ITERATIVE SOLUTIONS OF $K$-POSITIVE DEFINITE OPERATOR EQUATIONS IN REAL UNIFORMLY SMOOTH BANACH SPACES

ZEQING LIU, SHIN MIN KANG, and JEONG SHEOK UME

(Received 2 October 2000)

Abstract. Let $X$ be a real uniformly smooth Banach space and let $T : D(T) \subseteq X \to X$ be a $K$-positive definite operator. Under suitable conditions we establish that the iterative method by Bai (1999) converges strongly to the unique solution of the equation $Tx = f$, $f \in X$. The results presented in this paper generalize the corresponding results of Bai (1999), Chidume and Aneke (1993), and Chidume and Osilike (1997).

2000 Mathematics Subject Classification. 47H06, 47H07, 47H14.

1. Introduction and preliminaries. Let $X$ be a real uniformly smooth Banach space with a dual space $X^*$. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{ f \in X^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \}, \quad x \in X. \quad (1.1)$$

It is known that $X$ is uniformly smooth (equivalently, $X^*$ is uniformly convex) if and only if $J$ is single-valued and uniformly continuous on any bounded subset of $X$.

Chidume and Aneke [3] introduced the concept of $K$-positive definite operators and established the existence of the unique solution of the equation $Tx = f$ for that operator in real separable Banach spaces. Meanwhile they constructed, in $L_p$ (or $l_p$) spaces with $p \geq 2$, an iteration method which converges strongly to the unique solution, provided that $T$ and $K$ commute. Chidume and Osilike [5] gave a new iteration scheme, in separable $q$-uniformly smooth Banach spaces, which converges strongly to the unique solution of the equation $Tx = f$, $f \in X$.

Recently, Bai [1] constructed a more general iteration procedure and improved the results of [3, 5] to separable uniformly smooth real Banach spaces.

Very recently, Zhou et al. [7] established the following excellent result, which is a generalization of the main result of Chidume and Aneke [3].

Lemma 1.1 (see [7]). Let $X$ be a real Banach space and let $T$ be a $K$-positive definite operator with $D(T) = D(K)$. Then there exists a constant $\alpha > 0$ such that

$$\|Tx\| \leq \alpha \|Kx\|, \quad x \in D(T). \quad (1.2)$$

Moreover, the operator $T$ is closed, $R(T) = X$, and the equation $Tx = f$ for each $f \in X$, has a unique solution.

The purpose of this paper is to study the convergence problem of the iteration procedure introduced in [1] for $K$-positive definite operators in real uniformly smooth real
Banach spaces. Our results extend the corresponding results due to Bai [1], Chidume and Aneke [3], and Chidume and Osilike [5].

In what follows, we will also need the following concepts and results.

**Definition 1.2** (see [3, 7]). Let \( X \) be a real Banach space and \( X_1 \) a subspace of \( X \). An operator \( T \) with domain \( D(T) \supseteq X_1 \) is called **continuously \( X_1 \)-invertible** if \( T \), as an operator restricted to \( X_1 \), has a bounded inverse on \( R(T) \). A linear unbounded operator \( T \) with domain \( D(T) \) in \( X \) and range \( R(T) \) in \( X \) is called **\( K \)-positive definite** if there exist a continuously \( D(T) \)-invertible closed linear operator \( K \) with \( D(A) \subseteq D(K) \) and a constant \( c > 0 \) such that

\[
\langle Tu, j(Ku) \rangle \geq c \| Ku \|^2, \quad u \in D(T), \ j(Ku) \in J(Ku). \tag{1.3}
\]

Let \( X \) be a real Banach space. Recall that the modulus of smoothness of \( X \) is defined by

\[
\rho_X(t) = \sup \left\{ \frac{1}{2}(\| x + y \| + \| x - y \|) - 1 : x, y \in X, \| x \| = 1, \| y \| \leq t \right\}, \quad t \geq 0. \tag{1.4}
\]

\( X \) is said to be **uniformly smooth** if \( \lim_{t \to 0} \rho_X(t)/t = 0 \). Let \( p > 1 \) be a real number. \( X \) is called **\( p \)-uniformly smooth** if there exists a constant \( r > 0 \) such that

\[
\rho_X(t) \leq rtp, \quad t > 0. \tag{1.5}
\]

Hilbert spaces, \( L_p \) (or \( l_p \)) spaces, \( 1 < p < \infty \), and the Sobolev spaces \( W^{m,p} \), \( 1 < p < \infty \), are all \( p \)-uniformly smooth. It is well known that the class of \( p \)-uniformly smooth real Banach spaces is a proper subclass of that of uniformly smooth real ones.

