FINITE-RANK INTERMEDIATE HANKEL OPERATORS
ON THE BERGMAN SPACE

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ABSTRACT. Let \( L^2 = L^2(D, r \, dr \, d\theta / \pi) \) be the Lebesgue space on the open unit disc and let \( L^2_\alpha = L^2 \cap \mathcal{H}ol(D) \) be the Bergman space. Let \( P \) be the orthogonal projection of \( L^2 \) onto \( L^2_\alpha \) and let \( Q \) be the orthogonal projection onto \( L^2_\alpha^\perp = \{ g \in L^2; \ g \in L^2_\alpha, \ g(0) = 0 \} \). Then \( I - P \simeq Q \). The big Hankel operator and the small Hankel operator on \( L^2_\alpha \) are defined as:

- For \( \phi \) in \( L^\infty \), \( H^\text{big}_\phi (f) = (I - P)(\phi f) \) and \( H^\text{small}_\phi (f) = Q(\phi f)(f \in L^2_\alpha) \).

In this paper, the finite-rank intermediate Hankel operators between \( H^\text{big}_\phi \) and \( H^\text{small}_\phi \) are studied. We are working on the more general space, that is, the weighted Bergman space.

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1. Introduction. Let \( D \) be the open unit disc in \( \mathbb{C} \) and let \( d\mu \) be the finite positive Borel measure on \( D \). Let \( L^2 = L^2(\mu) = L^2(\mu, d\mu) \) and \( \mathcal{H}ol(D) \) be the set of all holomorphic functions on \( D \). The weighted Bergman space \( L^2_\alpha = L^2_\alpha(\mu) \) is the intersection of \( L^2 \) and \( \mathcal{H}ol(D) \). In general, \( L^2_\alpha \) is not closed. In [6, Theorem 8], when \( (\text{supp } \mu) \cap D \) is a uniqueness set for \( \mathcal{H}ol(D) \), the first author and M. Yamada gave a necessary and sufficient condition for that \( L^2_\alpha \) is closed. Throughout this paper, we assume that \( L^2_\alpha \) is closed. When \( d\mu = r \, dr \, d\theta / \pi \), \( L^2_\alpha \) is the usual Bergman space.

For \( \mu \) such that \( L^2_\alpha(\mu) \) is closed, when \( M \) is the closed subspace of \( L^2(\mu) \) and \( z \cdot M \subseteq M \), \( M \) is called an invariant subspace. Suppose that \( M \supseteq zL^2_\alpha \). \( P^\mu \) denotes the orthogonal projection from \( L^2 \) onto \( M \). For \( \phi \) in \( L^\infty = L^\infty(\mu) = L^\infty(D, d\mu) \), the intermediate Hankel operator \( H^\mu_\phi \) is defined by

\[
H^\mu_\phi f = (I - P^\mu)(\phi f) \quad (f \in L^2_\alpha).
\]

(1.1)

When \( M = L^2_\alpha \), \( H^\mu_\phi \) is called a big Hankel operator \( H^\text{big}_\phi \) and when \( M = (zL^2_\alpha)^\perp \), \( H^\mu_\phi \) is called a small Hankel operator \( H^\text{small}_\phi \). Note that \( H^\mu_\phi \) is called a little Hankel operator when \( M = (L^2_\alpha)^\perp \).

For arbitrary symbol \( \phi \) in \( L^\infty \), in the case of \( d\mu = r \, dr \, d\theta / \pi \), both \( H^\text{big}_\phi \) and \( H^\text{small}_\phi \) were studied when they are compact operators or Schatten class operators (see [12]). However it seems to have not been studied when they are finite-rank operators. When \( \hat{\phi} \) is in \( L^2_\alpha \), it is known (see [12, page 155]) that if \( H^\text{big}_\phi \) is a finite-rank operator, then \( H^\text{big}_\phi = 0 \) and if \( \hat{\phi} \) is a polynomial, then \( H^\text{small}_\phi \) is a finite-rank operator. In this paper, for arbitrary symbol \( \phi \) in \( L^\infty \) we show that if \( H^\text{big}_\phi \) is a finite-rank operator, then \( H^\text{big}_\phi = 0 \), and we study when \( H^\text{small}_\phi \) is a finite-rank operator. In fact, we study such problems for the intermediate Hankel operators \( H^\mu_\phi \) on the weighted Bergman space \( L^2_\alpha(\mu) \).
In [2, 7, 9, 10], intermediate Hankel operators were studied in special weights, \(d\mu = (\alpha + 1)(1 - r^2)\sigma_r \, dr \, d\theta / \pi\) for \(-1 < \alpha < \infty\). In particular, Strouse [9] studied finite-rank intermediate Hankel operators.

Let \(d\mu = d\sigma(r) \, d\theta\) be a Borel measure on \(D\), where \(d\sigma(r)\) is a positive measure on \([0, 1)\) with \(d\sigma([0, 1)) = 1/2\pi\) and \(d\theta\) is the Lebesgue measure on \(\partial D\). \(L^2_\sigma(\mu)\) is closed if \(d\sigma([t, 1)) > 0\) for any \(t > 0\) (see [6]). For this type measures, it is possible to study more precisely the intermediate Hankel operators. In fact, \(L^2\) has the following orthogonal decomposition:

\[
L^2 = \sum_{j=-\infty}^{\infty} \oplus \mathcal{L}^2 e^{ij\theta},
\]

where \(\mathcal{L}^2 = L^2(d\sigma) = L^2([0,1), d\sigma)\). Set

\[
\mathcal{H}^2 = \sum_{j=0}^{\infty} \oplus \mathcal{L}^2 e^{ij\theta},
\]

then \(L^2_\sigma \subset \mathcal{H}^2 \subset (2L^2_\sigma)^\perp\) and \(L^2 = \mathcal{H}^2 \oplus e^{-i\theta}\mathcal{H}^2\). If \(\mathcal{M} = \mathcal{H}^2\), it is easy comparatively to determine finite-rank Hankel operators \(H^\perp_\phi\) and we can do it completely in Section 5.

We can expect that there does not exist any nonzero finite-rank Hankel operators \(H^\perp_\phi\) in case \(\mathcal{M} \in \mathcal{H}^2\) (see Section 5) and \(H^\perp_\phi\) is closed in case \(\mathcal{M} \not\in \mathcal{H}^2\) (see Section 6).

In Section 2, we describe an invariant subspace in \(L^2_\sigma\) whose codimension is of finite. Moreover we show that there does not exist an invariant subspace which contains \(L^2_\sigma\) properly and in which Hankel operators are studied in this paper. In Section 3, we describe finite-rank intermediate Hankel operators for arbitrary measure \(\mu\) such that \(L^2_\sigma(\mu)\) is closed. Moreover, we show that there does not exist any nonzero finite-rank Hankel operators \(H^\perp_\phi\) and there exists a nonzero finite-rank Hankel operator \(H^\perp_\phi\). In fact, we give two necessary and sufficient conditions for that if \(H^\perp_\phi\) is of finite rank \(\leq \ell\), then \(H^\perp_\phi = 0\). In Sections 3, 4, and 5, we use the Fourier coefficients \(\{M_j\}_{j=-\infty}^{\infty}\) of \(\mathcal{M}\) and so we assume \(d\mu = d\sigma(r) \, d\theta\). Using the Fourier coefficients of \(\phi\) and \(\mathcal{M}\), we give a necessary and sufficient condition for that \(H^\perp_\phi\) is of finite rank \(\leq \ell\). Assuming that \(\phi\) is a harmonic function, we can get a better necessary and sufficient condition. When \(\mathcal{M} \in \mathcal{H}^2\), using the Fourier coefficients \(\{M_j\}_{j=-\infty}^{\infty}\), we give a necessary condition and a sufficient condition for that if \(H^\perp_\phi\) is of finite rank \(\leq \ell\), then \(H^\perp_\phi = 0\). Two conditions are very similar but are a little different. Applications are given to examples in Section 2.

