inequality involves the principle of the mathematical induction and other analytical methods.

Based on the mathematical induction, Elezović and Pečarić [2] generalized (1.1) and proved the following theorem.

**Theorem 1.1.** If the sequence \( \{a_n\}_{n=1}^{\infty} \) of positive real numbers satisfies the inequality

\[
1 \leq \left( \frac{a_{n+2}}{a_{n+1}} \right)^r \left[ \frac{a_{n+2}}{a_{n+1}} - 1 + \left( \frac{a_n}{a_{n+1}} \right)^{r+1} \right], \quad n \geq 0, \quad a_0 = 0,
\]

then the following inequality holds:

\[
\frac{a_n}{a_{n+1}} \leq \left( \frac{1/a_n \sum_{i=1}^{n} a_i^r}{(1/a_{n+1}) \sum_{i=1}^{n+1} a_i^r} \right)^{1/r}.
\]

Recently, Qi [4] proved a generalized version of (1.1). The reader is referred to [4, Corollary 2].

The main purpose of this paper is to further generalize inequalities (1.1) and (1.3).
2. Main Results

**Theorem 2.1.** Let $n$ and $m$ be natural numbers. Suppose \( \{a_1, a_2, \ldots \} \) is a positive and increasing sequence satisfying

\[
\frac{(k + 2)a_{k+2}^r - (k + 1)a_{k+1}^r}{(k + 1)a_{k+1}^r - ka_k^r} \geq \left( \frac{a_{k+2}}{a_{k+1}} \right)^r
\]

(2.1)

for any given positive real number $r$ and $k \in \mathbb{N}$, then we have the inequality

\[
\frac{a_n}{a_{n+m}} \leq \left( \frac{(1/n) \sum_{i=1}^{n} a_i^r}{(1/(n+m)) \sum_{i=1}^{n+m} a_i^r} \right)^{1/r}
\]

(2.2)

The lower bound of (2.2) is best possible.

**Proof.** The inequality (2.2) is equivalent to

\[
\frac{a_n^r}{a_{n+m}^r} \leq \left( \frac{(1/n) \sum_{i=1}^{n} a_i^r}{(1/(n+m)) \sum_{i=1}^{n+m} a_i^r} \right)^{1/r}
\]

(2.3)

that is,

\[
\frac{1}{na_n^r} \sum_{i=1}^{n} a_i^r \geq \frac{1}{(n+m)a_{n+m}^r} \sum_{i=1}^{n+m} a_i^r.
\]

(2.4)

This is also equivalent to

\[
\frac{1}{na_n^r} \sum_{i=1}^{n} a_i^r \geq \frac{1}{(n+1)a_{n+1}^r} \sum_{i=1}^{n+1} a_i^r.
\]

(2.5)

Since

\[
\sum_{i=1}^{n+1} a_i^r = \sum_{i=1}^{n} a_i^r + a_{n+1}^r,
\]

(2.6)

inequality (2.5) reduces to

\[
\sum_{i=1}^{n} a_i^r \geq \frac{na_n^r a_{n+1}^r}{(n+1)a_{n+1}^r - na_n^r}.
\]

(2.7)

It is easy to see that inequality (2.7) holds for $n = 1$.

Assume that inequality (2.7) holds for $n > 1$. Using the principle of induction, it is easy to show that (2.7) holds for $n + 1$. Using equality (2.6), the induction can be written as (2.1) for $k = n$. Thus, inequality (2.7) holds.

It can easily be shown that

\[
\lim_{r \to +\infty} \left( \frac{(1/n) \sum_{i=1}^{n} a_i^r}{(1/(n+m)) \sum_{i=1}^{n+m} a_i^r} \right)^{1/r} = \frac{a_n}{a_{n+m}}.
\]

(2.8)

Hence, the lower bound of (2.2) is best possible. The proof is complete. \(\square\)
Corollary 2.2. Let \( n \) and \( m \) be natural numbers. Suppose \( a = \{a_1, a_2, \ldots\} \) is a positive and increasing sequence satisfying

\[
\frac{a_{k+1}^2}{a_k a_{k+2}} \geq a_k a_{k+2},
\]

(2.9)

\[
\frac{a_{k+1} - a_k}{a_{k+1}^2 - a_k a_{k+2}} \geq \max \left\{ \frac{k+1}{a_{k+1}}, \frac{k+2}{a_{k+2}} \right\}, \quad k \in \mathbb{N}.
\]

(2.10)

Then, for any given positive real number \( r \), we have the inequality (2.2). The lower bound of (2.2) is best possible.