**Lemma 1.3** (see [4, 6]). Let \( X \) be a real uniformly smooth Banach space. Then

(i) there exist some positive constants \( A \) and \( B \) such that

\[
\| x + y \|^2 \leq \| x \|^2 + 2 \langle y, J(x) \rangle + A \max \{ \| x \|, B \} \rho_X(\| y \|), \quad x, y \in X. \tag{1.6}
\]

(ii) there exists a continuous nondecreasing function \( b : [0, \infty) \to [0, \infty) \) such that

\[
\rho_X(t) \leq rt^p, \quad t > 0. \tag{1.7}
\]

**Lemma 1.4** (see [2]). Suppose that \( \{ \alpha_n \}_{n=0}^{\infty}, \{ \beta_n \}_{n=0}^{\infty}, \) and \( \{ \omega_n \}_{n=0}^{\infty} \) are nonnegative sequences such that

\[
\alpha_{n+1} \leq (1 - \omega_n) \alpha_n + \beta_n \omega_n, \quad n \geq 0, \tag{1.8}
\]

with \( \{ \omega_n \}_{n=0}^{\infty} \subset [0, 1], \sum_{n=0}^{\infty} \omega_n = \infty \) and \( \lim_{n \to \infty} \beta_n = 0 \). Then \( \lim_{n \to 0} \alpha_n = 0 \).

**Lemma 1.5** (see [6]). Let \( X \) be a real Banach space. Then

(i) \( \rho_X(0) = 0, \rho_X(t) \leq t, \quad t > 0; \)

(ii) \( \rho_X(t) \) is convex, continuous, and nondecreasing on \( [0, \infty) \);

(iii) \( \rho_X(t)/t \) is nondecreasing on \( (0, \infty) \).
2. Main results

**Theorem 2.1.** Let $X$ be a real uniformly smooth Banach space and let $T : D(T) \subseteq X \to X$ be a $K$-positive definite operator with $D(T) = D(K)$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ iteratively from any $f \in X$ and $x_0 \in D(T)$ by

$$
y_n = x_n + b_n v_n, \quad x_{n+1} = y_n + a_n u_n, \quad n \geq 0;
$$

$$
u_n = K^{-1} f - K^{-1} T x_n, \quad u_n = K^{-1} f - K^{-1} T y_n, \quad n \geq 0, \tag{2.2}
$$

where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are arbitrary nonnegative sequences such that

$$
\sum_{n=0}^{\infty} (a_n + b_n) = \infty; \tag{2.3}
$$

$$
\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = 0; \tag{2.4}
$$

$$
\max\{a_n, b_n\} \leq \frac{1}{2c}, \quad n \geq 0; \tag{2.5}
$$

$$
\alpha_A \max\{(1 + \alpha a_n) \|Kv_0\|, (1 + \alpha b_n) \|Kv_0\|, B\} \leq 2c \|Kv_0\|, \quad n \geq 0, \tag{2.6}
$$

where $c, \alpha, A$ and $B$ are the constants appearing in (1.2), (1.3), and (1.6), respectively. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of the equation $Tx = f$.

**Proof.** It follows from Lemma 1.1 that the equation $Tx = f$ has a unique solution in $X$. Note that $T$ and $K$ are linear. From (2.1) and (2.2) we have

$$
Kv_{n+1} = f - Tx_{n+1} = Ku_n - a_n Tu_n, \quad n \geq 0; \tag{2.7}
$$

$$
Ku_n = f - Ty_n = Kv_n - b_n Tv_n, \quad n \geq 0. \tag{2.8}
$$

In view of (2.8) and (1.2), (1.3), and (1.6), we conclude that

$$
\|Ku_n\|^2 = \|Kv_n - b_n Tv_n\|^2
\leq \|Kv_n\|^2 - 2b_n \langle Tv_n, J(Kv_n) \rangle
+ A \max\{\|Kv_n\| + b_n \|Tv_n\|, B\} \rho_X (b_n \|Tv_n\|)
\leq (1 - 2c b_n) \|Kv_n\|^2
+ A \max\{(1 + \alpha b_n) \|Kv_n\|, B\} \rho_X (\alpha b_n \|Kv_n\|) \tag{2.9}
$$