2. Invariant subspaces. In this section, we assume that \(d\mu = d\sigma(r) \, d\theta\) and \(d\sigma([t, 1)) > 0\) for any \(t > 0\), except Propositions 2.1 and 2.2. For our purpose, the invariant subspace \(\mathcal{M}\) must contain \(z L^2_\sigma\) but \(\ker H^\perp_\phi\) is an invariant subspace in \(L^2_\sigma\). If \(H^\perp_\phi\) is of finite rank, then the codimension of \(\ker H^\perp_\phi\) in \(L^2_\sigma\) is finite. In order to study finite-rank intermediate Hankel operators, we need the generalization of a result of Axler and Bourdon [1] which determines finite codimensional invariant subspaces in \(L^2_\sigma\) when \(d\mu = r \, dr \, d\theta / \pi\). In Propositions 2.1 and 2.2, the measure \(\mu\) is an arbitrary finite positive Borel measure such that \(L^2_\sigma\) is closed and \((\text{supp} \mu) \cap D\) is a uniqueness set for \(\mathcal{H}0(D)\). Since \(\mathcal{H}^2 \cap L^\infty\) is an extended weak-* Dirichlet algebra in \(L^\infty\),
Proposition 2.3 is a corollary of [4, Theorem 1]. We will give several examples of invariant subspaces which contain $zL^2_a$.

PROPOSITION 2.1. Suppose $M$ is an invariant subspace in $L^2_a$ and $\ell$ is a positive integer. The codimension of $M$ in $L^2_a$ is $\ell$, if and only if $M = qL^2_a$, where $q = \prod_{j=1}^{\ell} (z - a_j)$ and $a_j \in D$ ($1 \leq j \leq \ell$).

PROOF. The proof is almost parallel to that in [1, Theorem 1]. We will give a sketch of it. Suppose $M^\perp = L^2_a \ominus M$ and $\dim M^\perp = \ell$. Put

$$S_z f = P(z f) \quad (f \in M),$$

(2.1)

where $P$ is an orthogonal projection. Since $\ell < \infty$, there exists an analytic polynomial $b$ such that $b(S_z) = S_{b(z)} = 0$ and the degree of $b$ is less than or equal to $\ell$. Hence $bM^\perp \subseteq M$ and so $bL^2_a \subseteq M$. We show that the zeros of $b$ are only in $D$ and the degree of $b = \ell$. Then $M = bL^2_a$. It is clear that the degree of $b = \ell$. In this direction, we did not need the condition such that $(\text{supp} \mu) \cap D$ is a uniqueness set.

If $a \notin D$, $(z - a) L^2_a$ is dense in $L^2_a$. Assuming $a \geq 1$ and so $a = 1$ without a loss of generality, if $\varepsilon > 0$, then $(z - 1)L^2_a = (z - 1)\{z - (1 + \varepsilon)\}^{-1}L^2_a$. For any $f \in L^2_a$, it is easy to see that

$$\int_D \left| \frac{z - 1}{z - (1 + \varepsilon)} f - f \right|^2 d\mu \to 0 \quad (\varepsilon \to 0).$$

(2.2)

This implies that $(z - 1)L^2_a$ is dense in $L^2_a$. Thus all zeros of $b$ must be in $D$. The “if” part is clear because any point $a \in D$ gives a bounded evaluation functional. Here we used the condition such that $(\text{supp} \mu) \cap D$ is a uniqueness set (see [6, (1) of Theorem 8]).

PROPOSITION 2.2. Suppose that $(z - a)^{-1}$ does not belong to $L^2$ for each $a \in D$. If $M$ is an invariant subspace which contains $L^2_a$ properly, then the codimension of $L^2_a$ in $M$ is infinite.

PROOF. If $\dim M \ominus L^2_a = \ell < \infty$, by the proof of Proposition 2.1, there exists a polynomial $b = \prod_{j=1}^{\ell} (z - a_j)$ such that $bM \subseteq L^2_a$ and $a_j \in D$ ($1 \leq j \leq \ell$). Hence there exists a function $\phi$ in $M$ such that $\phi \notin L^2_a$ and $g = b\phi \in L^2_a$. If $g(a_k) \neq 0$ for some $k$, then $g/(z - a_k) = \phi \prod_{j \neq k} (z - a_j)$ cannot belong to $L^2$ because $(z - a_k)^{-1} \notin L^2$. Hence $g(a_j) = 0$ for any $j$. By [6, the proof in (1) of Theorem 8], $g \in bL^2_a$ and so $\phi = g/b$ belongs to $L^2_a$. This contradiction implies that $\dim M \ominus L^2_a = \infty$.

For an invariant subspace $M$, set

$$M_j = \left\{ f_j \in L^2; f \in M, f(z) = \sum_{j=-\infty}^{\infty} f_j(r)e^{ij\theta} \right\}. \quad (2.3)$$

Then $M_j$ is a subspace of $L^2$, $rM_j \subseteq M_{j+1}$ and hence $\dim M_{j+1} \geq \dim M_j$. We call $\{M_j\}_{j=-\infty}^{\infty}$ the Fourier coefficients of $M$. $M_je^{ij\theta}$ may not belong to $M$. If $M_je^{ij\theta}$ belongs to $M$ for any $j$, then $M$ has the following decomposition:

$$M = \sum_{j=-\infty}^{\infty} \Phi M_j e^{ij\theta}. \quad (2.4)$$
This decomposition is called the Fourier decomposition of $\mathcal{M}$. In general, $\mathcal{M}$ does not have the Fourier decomposition but we can get an extension $\hat{\mathcal{M}}$ of $\mathcal{M}$ which has the following Fourier decomposition:

$$\hat{\mathcal{M}} = \sum_{j=-\infty}^{\infty} \oplus(\text{closure of } \mathcal{M}_j)e^{ij\theta}. \quad (2.5)$$

**Proposition 2.3.** If $\mathcal{M}$ is an invariant subspace which contains $L^2_\delta$ and $e^{i\theta} \mathcal{M} \subseteq \mathcal{M}$, then $\mathcal{M} = \chi_E \tilde{q} \mathcal{H}^2 \oplus \chi_F L^2$, where $\chi_E$ is a characteristic function in $L^2$ and $q$ is a unimodular function in $\mathcal{H}^2$. Hence $\mathcal{M} \subseteq \mathcal{H}^2$. If $\bigcap_{j=0}^{\infty} e^{ij\theta} \mathcal{M} = \{0\}$, then $\mathcal{M} = \tilde{q} \mathcal{H}^2$.