Proof. For \( x \in [n, n+1] \), let

\[
f(x) = (n + 1 - x)a_{n+1} + (x - n)a_{n+2},
\]

(2.11)

\[
g(x) = (n + 1 - x)a_n + (x - n)a_{n+1}.
\]

(2.12)

Further, we define

\[
F(x) = (x + 1) f'(x), \quad G(x) = x g'(x), \quad x \in [n, n+1].
\]

(2.13)

Direct calculation yields

\[
F(n) = (n + 1)a_{n+1}^r, \quad F(n+1) = (n + 2)a_{n+2}^r;
\]

(2.14)

\[
G(n) = n a_n^r, \quad G(n+1) = (n + 1)a_{n+1}^r;
\]

(2.15)

\[
F'(x) = f^{r-1}(x) \left[ f(x) + r(x+1)(a_{n+2} - a_{n+1}) \right];
\]

(2.16)

\[
G'(x) = g^{r-1}(x) \left[ g(x) + rx(a_{n+1} - a_n) \right].
\]

(2.17)

Therefore, using the inequality (2.10) and standard arguments gives

\[
\frac{F'(x)}{G'(x)} = \frac{(n + 1 - x)a_{n+1} + (x - n)a_{n+2}}{(n + 1 - x)a_n + (x - n)a_{n+1}}^r \times \frac{1 + r(x+1)(a_{n+2} - a_{n+1})/[\{(n + 1 - x)a_{n+1} + (x - n)a_{n+2}\}/\{(n + 1 - x)a_n + (x - n)a_{n+1}\}]}{1 + rx(a_{n+1} - a_n)/[\{(n + 1 - x)a_n + (x - n)a_{n+1}\}]} \geq \left( \frac{n + 1 - x}{n+1}a_{n+1} + (x - n)a_{n+2} \right)^r
\]

(2.18)

Applying the Cauchy's mean-value theorem to the left side of inequality (2.1), it turns out that there exists one point \( \zeta \in (n, n+1) \) such that

\[
\frac{(n+2)a_{n+2} - (n+1)a_{n+1}}{(n+1)a_{n+1} + na_n} = \frac{F'(|\zeta|)}{G'(|\zeta|)} \geq \left( \frac{(n + 1 - \zeta)a_{n+1} + (\zeta - n)a_{n+2}}{(n + 1 - \zeta)a_n + (\zeta - n)a_{n+1}} \right)^r \geq \left( \frac{a_{n+2}}{a_{n+1}} \right)^r,
\]

(2.19)

in which the logarithmic convexity of the sequence \( \{a_n\}_{n=1}^\infty \) is used. Thus, the inequality (2.1) is proved.

\[\square\]
**Corollary 2.3** [4]. Let $n$ and $m$ be natural numbers and $k$ a nonnegative integer. Then

\[
\frac{n + k}{n + m + k} \leq \left( \frac{(1/n) \sum_{i=k+1}^{n+k} i^r}{(1/(n+m)) \sum_{i=k+1}^{n+m+k} i^r} \right)^{1/r},
\]

where $r$ is any given positive real number. The lower bound is best possible.

**Proof.** This follows from Corollary 2.2 applied to $a = (k + 1, k + 2, \ldots)$.

**Note.** When $k = 0$ and $m = 1$, inequality (2.20) reduces to (1.1).

**Note.** Recently, some inequalities related to Alzer’s inequality and the sum of powers of positive integers or sequences have been proved. For details, see Qi [6, 5, 3], Sándor [9], and Qi and Luo [7].

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This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

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