for all $n \geq 0$. Using (2.7) and (1.2), (1.3), and (1.6), we have

$$
\|Kv_{n+1}\|^2 = \|Ku_n - a_n Tu_n\|^2
\leq \|Ku_n\|^2 - 2a_n \langle Tu_n, J(Ku_n) \rangle
+ A \max\{\|Ku_n\| + a_n \|Tu_n\|, B\} \rho_X (a_n \|Tu_n\|)
\leq (1 - 2c a_n) \|Ku_n\|^2
+ A \max\{(1 + \alpha a_n) \|Ku_n\|, B\} \rho_X (\alpha a_n \|Ku_n\|) \tag{2.10}
$$
for all \( n \geq 0 \). Set \( M = \| K v_0 \| \). We claim that
\[
\max \{ \| K v_n \|, \| K u_n \| \} \leq M, \quad n \geq 0.
\] (2.11)

By virtue of (2.6), (2.9), and Lemma 1.5, we get that
\[
\| K u_0 \| \leq (1 - 2 c b_0) \| K v_0 \|^2 + A \max \{ (1 + \alpha b_0) \| K v_0 \|, B \} \rho_X (\alpha b_0 \| K v_0 \|)
\leq (1 - 2 c b_0) M^2 + A \max \{ (1 + \alpha b_0) M, B \} \alpha b_0 M
\leq M^2.
\] (2.12)

That is, (2.11) is true for \( n = 0 \). Suppose that (2.11) holds for some \( n \geq 0 \). Using (2.10), (2.6), and Lemma 1.5, we infer that
\[
\| K v_{n+1} \|^2 \leq (1 - 2 c a_n) \| K u_n \|^2 + A \max \{ (1 + \alpha a_n) \| K u_n \|, B \} \rho_X (\alpha a_n \| K u_n \|)
\leq (1 - 2 c a_n) M^2 + A \max \{ (1 + \alpha a_n) M, B \} \alpha a_n M
\leq M^2.
\] (2.13)

From (2.6), (2.9), (2.13), and Lemma 1.5, we have
\[
\| K u_{n+1} \|^2 \leq (1 - 2 c b_{n+1}) \| K v_{n+1} \|^2 + A \max \{ (1 + \alpha b_{n+1}) \| K v_{n+1} \|, B \} \rho_X (\alpha b_{n+1} \| K v_{n+1} \|)
\leq (1 - 2 c b_{n+1}) M^2 + A \max \{ (1 + \alpha b_{n+1}) M, B \} \alpha b_{n+1} M
\leq M^2.
\] (2.14)

Therefore (2.11) holds for all \( n \geq 0 \). Since \( X \) is uniformly smooth, by (2.4) and Lemma 1.5 we conclude that there exist nonnegative sequences \( \{ s_n \}_{n=0}^{\infty} \) and \( \{ t_n \}_{n=0}^{\infty} \) such that \( \rho_X (\alpha Ma_n) = s_n a_n, \rho_X (\alpha Mb_n) = t_n b_n \) for all \( n \geq 0 \) and
\[
\lim_{n \to \infty} s_n = \lim_{n \to \infty} t_n = 0.
\] (2.15)

It follows from (2.5), (2.9), (2.10), and (2.11) that
\[
\| K v_{n+1} \|^2 \leq (1 - 2 c a_n) (1 - 2 c b_n) \| K v_n \|^2 + (1 - 2 c a_n) A \max \{ (1 + \alpha b_n) \| K v_n \|, B \} \rho_X (\alpha b_n \| K v_n \|)
\leq (1 - 2 c (a_n + b_n) + 4 c^2 a_n b_n) \| K v_n \|^2 + A \max \{ (1 + \alpha) M, B \} \rho_X (\alpha Ma_n) (\alpha Mb_n)
\leq [1 - c (a_n + b_n)] \| K v_n \|^2 + L (a_n s_n + b_n t_n)
\] (2.16)

for all \( n \geq 0 \), where \( L = A \max \{ (1 + \alpha) M, B \} \). Let
\[
\alpha_n = \| K v_n \|^2, \quad \omega_n = c (a_n + b_n), \quad \beta_n = \frac{L}{c} r_n, \quad n \geq 0.
\] (2.17)
where

\[ r_n = \begin{cases} 
0, & a_n + b_n = 0, \\
\frac{a_n}{a_n + b_n} s_n + \frac{b_n}{a_n + b_n} t_n, & a_n + b_n \neq 0.
\end{cases} \]  

(2.18)

It follows from (2.15) that \( \lim_{n \to \infty} r_n = 0 \). That is, \( \lim_{n \to \infty} \beta_n = 0 \). Thus (2.15) can be rewritten in the form

\[ \alpha_{n+1} \leq (1 - \omega_n) \alpha_n + \omega_n \beta_n, \quad n \geq 0. \]  