**Proof.** Suppose $S_0 = \mathcal{M} \ominus e^{i\theta} \mathcal{M}$, then $\mathcal{M} = (\bigcup_{j=0}^{\infty} S_0 e^{ij\theta}) \ominus \mathcal{M}_{-\infty}$, where $\mathcal{M}_{-\infty} = \bigcap_{j=0}^{\infty} e^{ij\theta} \mathcal{M}$, and $rS_0 \subseteq S_0$ because $r \mathcal{M}_j \subseteq \mathcal{M}_{j+1}$. It is well known that $\mathcal{M}_{-\infty} = \chi_F L^2$ for a characteristic function $\chi_F$ of some measurable subset in $D$. Put $E = G^c$ then there exists a function $f$ in $S_0$ such that

$$|f| > 0 \quad \text{on } E \quad \text{and} \quad f = 0 \quad \text{on } F. \quad (2.6)$$

Since $f$ is orthogonal to $f^* e^{ij\theta}$ for all $j \geq 0$, $|f|^2$ belongs to $L^1 = L^1(d\sigma) = L^1([0,1],d\sigma)$ and so $|f|$ belongs to $L^2$. Hence $\chi_E$ belongs to $L^2$. Set

$$F(r e^{i\theta}) = \begin{cases} \frac{f(r e^{i\theta})}{|f(r e^{i\theta})|} & \text{if } f \neq 0, \\ 1 & \text{if } f = 0, \end{cases} \quad (2.7)$$

then $F$ is a unimodular function in $L^2$. Since $rS_0 \subseteq S_0$, we can show that $\chi_E F$ belongs to $S_0$ and so $S_0 = \chi_E F L^2$. Hence $\mathcal{M} \ominus \mathcal{M}_{-\infty} = \chi_E \tilde{q} \mathcal{H}^2$. Since $1 \in \mathcal{M}$, $\chi_E \tilde{F} \in \mathcal{H}^2$ and $q = \tilde{F} \in \mathcal{H}^2$, $\mathcal{M} = \tilde{q} \mathcal{H}^2$. $\Box$

**Example 2.4.** (i) For $0 < \beta < 1$, put

$$T_\beta = \text{span}\{z^n z^m; \beta n \geq m \geq 0\}. \quad (2.8)$$

Then $T_\beta$ is an invariant subspace and $T_\beta \supseteq L^2_\delta$, Put $T_\beta = L^2_\delta$ for $\beta = 0$ and $T_\beta = \mathcal{H}^2$ for $\beta = 1$. In general, $L^2_\delta \subseteq T_\beta \subseteq \mathcal{H}^2$ and $T_\beta (0 \leq \beta < 1)$ has the following Fourier decomposition:

$$T_\beta = \sum_{j=0}^{\infty} \oplus(T_\beta)_j e^{ij\theta}, \quad (2.9)$$

where $(T_\beta)_j = \text{span}\{r^j p_j(r^2); \ p_j \text{ is a polynomial of degree at most } \beta j/(1 - \beta)\}$. Janson and Rochberg [2] studied $H^\alpha_\phi$ when $\mathcal{M} = (T_\beta)^{\perp}$. Then $(T_\beta)^{\perp} = e^{i\theta} \mathcal{H}^2 \oplus \sum_{j=0}^{\infty} \oplus \{L^2 \ominus (T_\beta)_j\} e^{-ij\theta}$.

(ii) For $k \geq 0$, put $E^k = \text{span}\{z^m z^n; \ m = 0,1,\ldots,k; \ n = m,m+1,\ldots\}$. $E^k$ is an invariant subspace and $L^2_\delta \subseteq E^k \subseteq \mathcal{H}^2$. $E^k$ has the following Fourier decomposition:

$$E^k = \sum_{j=0}^{\infty} \oplus(E^k)_j e^{ij\theta}, \quad (2.10)$$

where $(E^k)_j = \text{span}\{r^j,\ldots,r^{j+2k}\}$. Strouse [9] studied $H^\alpha_\phi$ when $\mathcal{M} = (E^k)^{\perp}$. Then $(E^k)^{\perp} = e^{i\theta} \mathcal{H}^2 \oplus \sum_{j=0}^{\infty} \oplus \{L^2 \ominus (E^k)_j\} e^{-ij\theta}$.\]
(iii) Fix a polynomial $p$ of degree $k$, that is, $p = \sum_{j=0}^{k} a_j z^j$. Put

\[ Y(p) = \text{span}\{z^n, z^m p; \ n \geq 0, \ m \geq 0\}, \]

\[ Y^k = \text{span}\{z^\ell z^j; \ \ell \geq 0, \ 0 \leq j \leq k\}. \quad (2.11) \]

Both $Y(p)$ and $Y^k$ are invariant subspaces and $L^2_a \subseteq Y(p) \subseteq Y^k$, and $Y^k$ has the following Fourier decomposition:

\[ Y^k = \sum_{j=-k}^{\infty} \Phi(Y^k)_j e^{ij\theta}, \quad (2.12) \]

where $Y^k_0 = \text{span}\{1, r^2, \ldots, r^{2k}\}$ and $(Y^k)_j = r^j(Y^k_0)$ for $j \geq 0$, and $(Y^k)_{-j} = \text{span}\{r^{2k-j}; j \leq \ell \leq k\}$ for $1 \leq j \leq k$. $(Y(p))_j \subseteq (Y^k)_j$ for any $j$ but $Y(p)$ does not have a Fourier decomposition. If $a_j \neq 0$ for $1 \leq j \leq k$, $(Y(p))_j = (Y^k)_j$ for any $j$ and so $\tilde{Y}(p) = Y^k$. Peng, Rochberg, and Wu [7] and Wang and Wu [10] studied $H/\mathcal{H}$ when $\mathcal{H} = (\overline{Y^k})^\perp$. In general, we can define $Y(g)$ for any function $g$ in $L^2$. Usually, $Y(g)$ does not have the Fourier decomposition.

(iv) For a unimodular function $q$ in $H^2$, put $\mathcal{M} = \overline{qH^2}$. Then $\mathcal{M}$ is an invariant subspace which contains $H^2$. In general, $\overline{qH^2}$ may not have the Fourier decomposition but for $q = e^{i\ell\theta}$, for some $\ell \geq 0$,

\[ \mathcal{M} = \sum_{j=-\ell}^{\infty} \Phi L^2 e^{ij\theta}. \quad (2.13) \]

There are a lot of invariant subspaces between $H^2$ and $e^{-i\ell\theta}H^2$ even if $\ell = 1$.

(v) For arbitrary closed subspaces $S$ in $L^2$, put $\mathcal{M} = H^2 \oplus S e^{-i\theta}$. Then $\mathcal{M}$ is an invariant subspace between $H^2$ and $e^{-i\ell\theta}H^2$.

3. **Kronecker’s theorem.** In this section, the measure $\mu$ is an arbitrary finite positive Borel measure such that $L^2_a$ is closed. We will write

\[ \mathcal{M}^\infty = \mathcal{M} \cap L^\infty \]

and, for each positive integer $\ell$,

\[ \mathcal{M}^{\infty,\ell} = \left\{ \phi \in L^\infty; \ \phi(z) = g(z) \prod_{j=1}^\ell (z-a_j)^{-1} \text{ a.e. } \mu \text{ on } D, g \in \mathcal{M}^\infty \text{ and } a_1, \ldots, a_\ell \in D \right\}. \quad (3.1) \]

Then $\mathcal{M}^\infty \subseteq \mathcal{M}^{\infty,1} \subseteq \mathcal{M}^{\infty,2} \subseteq \cdots$.

Kronecker (cf. [11, page 210]) described finite-rank Hankel operators on the Hardy space. Theorem 3.1 describes finite-rank intermediate Hankel operators on the (weighted) Bergman space. However the situation is very different from that of Kronecker because $\mathcal{M}^\infty = \mathcal{M}^{\infty,\ell}$ may happen for some $\ell > 0$. See Corollaries 3.3 and 3.4.

**Theorem 3.1.** Suppose $\mathcal{M}$ is an invariant subspace which contains $zL^2_a$, and $\phi$ is a function in $L^\infty$. $H^2_\phi$ is of finite rank $\leq \ell$ if and only if $\phi$ belongs to $\mathcal{M}^{\infty,\ell}$.

**Proof.** Note that $\ker H^2_\phi = \{f \in L^2_a; \ \phi f \in \mathcal{M}\}$. Since $\mathcal{M}$ is an invariant subspace, $\ker H^2_\phi$ is also an invariant subspace. Proposition 2.1 implies the theorem. □
**Theorem 3.2.** Suppose \( \mathcal{M} \) is an invariant subspace which contains \( L^2_a \), and \( \phi \) is a function in \( L^\infty \). Then the following are equivalent:

1. If \( H^R_0 \) is of finite rank, then \( H^R_0 = 0 \).
2. \( \mathcal{M}_0 = \mathcal{M}_{0,\ell} \) for any \( \ell > 0 \).
3. If \( g \in \mathcal{M}_0 \), \( a \in D \) and \( (g(z) - g(a))/(z - a) \in L^\infty \), then \( (g(z) - g(a))/(z - a) \) belongs to \( \mathcal{M}_0 \).
4. If \( \mathcal{M}' \) is an invariant subspace and \( (\mathcal{M}')^\infty \not\supseteq \mathcal{M}_0 \), then there does not exist a nonzero polynomial \( b \) such that \( b(\mathcal{M}')^\infty \subseteq \mathcal{M}_0 \).

**Proof.** By Theorem 3.1, (1) \( \Leftrightarrow \) (2) is clear.

(1) \( \Rightarrow \) (3). If there exists \( g \in \mathcal{M}_0 \) such that \( (g - g(a))/(z - a) \in L^\infty \) does not belong to \( \mathcal{M}_0 \), put \( \phi = (g - g(a))/(z - a) \), then \( H^R_0 \) is of rank 1 and \( H^R_0 \neq 0 \).

(3) \( \Rightarrow \) (4). If (4) is not true, there exists \( \psi \) such that \( \psi \notin \mathcal{M}_0 \), \( \psi \in (\mathcal{M}')^\infty \) and \( b \psi \in \mathcal{M}_0 \) for some polynomial: \( b = \prod_{j=1}^\ell (z - a_j) \) and \( a_j \in D (1 \leq j \leq \ell < \infty) \). We may assume that \( \phi = \psi \prod_{j=1}^{\ell-1} (z - a_j) \notin \mathcal{M}_0 \) and \( g = (z - a_\ell) \phi \in \mathcal{M}_0 \). Then

\[
\frac{g - g(a_\ell)}{z - a_\ell} = \phi \in L^\infty, \quad \phi \notin \mathcal{M}_0.
\]  

(4) \( \Rightarrow \) (1). By Theorem 3.1, if \( H^R_0 \) is of finite rank \( \leq \ell \), then \( \phi \in \mathcal{M}_{0,\ell} \). If \( \phi \notin \mathcal{M}_0 \), suppose \( \mathcal{M}' \) is an invariant subspace generated by \( \phi \) and \( \mathcal{M} \), then \( (\mathcal{M}')^\infty \not\supseteq \mathcal{M}_0 \) but there does not exist a nonzero polynomial \( b \) such that \( b(\mathcal{M})^\infty \subseteq \mathcal{M}_0 \). Since \( \phi \in \mathcal{M}' \), this contradicts that \( \phi \in \mathcal{M}_{0,\ell} \).

**Corollary 3.3.** Suppose \( (\text{supp} \, \mu) \cap D \) is a uniqueness set for \( \Re \text{ol}(D) \). If \( H^\text{big}_{\phi} \) is of finite rank, then \( H^\text{big}_{\phi} = 0 \).

**Proof.** Theorem 3.2(3) implies the corollary. In fact, if \( g \in L^2_\mathcal{M} \cap L^\infty \), then \( g(z) - g(a) \in (z - a)L^2_\mathcal{M} \) by [6, the proof in (1) of Theorem 5.4]. Thus \( (g(z) - g(a))/(z - a) \) belongs to \( L^2_\mathcal{M} \cap L^\infty \).

**Corollary 3.4.** Suppose \( d\mu = r \, dr \, d\theta / \pi \). Let \( D_0 \) be an open subset of \( D \) and \( \mathcal{M} = \{ f \in L^2; f \text{ is analytic on } D_0 \} \). Then \( \mathcal{M} \) is an invariant subspace and if \( H^R_0 \) is of finite rank then \( H^R_{\phi} = 0 \).

**Proof.** It is easy to see that \( \mathcal{M}_0 \) satisfies Theorem 3.2(3).

**Corollary 3.5.** Suppose that if \( H^R_0 \) is of finite rank then \( H^R_0 = 0 \). If \( \mathcal{M}' \) is an invariant subspace which contains \( \mathcal{M} \) properly, then the codimension of \( \mathcal{M} \) in \( \mathcal{M}' \) is infinite or \( (\mathcal{M}')^\infty = \mathcal{M}_0 \).

**Proof.** If \( \dim \mathcal{M}' / \mathcal{M} < \infty \), as in the proof of Proposition 2.2, then there exists a nonzero polynomial \( b \) such that \( b \mathcal{M}' \subseteq \mathcal{M} \). Hence \( b(\mathcal{M}')^\infty \subseteq \mathcal{M}_0 \). If \( (\mathcal{M}')^\infty \not\supseteq \mathcal{M}_0 \), by Theorem 3.2, this contradicts that if \( H^R_0 \) is of finite rank, then \( H^R_{\phi} = 0 \).

4. **General case.** In this section, we assume that \( d\mu = d\sigma (r) \, dr \) and \( d\sigma ([t, 1)) > 0 \) for any \( t > 0 \). Hence we can define the Fourier coefficients \( \{ \mathcal{M}_j \}_{j=-\infty}^\infty \) of \( \mathcal{M} \). We assume \( \mathcal{M} = \hat{\mathcal{M}} \), that is, \( \mathcal{M} \) has the Fourier decomposition.
**Theorem 4.1.** Suppose $M$ is an invariant subspace which contains $zL^2_d$ and $\phi = \sum_{j=-\infty}^{\infty} \phi_j(r)e^{ij\theta}$ is a function in $L^\infty$. Then $H^u_\phi$ is of finite rank $\leq \ell$ if and only if there exist complex numbers $b_0, \ldots, b_\ell$ such that $b_\ell = 1$ and, for any integer $n$,

$$
\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) \in M_n.
$$

(4.1)

If $\ell$ is the minimum number of complex numbers $b_1, \ldots, b_\ell$ such that $\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) \in M_n$ for all $n$, then $H^u_\phi$ is of rank $\ell$.

**Proof.** If $H^u_\phi$ is of rank $\leq \ell$, by Theorem 3.1 there exists a polynomial $b = \sum_{j=0}^{\ell} b_j z^j$ such that $b\phi \in M$. Then

$$
\left( \sum_{j=-\infty}^{\infty} \phi_j(r)e^{ij\theta} \right) \left( \sum_{j=0}^{\ell} b_j r^j e^{ij\theta} \right) = \sum_{n=-\infty}^{\infty} \left( \sum_{j=0}^{\ell} \phi_{n-j}(r)b_j r^j \right) e^{in\theta}
$$

and so $\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) \in M_n$ for any $n$. The converse and the second statement are clear by Theorem 3.2.

**Corollary 4.2.** Let $\phi = \phi_t(r)e^{it\theta}$ for some integer $t$ in Theorem 4.1. Then $H^u_\phi$ is of finite rank $\leq \ell$ if and only if there exist complex numbers $b_0, \ldots, b_\ell$ such that $b_\ell = 1$ and for $t \leq n \leq \ell + t$, $b_n - r^{n-t} \phi_t(r) \in M_n$.

**Proof.** Since $\phi_j(r) = 0$ for $j \neq t$, if $n < t$ or $n > \ell + t$, then $\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) = 0$. For $t \leq n \leq \ell + t$, $\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) = b_n - r^{n-t} \phi_t(r)$, thus the corollary follows.

**Corollary 4.3.** Let $\phi = \sum_{j=1}^{\infty} a_j z^j + \sum_{j=0}^{\infty} a_{-j} z^{-j}$ in Theorem 4.1. Then $H^u_\phi$ is of rank $\leq \ell$ if and only if there exist complex numbers $b_0, \ldots, b_\ell$ such that $b_\ell = 1$ and for any nonpositive integer $n$ $\sum_{j=0}^{\ell} b_j a_{n-j} z^{2j-n} \in M_n$ and, for $0 < n < \ell$, $\sum_{j=n}^{\ell} b_j a_{n-j} z^{2j-n} \in M_n$.

**Proof.** If $n \geq \ell$ and $n \neq 0$, then

$$
\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) = \sum_{j=0}^{\ell} b_j a_{n-j} r^{j+n-j} = \left( \sum_{j=0}^{\ell} b_j a_{n-j} \right) r^n
$$

(4.3)

and hence $\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) \in M_n$ because $zL^2_d \subseteq M$. Now Theorem 4.1 implies the corollary.

Theorem 4.1 does not give an exact relation between the rank of $H^u_\phi$ and the number $\ell$ of complex numbers $b_0, \ldots, b_\ell$ such that $b_\ell = 1$. However, we can show the following: if $H^u_\phi$ is of rank $\ell$, then there exist complex numbers $b_0, \ldots, b_\ell$ such that $b_\ell = 1$, $\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) \in M_n$ for any $n$ and $b = \sum_{j=0}^{\ell} b_j z^j$ has just $\ell$ zeros in $D$. That is, if $\ell = 1$, then $|b_0| < 1$.

By Theorem 4.1, $H^u_\phi = 0$ if and only if $\phi_n \in M_n$ for any $n$ (i.e., $\phi \in M$). Moreover, $H^u_\phi$ is of rank $\leq 1$ if and only if there exist complex numbers $(b_0, b_1) \neq (0, 0)$ such that $b_1 = 1$ and $b_0 \phi_n + b_1 r \phi_{n-1} \in M_n$ for any $n$.
5. Big Hankel operator and $\mathcal{M} \subset H^2$. In this section, we assume that $d\mu = d\sigma(r)\,d\theta$ and $d\sigma([t,1)) > 0$ for any $t > 0$. Hence we can define the Fourier coefficients $\{\mathcal{M}_j\}_{j=-k}^\infty$ of $\mathcal{M}$ and we assume $\mathcal{M} = \mathcal{M}_k$. In this case, $\mathcal{H}_\phi^\mu$ is close to $\mathcal{H}_\phi^{\text{big}}$. Recall examples in Section 2, that is, $T_\phi$, $\hat{E}_k$, $Y(p)$, and $Y^k$.

**Corollary 5.1.** Suppose $\mathcal{M}$ is an invariant subspace between $zL^2_{\alpha}$ and $H^2$, and $\phi = \sum_{j=1}^\infty a_j z^j + \sum_{j=0}^\infty a_{-j} \bar{z}^j$. Then $\mathcal{H}_\phi^{\mu}$ is of finite rank $\leq \ell$ if and only if $a_{-n} = 0$ for $n > \ell$ and there exists complex numbers $b_0, \ldots, b_\ell$ such that $b_\ell = 1$ and $\sum_{j=n}^\ell b_j a_{n-j} r^{2j-n} \in \mathcal{M}$ for $0 \leq n \leq \ell$ and $\sum_{j=0}^\ell b_j a_{n-j} r^{2j-n} = 0$ for $-\ell < n < 0$.

**Proof.** Since $\mathcal{M} \subset H^2$, by Corollary 4.3 $\mathcal{H}_\phi^{\mu}$ is of finite rank $\leq \ell$ if and only if there exist complex numbers $b_0, \ldots, b_\ell$ such that $b_\ell = 1$ and $\sum_{j=0}^\ell b_j a_{n-j} r^{2j-n} = 0$ for $n < 0$ and $\sum_{j=n}^\ell b_j a_{n-j} r^{2j-n} \in \mathcal{M}$ for $0 \leq n \leq \ell$. If $\sum_{j=0}^\ell b_j a_{n-j} r^{2j-n} = 0$ for $n < 0$, then $b_j a_{n-j} = 0$ for $0 \leq j \leq \ell$ and $n < 0$. Hence for each $j$ $(0 \leq j \leq \ell)$, $b_j a_{-t} = 0$ if $t > j$. Thus $a_{-t} = 0$ if $t > \ell$.

**Proposition 5.2.** Suppose $\mathcal{M}$ is an invariant subspace between $zL^2_{\alpha}$ and $e^{-ik\theta}H^2$ where $k \geq 0$, and $\phi = \sum_{j=0}^\infty \phi_j(r) e^{-ij\theta}$ is a function in $L^\infty$. Then $\mathcal{H}_\phi^{\mu}$ is of finite rank $\leq \ell$ if and only if

$$
\phi(z) = \frac{\sum_{j=-k}^\ell \psi_j(r) e^{ij\theta}}{\sum_{j=0}^\ell b_j r^j e^{ij\theta}},
$$

(5.1)

where $\psi_n = \sum_{j=0}^\ell b_j r^j \phi_{n-j} \in \mathcal{M}$, for $-k \leq n \leq \ell$, and $(b_0, \ldots, b_\ell) \in \mathbb{C}^\ell$.

**Proof.** Note that $\mathcal{M} \subset e^{-ik\theta}H^2$ and $\phi_j(r) = 0$ for $j > 0$. If $\mathcal{H}_\phi^{\mu}$ is of finite rank $\leq \ell$, then, by Theorem 4.1,

$$
\left( \sum_{j=0}^\ell b_j r^j e^{ij\theta} \right) \left( \sum_{j=0}^\infty \phi_{-j}(r) e^{-ij\theta} \right) = \sum_{n=-k}^\ell \psi_n(r) e^{in\theta}
$$

(5.2)

and $\psi_n = \sum_{j=0}^\ell b_j r^j \phi_{n-j} \in \mathcal{M}$, for $-k \leq n \leq \ell$. The converse is also a result of Theorem 3.1.

**Corollary 5.3.** Suppose $\mathcal{M}$ is an invariant subspace in Proposition 5.2. If $\phi = \phi_+ + \phi_- = \sum_{j=1}^\infty a_j z^j + \sum_{j=0}^\infty a_{-j} \bar{z}^j$ and $\phi_- \in L^\infty$, then $\mathcal{H}_\phi^{\mu}$ is of finite rank $\leq \ell$ if and only if

$$
\phi(z) = \phi_+ + \frac{\sum_{j=-k}^\ell \psi_j(r) e^{ij\theta}}{\sum_{j=0}^\ell b_j r^j e^{ij\theta}},
$$

(5.3)

where $\psi_n = \sum_{j=0}^\ell b_j a_{n-j} r^{j+|n-j|} \in \mathcal{M}$, for $-k \leq n \leq \ell$, and $(b_0, \ldots, b_\ell) \in \mathbb{C}^\ell$. If $(b_0, \ldots, b_\ell) = (0, \ldots, 0)$, then $\psi_n = 0$ and so $\phi = \phi_+$.

**Theorem 5.4.** Suppose $\mathcal{M}$ is an invariant subspace between $zL^2_{\alpha}$ and $e^{-ik\theta}H^2$ where $k \geq 0$, and $\phi = \sum_{j=1}^\infty \phi_j(r) e^{ij\theta}$ is a function in $L^\infty$.

1. If $\mathcal{M}_j \cap r^{j+1} \mathcal{L}^2 = \{0\}$ for any $j \geq 0$, then there does not exist any finite rank $\mathcal{H}_\phi^{\mu}$ except $\mathcal{H}_\phi^{\mu} = 0$.

2. If there does not exist any finite rank $\mathcal{H}_\phi^{\mu}$ except $\mathcal{H}_\phi^{\mu} = 0$, then $\mathcal{M}_{-(-k)} \cap r^{j+1} \mathcal{L}^\infty = \{0\}$ for any $j \geq 0$. 


PROOF. (1) If $H^\mu_\phi$ is of finite rank $\ell$, by Proposition 5.2,

$$\psi_n = \sum_{j=n}^\ell b_j r^j \phi_{n-j} \in M_n,$$

(5.4)

for $0 \leq n \leq \ell$ because $\phi_{n-j}(r) = 0$ for $0 \leq j \leq n - 1$. We may assume $b_\ell = 1$. As $n = \ell - 1$, $r^\ell \phi_{\ell-1}(r) \in M_{\ell-1}$. Since $M_{\ell-1} \cap r^\ell L^2 = \{0\}$, $\phi_{\ell-1}(r) = 0$. As $n = \ell - 2$,

$$b_{\ell-1} r^{\ell-1} \phi_{\ell-2}(r) + r^\ell \phi_{\ell-2}(r) \in M_{\ell-2}. \quad (5.5)$$

Since $M_{\ell-2} \cap r^{\ell-1} L^2 = \{0\}$ and $\phi_{\ell-2}(r) = 0$, we can get $\phi_{\ell-1}(r) = 0$ for $j \leq \ell$. In Proposition 5.2, $\psi_n = 0$ for $0 \leq n \leq \ell$ and so $\phi = 0$.

(2) If $r^{j+1} \phi \in M_{-(k-j)} \cap r^{j+1} L^\infty$, then put $\phi = g e^{-i(k+1)\theta}$. If $g \neq 0$ then $\phi \in M$ and

$$z^{j+1} \phi = r^{j+1} g e^{-i(k+j)\theta} \in M_{-(k-j)} e^{-i(k+j)\theta}.$$ 

(5.6)

Since $M$ has the Fourier decomposition, $M_j e^{ij\theta} \subseteq M$ and so $z^{j+1} \phi \in M$. Theorem 3.1 gives a contradiction.

We will apply results in this section to Example 2.4 in Section 2.

EXAMPLE 5.5. (i) Suppose $M = T_\beta (0 \leq \beta < 1)$.

(1) When $\phi = \sum_{j=1}^\infty \phi_{-j}(r) e^{-ij\theta}$ is a function in $L^\infty$, there does not exist any finite rank $H^\mu_\phi$ except $H^\mu_\phi = 0$ if and only if $\beta = 0$.

(2) When $\phi = \sum_{j=0}^\infty a_j z^j + \sum_{j=1}^\infty a_j \bar{z}^j$ is a function in $L^\infty$, there does not exist any finite rank $H^\mu_\phi$ except $H^\mu_\phi = 0$ if and only if $\beta = 0$.

PROOF. Recall that $T_\beta = \sum_{j=0}^\infty \theta (T_\beta)_j e^{ij\theta}$ and $(T_\beta)_j = \text{span}\{r^j p_j (r^2); p_j \text{ is a polynomial of degree at most } j/1 - \beta\}$.

(1) If $\beta = 0$, then $(T_\beta)_j \cap r^{j+1} L^2 = \{0\}$ for any $j \geq 0$ and if $\beta \neq 0$, then $(T_\beta)_j \cap r^{j+1} L^\infty \neq \{0\}$ for enough large $j$. Theorem 5.4 implies (1).

(2) If $\beta \neq 0$, then there exists $n$ such that $1 - \beta \leq \beta (n-1)$. Hence $(T_\beta)_n \supseteq r^{n+1}$.

Suppose $\phi = 2$, then $z^n \phi = r^{n+1} e^{i(n-1)\theta}$ and so $z^n \phi \in (T_\beta)_n e^{i(n-1)\theta} \subseteq T_\beta$. By Theorem 3.1, $H^\mu_\phi$ is of rank $\lesssim n$ and $H^\mu_\phi \neq 0$.

(ii) Suppose $M = \tilde{E}^m$ ($0 < m < \infty$).

(1) When $\phi = \sum_{j=1}^\infty \phi_{-j}(r) e^{-ij\theta}$, there does not exist any finite rank $H^\mu_\phi$ except $H^\mu_\phi = 0$ if and only if $m = 0$.

(2) When $\phi = \sum_{j=0}^\infty a_j z^j + \sum_{j=1}^\infty a_j \bar{z}^j$ is a function in $L^\infty$, there does not exist any finite rank $H^\mu_\phi$ except $H^\mu_\phi = 0$ if and only if $m = 0$ or 1.

PROOF. We recall that $(\tilde{E})^m = \sum_{j=0}^\infty \theta (\tilde{E}^m)_j e^{ij\theta}$ and $(\tilde{E}^m)_j = \text{span}\{r^j, \ldots, r^{j+2m}\}$.

(1) If $m = 0$, then $(\tilde{E}^m)_j \cap r^{j+1} L^2 = \{0\}$ for any $j \geq 0$ and if $m \neq 0$, then $(\tilde{E}^m)_j \cap r^{j+1} L^\infty \neq \{0\}$ for any $j \geq 0$. Theorem 5.4 implies (1).

(2) If $m = 0$, by (1) there does not exist any finite rank $H^\mu_\phi$ except $H^\mu_\phi = 0$. If $m = 1$, then $(\tilde{E}^m)_n = \text{span}\{r^n, r^{n+2}\}$ for $n \geq 0$. When $H^M_\phi$ is of finite rank $\ell$, by Corollary 5.1, $a_{-n} = 0$ for $n > \ell$ and if $0 \leq n \leq \ell$,

$$\sum_{j=n}^\ell b_j a_{n-j} r^{2j-n} = cr^n + dr^{n+2}$$

(5.7)
for complex constants $c, d$. Hence, for $0 \leq n \leq \ell$,  
\[ b_j a_{n-j} = 0 \quad \text{for} \quad n + 2 \leq j \leq \ell. \]  
(5.8)
Since $b_\ell = 1$, $a_{n-\ell} = 0$ for $0 \leq n \leq \ell$ and so $a_j = 0$ for $0 \leq j \leq \ell$. When $m \geq 2$, if $\phi = \bar{z}$, then $z \phi = r^2 \in (E^m)_0 = \text{span}\{1, r^2, \ldots, r^{2m}\}$ and $z \phi \in E^m$ because $(E^m)_0 \subset E^m$. However $H_\phi^u \neq 0$.

(iii) Suppose $M = Y^k$.

1. When $\phi = \sum_{j=1}^\infty \phi_j(r) e^{ij\theta}$, there does not exist any finite rank $H_\phi^u$ except $H_\phi^u = 0$ if and only if $k = 0$.

2. When $\phi = \phi_+ + \phi_-$, it is sufficient to prove (1). We recall that $Y^k = \sum_{j=-\infty}^\infty \phi_j(r) e^{ij\theta}$, where $Y_j^0 = \text{span}\{1, r^2, \ldots, r^{2j}\}$ and $(Y^k)_j = r^j (Y^k)_0$ for $j \geq 0$, and $(Y^k)_{-j} = \text{span}\{r^{2j-\ell}, j \leq \ell \leq k\}$ for $0 \leq j \leq k$. If $k = 0$, then $Y^k = L^1_z$. If $k \geq 1$, $(Y^k)_{-k} = \text{span}\{r^k\}$. Theorem 5.4(2) implies that there exists a nonzero finite rank $H_\phi^u$.

**Proof.** Since $H_\phi^u = H_{\phi^u}^u$, it is sufficient to prove (1). We recall that $Y^k = \sum_{j=-\infty}^\infty \phi_j(r) e^{ij\theta}$, where $Y^0_j = \text{span}\{1, r^2, \ldots, r^{2j}\}$ and $(Y^k)_j = r^j (Y^k)_0$ for $j \geq 0$, and $(Y^k)_{-j} = \text{span}\{r^{2j-\ell}, j \leq \ell \leq k\}$ for $0 \leq j \leq k$. If $k = 0$, then $Y^k = L^1_z$. If $k \geq 1$, $(Y^k)_{-k} = \text{span}\{r^k\}$. Theorem 5.4(2) implies that there exists a nonzero finite rank $H_\phi^u$.

6. **Small Hankel operator and $M \supseteq H^2$.** In this section, we assume that $d\mu = d\sigma(r) d\theta$ and $d\sigma((t, 1)) > 0$ for any $t > 0$. Hence we can define the Fourier coefficients $\{M_j\}_{j=-\infty}^\infty$ of $M$. In this case, $H_\phi^u$ is close to $H_{\phi^u}^u$ and far from $H_{\phi^u}^u$. Note that if $M'$ is an invariant subspace and $M' \subseteq e^{it\theta} H^2$, then $M = (M')^\perp$ is an invariant subspace and $M \supseteq e^{it\theta} H^2$.

**Proposition 6.1.** Suppose $M$ is an invariant subspace which contains $e^{ik\theta} H^2$ for some nonnegative integer $k$. If $M \neq L^2$, there exists at least a nonzero finite rank $H_\phi^u$.

**Proof.** If $z^n \in M$ for all $n \geq 1$, then $z^\ell z^n \in M$ for all $\ell \geq 1$ because $z M \subseteq M$. Let $E$ be the closed linear span of $\{z^\ell z^n; n \geq 1, \ell \geq 0\}$, then $E \subseteq M$ and $g E \subseteq E$ for arbitrary polynomial $g$ of $z$ and $\bar{z}$. It is well known that $\|E\| = L^2$. This contradiction implies that there exists at least $n$ such that $z^n \notin M$ and $n \geq 1$. If $\phi = z^n$, then $z^{n+k} \phi \notin M$. Then $H_\phi^u \neq 0$ but $H_\phi^u$ is of finite rank $n+k$, by Theorem 3.1.

**Proposition 6.2.** Suppose $M$ is an invariant subspace which contains $e^{ik\theta} H^2$ for some nonnegative integer $k$. The following statements are valid.

1. If $\phi = \sum_{j=-\infty}^{\infty} \phi_j(r) e^{ij\theta}$ is a function in $L^\infty$, then there exists a function $\phi'$ in $L^2$ such that $\phi' = \sum_{j=0}^{k-1} \phi_j(r) e^{ij\theta} + \sum_{j=k}^{\infty} \phi_j(r) e^{ij\theta}$ and $H_\phi^u = H_{\phi'}^u$.

2. If $\phi = \sum_{j=-\infty}^{\infty} \phi_j(r) e^{ij\theta}$ is a function in $L^\infty$, then $H_\phi^u = 0$.

3. If $\phi = \sum_{j=-k}^{\infty} \phi_j(r) e^{ij\theta}$ is a function in $L^\infty$, then $H_\phi^u$ is of rank $\leq \ell + k < \infty$.

Conversely, if one of (1) or (2) is valid, then $M$ contains $e^{it\theta} H^2$.

**Proof.** Both (1) and (2) are clear because $M \supseteq e^{ik\theta} H^2$. (3) is a result of Theorem 3.1. The converse is also clear.

We will consider Example 2.4 in Section 2.

**Example 6.3.** (i) Suppose $M = (E^k)^\perp (0 \leq k < \infty)$ and $\phi = \sum_{j=-\infty}^{\infty} \phi_j(r) e^{ij\theta}$ is a function in $L^\infty$. 


(1) $H_{\Phi}^H = 0$ if and only if
\[
\int_0^1 \varphi_{-j}(r) r^{j+2t} d\sigma = 0 \quad (j \geq 0, \ 0 \leq t \leq k).
\] (6.1)

(2) $H_{\Phi}^H$ is of rank $\leq 1$ if and only if there exist complex numbers $(b_0, b_1) \neq (0,0)$ such that
\[
b_0 \int_0^1 \varphi_{-j}(r) r^{j+2t} d\sigma = -b_1 \int_0^1 \varphi_{-j-1}(r) r^{j+2t+1} d\sigma
\] for $j \geq 0, \ 0 \leq t \leq k$.

(3) Suppose $d\sigma = r \, dr / 2\pi$. When $\varphi = \sum_{j=0}^{\infty} a_j z^j + \sum_{j=1}^{\infty} a_{-j} \bar{z}^j$, if $H_{\Phi}^H$ is of rank $\leq 1$, then $H_{\Phi}^H = 0$.

**Proof.** From the remark in the last part of Section 4, (1) and (2) follows. (3) By (2), $H_{\Phi}^H$ is of rank $\leq 1$ if and only if there exist complex numbers $(b_0, b_1) \neq (0,0)$ such that
\[
b_0 a_{-j} \frac{1}{2j+2t+1} = -b_1 a_{-j-1} \frac{1}{2j+2t+3}
\] for $j \geq 0, \ 0 \leq t \leq k$. When $k \neq 0$, for each $j$, as $t = 0$,
\[
b_0 a_{-j} \frac{1}{2j+1} = -b_1 a_{-j-1} \frac{1}{2j+3},
\]
\[
b_0 a_{-j} \frac{1}{2j+3} = -b_1 a_{-j-1} \frac{1}{2j+5}.
\] (6.4)

This implies that $a_{-j} = a_{-j-1} = 0$, for $j \geq 0$, and so $\varphi = \sum_{j=1}^{\infty} a_j z^j$. When $k = 0$, Corollary 3.3 implies (3).

(iv) Suppose $\mathcal{M} = q \mathcal{H}^2$ for some unimodular function $q$ in $\mathcal{H}^2$ and $\varphi$ is a function in $L^\infty$. $H_{\Phi}^H$ is of finite rank $\ell$ if and only if
\[
\varphi = \sum_{j=-\ell}^{\infty} \psi_j(r) e^{ij\theta},
\] (6.5)

where $\psi_{-\ell}(r) \neq 0$.

**Proof.** If $\varphi = \hat{q} \sum_{j=-\ell}^{\infty} \psi_j(r) e^{ij\theta}$, then $z^\ell \varphi \in \mathcal{M}$ and so, by Theorem 3.1, $H_{\Phi}^H$ is of finite rank $\leq \ell$. Since $\psi_{-\ell}(r) \neq 0$, $b \varphi \notin \mathcal{M}$ for any polynomial $b$ of degree $\leq \ell - 1$ and so $H_{\Phi}^H$ is of finite rank $\ell$. The converse is clear.

(v) Suppose $\mathcal{M} = \mathcal{H}^2 \oplus S e^{-i\theta}$ and $S$ is a closed subspace in $L^2$. Let $\varphi = \sum_{j=-\infty}^{\infty} \phi_j(r) e^{ij\theta}$ be a function in $L^\infty$. By Theorems 3.1 and 4.1, $H_{\Phi}^H$ is of finite rank $\leq \ell$ if and only if $\phi_j(r) = 0$ for $j \leq -(\ell + 2)$ and there exist complex numbers $b_0, \ldots, b_\ell$ such that $b_\ell = 1,$
\[
\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) = 0 \quad \text{for} \quad -(\ell + 1) \leq n < -1,
\]
\[
\sum_{j=0}^{\ell} b_j r^j \phi_{-1-j}(r) \in S.
\] (6.6)
7. Restricted shift operator and \( \mathcal{M} \subseteq L^2_{\alpha} \). In this section, we assume \( \mu = r \, dr \, d\theta / \pi \) for simplicity. Let \( \mathcal{M} \) be an invariant subspace in \( L^2_{\alpha} \) and \( \mathcal{K} = L^2_{\alpha} \cap \mathcal{M} \). For \( \phi \) in \( L^\infty_{\alpha} = L^2_{\alpha} \cap L^\infty \),

\[
S^\mathcal{K}_\phi f = (I - \mathcal{P}^\mathcal{K}) (\phi f) \quad (f \in \mathcal{K}),
\]

where \( \mathcal{P}^\mathcal{K} \) is the orthogonal projection from \( L^2_{\alpha} \) to \( \mathcal{K} \). \( S^\mathcal{K}_\phi \) is called a restricted shift operator. For any \( \phi \) in \( L^\infty_{\alpha} \), \( S^\mathcal{K}_\phi \) commutes with \( S_z \). We do not know whether if the bounded linear operator \( T \) on \( \mathcal{K} \) commutes with \( S_z \), then \( T = S_z \phi \) for some \( \phi \in L^\infty_{\alpha} \). If \( T S_z^\phi = S_z^\phi T \) and \( \phi = T \mathcal{P} \) is bounded, then it is easy to see that \( T = S_z^\phi \) (cf. [5, page 784]). In the Hardy space instead of the Bergman space, Sarason [8] showed that this is true without any condition and \( \|T\| = \|\phi\|_\infty \).

We can define the Hankel operator \( H_\phi^\mathcal{K} \) as in the introduction. However \( H_\phi^\mathcal{K} \) is not an intermediate Hankel operator. It is not so difficult to see the following: when \( \mathcal{K} = L^2_{\alpha} \supseteq \mathcal{M} \) and \( \phi \) in \( L^\infty_{\alpha} \),

\[
\|H_\phi^\mathcal{K}\| = \|S^\mathcal{K}_\phi\|.
\]

This is known for the Hardy space. In fact, for \( f \) in \( L^2_{\alpha} \),

\[
H_\phi^\mathcal{K} f = (I - \mathcal{P}_\mu) \phi f = \mathcal{P}_\mu^\mathcal{K} \phi \mathcal{P}_\mu f
\]

and so \( H_\phi^\mathcal{K} f = \mathcal{P}_\mu^\mathcal{K} \phi \mathcal{P}_\mu f \) for \( f \) in \( L^2_{\alpha} \). Hence \( H_\phi^\mathcal{K} \) is of finite rank \( n \) if and only if \( S^\mathcal{K}_\phi \) is of finite rank \( n \). It is easy to see that \( S_z^\phi \) is of finite rank \( \ell \leq n \) if and only if there exists an analytic polynomial \( p \) of degree \( \ell \leq n \) such that \( p(\phi) \in \mathcal{M} \). When \( \phi \) is in \( L^\infty \), Theorems 3.1 and 4.1 are true for \( H_\phi^\mathcal{K} \).

Suppose \( \phi \) is a function in \( L^\infty_{\alpha} \).

1. \( L^2_{\alpha} \supseteq \ker H_\phi^\mathcal{K} \supseteq \mathcal{M} \).

2. When the common zero set \( Z(\mathcal{M}) \) of \( \mathcal{M} \) in \( D \) is empty, if \( H_\phi^\mathcal{K} \) is of finite rank then \( H_\phi^\mathcal{K} = 0 \). This is a result of (1) and Proposition 2.1.

3. If \( Z(\mathcal{M}) \) is not empty, there exists a nonzero finite rank \( H_\phi^\mathcal{K} \).

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This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

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