A NOTE ON KAKUTANI TYPE FIXED POINT THEOREMS

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ABSTRACT. We present Kakutani type fixed point theorems for certain semigroups of self maps by relaxing conditions on the underlying set, family of self maps, and the mappings themselves in a locally convex space setting.

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1. Introduction. Using a technique of Tarafdar [9], we establish fixed point theorems by utilizing following semigroups under composition of self maps $T$ on a subset $M$ of a Hausdorff locally convex space

(i) $\mathcal{F} = C_T = \{f : M \to M \mid fT = Tf\}$,

(ii) $\mathcal{F} = \{T^n : n \in \mathbb{N} \cup \{0\}\}$,

(iii) $\mathcal{F} =$ identity map.

In the sequel $(E, \tau)$ will be a Hausdorff locally convex topological vector space. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on $E$ is said to be an associated family of seminorms for $\tau$ if the family $\{rU : r > 0\}$, where $U = \bigcap_{i=1}^{n} U_{\alpha_i}$ and $U_{\alpha_i} = \{x : p_{\alpha_i}(x) < 1\}$, forms a base of neighbourhoods of zero for $\tau$. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on $E$ is called an augmented associated family for $\tau$ if $\{p_\alpha : \alpha \in I\}$ is an associated family with the property that the seminorm $\max\{p_\alpha, p_\beta\} \in \{p_\alpha : \alpha \in I\}$ for any $\alpha, \beta \in I$. The associated and augmented associated families of seminorms shall be denoted by $A(\tau)$ and $A^*(\tau)$, respectively. It is well known that given a locally convex space $(E, \tau)$, there always exists a family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on $E$ such that $\{p_\alpha : \alpha \in I\} = A^*(\tau)$ (see [7, page 203]).

The following construction will be crucial. Suppose that $M$ is a $\tau$-bounded subset of $E$. For this set $M$ we can select a number $\lambda_\alpha > 0$ for each $\alpha \in I$ such that $M \subset \lambda_\alpha U_{\alpha}$, where $U_{\alpha} = \{x : p_{\alpha}(x) \leq 1\}$. Clearly, $B = \bigcap_{\alpha} \lambda_\alpha U_{\alpha}$ is $\tau$-bounded, $\tau$-closed, absolutely convex, and contains $M$. The linear span $E_B$ of $B$ in $E$ is $\bigcup_{n=1}^{\infty} nB$. The Minkowski functional of $B$ is a norm $\|\cdot\|_B$ on $E_B$. Thus $(E_B, \|\cdot\|_B)$ is a normed space with $B$ as its closed unit ball and $\sup_{\alpha} p_{\alpha}(x/\lambda_\alpha) = \|x\|_B$ for each $x \in E_B$.

A self map $T$ on $M$ is said to be

(i) $A^*(\tau)$-nonexpansive if for all $x, y \in M$,

$$p_{\alpha}(Tx - Ty) \leq p_{\alpha}(x - y) \quad \text{for each } p_{\alpha} \in A^*(\tau). \quad (1.1)$$

(ii) $A^*(\tau)$-asymptotically nonexpansive if for each $x, y \in M$, 

...
\[ p_\alpha(T^n x - T^n y) \leq k_n p_\alpha(x - y), \quad n = 1, 2, 3, \ldots, \text{ for each } p_\alpha \in A^*(\tau), \quad (1.2) \]

where \( \{k_n\} \) is a fixed sequence of real numbers such that \( k_n \to 1 \) as \( n \to \infty \).

In sequel, for simplicity, we shall call \( A^*(\tau) \)-nonexpansive (\( A^*(\tau) \)-asymptotically nonexpansive) maps to be nonexpansive (asymptotically nonexpansive).

Common fixed points of nonexpansive maps and best approximations have been considered in normed spaces (see [1, 3]). We prove common fixed point theorems for asymptotically nonexpansive maps in the setting of a locally convex space.

2. Results

**Lemma 2.1.** Let \( M \) be a \( \tau \)-bounded subset of a Hausdorff locally convex space \((E, \tau)\) and \( T : M \to M \) be asymptotically nonexpansive map. Then \( T \) is asymptotically nonexpansive on \( M \) with respect to \( \| \cdot \|_B \).

**Proof.** By hypothesis for \( x, y \in M \) and \( n = 1, 2, 3, \ldots, \)
\[ p_\alpha(T^n x - T^n y) \leq k_n p_\alpha(x - y) \quad \text{for each } p_\alpha \in A^*(\tau), \quad (2.1) \]

where \( \{k_n\} \) is a real sequence converging to 1,
\[
\sup_\alpha p_\alpha \left( \frac{T^n x - T^n y}{\lambda_\alpha} \right) \leq k_n \sup_\alpha p_\alpha \left( \frac{x - y}{\lambda_\alpha} \right),
\]
\[ \| T^n x - T^n y \|_B \leq k_n \| x - y \|_B, \quad (2.2) \]

where \( \{k_n\} \to 1 \) as \( n \to \infty \) and is a fixed real sequence. This completes the proof. \( \square \)

Note that \((E_B, \tau) \subset (E_B, \| \cdot \|_B)\) so a set compact in \((E_B, \tau)\) need not be compact in \((E_B, \| \cdot \|_B)\) (cf. [8, page 159, problem 3(c)]). To overcome this difficulty we use finite dimensionality to obtain following generalization of [9, Theorem 2.1].

**Theorem 2.2.** Let \( M \) be a nonempty convex \( \tau \)-bounded, \( \tau \)-sequentially closed and finite dimensional subset of a Hausdorff locally convex space \((E, \tau)\). Suppose \( \mathcal{F} \) is a commutative semigroup of asymptotically nonexpansive self maps of \( M \). Then there exists a point \( a \in M \) such that
\[ T(a) = a \quad \text{for all } T \in \mathcal{F}. \quad (2.3) \]

**Proof.** Since \( M \) is \( \tau \)-complete, it follows that \((E_B, \| \cdot \|_B)\) is a Banach space and \( M \) is complete in it. A closed, bounded and finite dimensional subset of a normed space is compact by [2, Theorem on page 10] so \( M \) is compact in \((E_B, \| \cdot \|_B)\). By Lemma 2.1, each \( T \in \mathcal{F} \) is \( \| \cdot \|_B \)-asymptotically nonexpansive. Hence \( \mathcal{F} \) is a commutative semigroup of asymptotically nonexpansive self maps of a compact convex subset \( M \) of the Banach space \((E_B, \| \cdot \|_B)\). The family \( \mathcal{F} \) has a common fixed point by [4, Theorem 3.1]. \( \square \)

We now prove another fixed point theorem for locally convex spaces by making use of Jungck and Sessa [6, Theorem 3]; see also [1, Corollary 2.3] and [5, Theorem 1].

**Theorem 2.3.** Let \( M \) be a \( \tau \)-bounded, \( \tau \)-sequentially closed and finite dimensional subset of a Hausdorff locally convex space \((E, \tau)\). Suppose that \( M \) is starshaped with
starcentre \( q \in M \) and \( T : M \to M \) is nonexpansive. Let \( \mathcal{F} \) be a family of affine nonexpansive self maps of \( M \) commuting with \( T \) and leaving \( q \) fixed. Suppose for each pair \((x, y) \in M^2\), there exists \( f = f(x, y) \) and \( g = g(x, y) \) in \( \mathcal{F} \) such that

\[
p_\alpha(Tx - Ty) \leq p_\alpha(fx - gy) \quad \text{for all } p_\alpha \in A^*(\tau).
\]

Then there exists \( a \in M \) such that

\[
a = T(a) = h(a) \quad \text{for all } h \in \mathcal{F}.
\]

**Proof.** Since \( \| \cdot \|_B \)-topology is finer than the relative \( \tau \)-topology on \( E_B, \| \cdot \|_B \)-cl\((M) \subset \tau\)-sequential-cl\((M) = M \). Therefore, \( M \) is \( \| \cdot \|_B \)-closed in the normed space \((E_B, \| \cdot \|_B)\). As above, \( M \) is a compact subset of \((E_B, \| \cdot \|_B)\). Moreover, \( T \) and each \( h \in \mathcal{F} \) is nonexpansive in \((E, \tau)\), which by Lemma 2.1 implies that \( T \) and each \( h \in \mathcal{F} \) is \( \| \cdot \|_B \)-nonexpansive—so certainly \( \| \cdot \|_B \)-continuous. And from (2.4) we obtain for \( x, y \in M \),

\[
\sup_\alpha p_\alpha\left(\frac{Tx - Ty}{\lambda_\alpha}\right) \leq \sup_\alpha p_\alpha\left(\frac{fx - gy}{\lambda_\alpha}\right).
\]

Thus

\[
\|Tx - Ty\|_B \leq \|fx - gy\|_B \quad \text{for } x, y \in M.
\]

A comparison of our hypothesis with that of [6, Theorem 3] tells us that we can now apply [6, Theorem 3] to \( M \) as a subset of \((E_B, \| \cdot \|_B)\) to conclude that there exists \( a \in M \) such that \( a = T(a) = h(a) \) for all \( h \in \mathcal{F} \).

**Corollary 2.4.** Let \( M \) be a \( \tau \)-bounded, \( \tau \)-sequentially closed, and finite dimensional subset of a Hausdorff locally convex space \((E, \tau)\). Assume \( M \) is starshaped with starcentre \( q \in M \). Suppose \( T, I : M \to M \) are nonexpansive, \( I \) is affine and leaving \( q \) fixed and \( TI = IT \). Suppose for \( x, y \in M \), there exist \( n = n(x, y) \), \( m = m(x, y) \) in \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \) such that

\[
p_\alpha(Tx - Ty) \leq p_\alpha(l^n x - I^n y) \quad \text{for each } p_\alpha \in A^*(\tau).
\]

Then \( T \) and \( I \) have a common fixed point.

**Proof.** Let \( \mathcal{F} = \{I^n : n \in \mathbb{N}_0\} \) \( (I^0 x = x) \). For each \( n \), \( I^n \) is affine, \( TI^n = I^n T \) and \( I^n : M \to M \) since \( I \) has these properties. Further (2.8) assures that \( \mathcal{F} \) and its members satisfy (2.4) and the hypotheses of Theorem 2.3; consequently, the conclusion of the corollary follows.

**Corollary 2.5.** Let \( M \) be a \( \tau \)-bounded, \( \tau \)-closed finite dimensional starshaped subset of a Hausdorff locally convex space \((E, \tau)\) and \( T \) a nonexpansive self map of \( M \). Then \( T \) has a fixed point.

Finally, we consider an application of Corollary 2.4 to best approximation theory. A related result for normed spaces was given in [6, Theorem 4]. For any \( \hat{x} \in E, C \subseteq E \)
and \( p_\alpha \in A^* (\tau) \), let
\[
d_{p_\alpha} (\bar{x}, C) = \inf \{ p_\alpha (y - \bar{x}) : y \in C \}
\] (2.9)

and let
\[
D = \{ y \in C : p_\alpha (y - \bar{x}) = d_{p_\alpha} (\bar{x}, C) \text{ for all } p_\alpha \in A^* (\tau) \}. \tag{2.10}
\]

**Theorem 2.6.** Let \( T \) and \( I \) be self maps of a Hausdorff locally convex space \((E, \tau)\) and let \( C \subseteq E \) be such that \( T : \partial C \to C \). Let \( T \) and \( I \) leave \( \bar{x} \in E \) fixed and satisfy (2.8) for all \( x, y \in D \cup \{ \bar{x} \} \). Suppose \( I \) is nonexpansive and affine, \( T \) is nonexpansive on \( D \), \( IT = TI \) on \( D \), and \( D \) is nonempty \( \tau \)-bounded, \( \tau \)-sequentially closed, finite dimensional and starshaped with respect to \( q \). If \( I \) leaves \( q \) invariant and \( I(D) \subseteq D \), then there exists \( a \in D \) such that \( a = I(a) = T(a) \).

**Proof.** Let \( y \in D \). Then \( I^n y \in D \) for \( n \in \mathbb{N}_0 \) since \( I(D) \subseteq D \). By definition of \( D \), \( y \in \partial C \) and since \( T : \partial C \to C \), it follows that \( Ty \in C \). By (2.8), for each \( p_\alpha \in A^* (\tau) \),
\[
p_\alpha (Ty - \bar{x}) = p_\alpha (Ty - T\bar{x}) \leq p_\alpha (I^n y - I^n \bar{x})
\] (2.11)

for some \( n, m \in \mathbb{N}_0 \). As \( I^n \bar{x} = \bar{x} \), we get
\[
p_\alpha (Ty - \bar{x}) \leq p_\alpha (I^n y - \bar{x}) \quad \text{for all } p_\alpha \in A^* (\tau). \tag{2.12}
\]

Again since \( Ty \in C \) and \( I^n y \in D \), the definition of \( D \) further implies that \( Ty \in D \). Consequently, \( T, I : D \to D \) and the conditions of Corollary 2.4 are satisfied. Hence there exists \( a \in D \) such that \( a = I(a) = T(a) \).

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**References**


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<th>Date</th>
</tr>
</thead>
<tbody>
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<td>Manuscript Due</td>
<td>December 1, 2008</td>
</tr>
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<td>First Round of Reviews</td>
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