(2.19)

Note that (2.3) and (2.5) mean that \( \sum_{n=0}^{\infty} \omega_n = \infty, \omega_n \in [0,1] \). Consequently, Lemma 1.4 ensures that \( \alpha_n \to 0 \) as \( n \to \infty \). That is,

\[ \|Kv_n\| \to 0 \quad \text{as} \quad n \to \infty. \]  

(2.20)

It follows from (2.2) and (2.20) that

\[ ||Tx_n - f|| = ||Kv_n|| \to 0 \quad \text{as} \quad n \to \infty. \]  

(2.21)

Note that \( T \) has a bounded inverse. Thus (2.21) means that \( x_n \to T^{-1} f \), the unique solution of \( Tx = f \). This completes the proof.

**Theorem 2.2.** Let \( X, T, K, f, \{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty} \) and \( \{u_n\}_{n=0}^{\infty} \) be as in Theorem 2.1. Suppose that \( \{a_n\} \) and \( \{b_n\}_{n=0}^{\infty} \) are any nonnegative sequences such that (2.3), (2.4), and (2.5) and

\[ \max \{ b(\alpha a_n), b(\alpha b_n) \} \leq \frac{2c}{\max \{1,\|Kv_0\|\}}, \quad n \geq 0, \]  

where \( b(t) \) is as in (1.7), \( \alpha \) and \( c \) are the constants appearing in (1.3) and (1.2), respectively. Then the sequence \( \{x_n\}_{n=0}^{\infty} \) converges strongly to the unique solution of the equation \( Tx = f \).

**Proof.** Set \( M = \max \{1,\|Kv_0\|\} \). As in the proof of Theorem 3 in [1] we have

\[ ||Kv_{n+1}||^2 \leq (1 - c(a_n + b_n))||Kv_n||^2 + M^3 \alpha (a_n b(\alpha a_n) + b_n b(\alpha b_n)), \quad n \geq 0. \]  

(2.23)

Let

\[ \alpha_n = ||Kv_n||^2, \quad \omega_n = c(a_n + b_n), \quad \beta_n = \frac{\alpha}{c} M^3 r_n, \quad n \geq 0, \]  

(2.24)

where

\[ r_n = \begin{cases} 
0, & a_n + b_n = 0, \\
\frac{a_n}{a_n + b_n} b(\alpha a_n) + \frac{b_n}{a_n + b_n} b(\alpha b_n), & a_n + b_n \neq 0.
\end{cases} \]  

(2.25)

It is easily seen that \( \lim_{n \to \infty} \beta_n = 0 \). The rest of the argument now follows as in the proof of Theorem 2.1 to yield that \( x_n \to T^{-1} f \) as \( n \to \infty \). This completes the proof.
Remark 2.3. Theorems 2.1 and 2.2 extend Theorem 3.3 of Bai [1], Theorem 2 of Chidume and Aneke [3] and Theorem of Chidume and Osilike [5], respectively, in the following ways:

(a) Condition (2.3) is much weaker than $\sum_{n=0}^{\infty} a_n = \infty$ of [1].

(b) $L_p$ (or $l_p$) spaces, $p \geq 2$, in [3] and $q$-uniformly smooth Banach space, $q > 1$, in [5] are replaced by the more general uniformly smooth Banach spaces.

(c) The commutativity condition of $T$ and $K$ in [3] is dropped.

(d) The iteration methods in [3, 5] are special cases of our iteration method.

Acknowledgements. The first author is grateful to the Research Institute of Natural Sciences, Gyeongsang National University, in Korea for giving him the opportunity to visit the institute, the second author was supported by Korea Research Foundation Grant (KRF-99-005-D00003), and the third author was supported by KOSEF research project No. (2001-1-10100-005-2).

References


Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the *Mathematical Problems in Engineering* aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at [http://www.hindawi.com/journals/mpe/](http://www.hindawi.com/journals/mpe/). Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at [http://mts.hindawi.com/](http://mts.hindawi.com/) according to the following timetable:

<table>
<thead>
<tr>
<th>Event</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>December 1, 2008</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>March 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>June 1, 2009</td>
</tr>
</tbody>
</table>

**Guest Editors**

**José Roberto Castilho Piqueira**, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

**Elbert E. Neher Macau**, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil; elbert@lac.inpe.br

**Celso Grebogi**, Center for Applied Dynamics Research, King’s College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk