SUBMODULES OF SECONDARY MODULES

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Let $R$ be a commutative ring with nonzero identity. Our objective is to investigate representable modules and to examine in particular when submodules of such modules are representable. Moreover, we establish a connection between the secondary modules and the pure-injective, the $\Sigma$-pure-injective, and the prime modules.

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1. Introduction. In this paper, all rings are commutative rings with identity and all modules are unital. The notion of associated prime ideals and the related one of primary decomposition are classical. In a dual way, we define the attached prime ideals and the secondary representation. This theory is developed in the appendix to Section 6 in Matsumura [6] and in Macdonald [5]. Now we define the concepts that we will need.

Let $R$ be a ring and let $0 \neq M$ be an $R$-module. Then $M$ is called a secondary module (second module) provided that for every element $r$ of $R$ the homothety $M \overset{r}{\rightarrow} M$ is either surjective or nilpotent (either surjective or zero). This implies that $\text{nilrad}(M) = P$ ($\text{Ann}(M) = P'$) is a prime ideal of $R$, and $M$ is said to be $P$-secondary ($P'$-second), so every second module is secondary (the concept of second module is introduced by Yassemi [14]). A secondary representation for an $R$-module $M$ is an expression for $M$ as a finite sum of secondary modules (see [5]). If such a representation exists, we will say that $M$ is representable.

If $R$ is a ring and $N$ is a submodule of an $R$-module $M$, the ideal $\{r \in R : rM \subseteq N\}$ will be denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of $M$, $\text{Ann}(M)$. A proper submodule $N$ of a module $M$ over a ring $R$ is said to be prime submodule (primary submodule) if for each $r \in R$ the homothety $M/N \overset{r}{\rightarrow} M/N$ is either injective or zero (either injective or nilpotent), so $(0 : M/N) = P$ (nilrad$(M/N) = P'$) is a prime ideal of $R$, and $N$ is said to be $P$-prime submodule ($P'$-primary submodule). So $N$ is prime in $M$ if and only if whenever $rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $rM \subseteq N$. We say that $M$ is a prime module (primary module) if zero submodule of $M$ is prime (primary) submodule of $M$, so $N$ is a prime submodule of $M$ if and only if $M/N$ is a prime module. Moreover, every prime module is primary.

Let $R$ be a ring, and let $N$ be an $R$-submodule of $M$. Then $N$ is pure in $M$ if for any finite system of equations over $N$ which is solvable in $M$, the system is also solvable in $N$. A module is said to be absolutely pure if every embedding of it into any other modules is pure embedding. A submodule $N$ of an $R$-module $M$ is called relatively divisible (or an RD-submodule) if $rN = N \cap rM$ for all $r \in R$. Every RD-submodule of a $P$-secondary module over a commutative ring $R$ is $P$-secondary (see [2, Lemma 2.1]).
A module $M$ is pure-injective if and only if any system of equations in $M$ which is finitely solvable in $M$, has a global solution in $M$ [7, Theorem 2.8]. The module $N$ is a pure-essential extension of $M$ if $M$ is pure in $N$ and for all nonzero submodules $L$ of $N$, if $M \cap L = 0$, then $(M \oplus L)/L$ is not pure in $N/L$. A pure-injective hull $H(M)$ of a module $M$ is a pure essential extension of $M$ which is pure-injective. Every module $M$ has a pure-injective hull which is unique to isomorphism over $M$ [12].

Given an $R$-module $M$ and index set $I$, the direct sum of the family $\{M_i : i \in I\}$ where $M_i = M$ for each $i \in I$ will be denoted by $M^{(I)}$. Given a module property $\mathcal{P}$, we will say that a module $M$ is $\sum \cdot \mathcal{P}$ if $M^{(I)}$ satisfies $\mathcal{P}$ for every index set $I$.

Let $R$ be a commutative ring. An element $a \in R$ is said to be regular if there exists $b \in R$ such that $a = a^2b$, and $R$ is said to be regular if each of its elements is regular. An important property of regular rings is that every module is absolutely pure (see [13, Theorem 37.6]).

Let $R$ be a ring and $M$ an $R$-module. A prime ideal $P$ of $R$ is called an associated prime ideal of $M$ if $P$ is the annihilator $\text{Ann}(x)$ of some $x \in M$. The set of associated primes of $M$ is written $\text{Ass}(M)$. For undefined terms, we refer to [6, 7].

2. Secondary submodules. In general, a nonzero submodule of a representable (even secondary) $R$-module is not representable (secondary), but we have the following results.

**Lemma 2.1.** Let $R$ be a commutative ring and let $0 \neq N$ be an RD-submodule of $R$-module $M$. Then $M$ is $P$-secondary if and only if $N$ and $M/N$ are $P$-secondary.

**Proof.** If $M$ is $P$-secondary, then $N$ and $M/N$ are $P$-secondary by [2, Lemma 2.1] and [5, Theorem 2.4], respectively. Conversely, suppose that $r \in R$. If $r \in P$, then $r^n(M/N) = 0$ and $r^nN = 0$ for some $n$, hence $r^nM \subseteq N$ and $0 = r^nN = r^nM \cap N = r^nM$. If $r \notin P$, then $rM + N = M$, $rN = N$, and $N = rN = rM \cap N$, so we have $rM = M$, as required.

**Corollary 2.2.** Let $R$ be a commutative regular ring, and let $0 \neq N$ be a submodule of $R$-module $M$. Then $M$ is $P$-secondary if and only if $N$ and $M/N$ are $P$-secondary.

**Proof.** This follows from Lemma 2.1.

**Theorem 2.3.** Let $R$ be a commutative regular ring. Then every nonzero submodule of a representable $R$-module is representable.

**Proof.** Let $M$ be a representable $R$-module and let $M = \sum_{i=1}^n M_i$ be a minimal secondary representation with $\text{nilrad}(M_i) = P_i$. There is an element $r_1 \notin \bigcup_{i=2}^n P_i$. Otherwise $P_1 \subseteq \bigcup_{i=2}^n P_i$, so by [10, Theorem 3.61], $P_1 \subseteq P_j$ for some $j$, and hence $P_1 = P_j$, a contradiction. Thus there exists a positive integer $m_1$ such that $r_1^{m_1} \in \text{Ann}(M_1)$ and the module $r_1^{m_1}M = \sum_{i=2}^n r_1^{m_1}M_i$ is representable. By using this process for the ideals $P_2, \ldots, P_{n-1}$, there are integers $m_2, \ldots, m_{n-1}$ and elements $r_2 \in P_2, \ldots, r_{n-1} \in P_{n-1}$ such that $s_nM = M_n$, where $0 \neq s_n = r_1^{m_1} r_2^{m_2} \cdots r_{n-1}^{m_{n-1}}$, $s_n \in \cap_{i=1}^{n-1} P_i$ and $s_n \notin P_n$. Therefore by a similar argument, there are elements $s_1, \ldots, s_{n-1}$
such that $M = \sum_{i=1}^{n} s_i M$, where for each $i$, where $i = 1, \ldots, n$, $s_i \in P_i$, $s_i M = M_i$, and $s_i \in \cap_{i=1}^{n} \text{Ann}(M_j)$.

Let $N$ be a nonzero submodule of $M$ and $0 \neq a \in N$. Then $a = s_1 b_1 + \cdots + s_n b_n$ for some $b_i \in M$, $i = 1, \ldots, n$. By assumption, there exists $t_1, \ldots, t_n \in R$ such that for each $i$, $s_i = s_i^2 t_i$. As $0 \neq a$, $s_i b_i \neq 0$ for some $i$ and $s_i t_i a = s_i^2 t_i b_i = s_i b_i$, so $s_i N \neq 0$. We can assume that $s_1 N \neq 0, \ldots, s_i N \neq 0$, where $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$. By a similar argument as above, if $a \in N$, then $a = \sum_{j=1}^{k} s_j t_j a = \sum_{j=1}^{k} s_j N$, and hence $N = \sum_{j=1}^{k} s_j N$. Since for each $j$, where $j = 1, \ldots, k$, $s_j N$ is pure in the $P_{ij}$-secondary module $M_{ij}$, it is $P_{ij}$-secondary by [2, Lemma 2.1], as required.

**Theorem 2.4.** Let $R$ be a commutative ring and let $N$ be a prime submodule of secondary $R$-module of $M$. Then $N$ is $(N : M)$-secondary.

**Proof.** Suppose that $M$ is a $P$-secondary module over $R$. Let $r \in R$. If $r \in P$, then $r^n N \subseteq r^n M = 0$ for some $n$. If $r \notin P$, then $r M = M$. Suppose that $n \in N$, so there is an element $m \in M$ such that $n = r m$. As $N$ is a prime submodule of $M$ and $N \neq r M = M$, $m \in N$, so $r N = N$, hence $N$ is $P$-secondary.

By [4, Lemma 1], the ideal $P' = (N : M) = \{r \in R : r M \subseteq N\}$ is prime. Clearly, $P' \subseteq P$. Let $s \in P$. Then $s^n N = s^n M = 0$ for some $n$. There is an element $m \in M$ such that $m \notin N$ and $s^n m = 0 \in N$, so $s^n \in P'$, hence $s \in P'$. Thus $P = P'$, as required.

**Proposition 2.5.** Let $R$ be a commutative ring and let $N$ be a prime submodule of $P$-second $R$-module of $M$. Then $N$ is an RD-submodule of $M$.

**Proof.** Let $r \in R$. If $r \in P$, then $r N \subseteq r M = 0$, so $r N = N \cap r M = 0$. If $r \notin P$, then $r M = M$, so the homothety $M/N \rightarrow M/N$ is not zero since $N$ is prime. It follows that the above homothety is injective. If $a \in N \cap r M$, then there is $b \in M$ such that $a = r b$. Since $r (b + N) = 0$, so $b \in N$, hence $r N = N \cap r M$, as required.

**Theorem 2.6.** Let $M$ be a $P$-second module over a commutative ring $R$, and let $N$ be a prime submodule of $M$Then every submodule of $M$ properly containing $N$ is an RD-submodule. In particular, it is $P$-second.

**Proof.** Let $K$ be a submodule of $M$ properly containing $N$. Then $K/N$ is a prime submodule of prime and $P$-second module $M/N$, so by Proposition 2.5, $K/N$ is an RD-submodule of $M/N$. Now the assertion follows from [3, Consequences 18.2.2(c)] and Proposition 2.5.

**Lemma 2.7.** Let $M$ be a nonzero module over a commutative domain $R$. Then $M$ is $(0)$-secondary if and only if $M$ is $(0)$-secondary.

**Proof.** The proof is completely straightforward.

By [3, Proposition 11.3.11] and [11, Proposition 12, page 506] (see also [14]), and the definitions of secondary and primary modules, we obtain the following corollary.

**Corollary 2.8.** Let $R$ be a commutative ring.

(i) Every Artinian primary module over $R$ is secondary.

(ii) Every Noetherian secondary module over $R$ is primary.

(iii) Every finitely generated secondary module is primary.
**Lemma 2.9.** Let $R$ be a commutative ring. Let $K$ and $N$ be submodules of an $R$-module $M$ such that $N$ is prime and $K$ is $P$-secondary. Then $N \cap K$ is $P$-secondary.

**Proof.** Let $r \in R$. If $r \in P$, then $r^n(N \cap K) \subseteq r^nK = 0$ for some $n$. Suppose $r \notin P$ and $t \in N \cap K$. Then $t = rs$ for some $s \in K$ since $K$ is $P$-secondary. As $N$ is prime, we have $s \in N$, and hence $t \in r(N \cap K)$. This gives, $N \cap K = r(N \cap K)$. \hfill \Box

**Theorem 2.10.** Let $M$ be a representable module over a commutative ring $R$, and let $N$ be a prime submodule of $M$ with $(N : M) = P$. Then the following hold:

(i) $N$ is representable;

(ii) $M/N$ is $P$-secondary.

**Proof.** (i) Let $M$ be a representable $R$-module and let $M = \sum_{i=1}^{m} M_i$ be a minimal secondary representation with nilrad $M_i = P_i$. For each $i$, $i = 1, 2, \ldots, m$, let $m_i \in M_i$ and $r_i \in P_i$. Then $r_i^{n_i}m_i = 0$ for some $n_i$, and we have $(r_i^{n_i} + P)(m_i + M_i) = 0$ and hence either $P_i \subseteq P$ or $M_i \subseteq N$ ($i = 1, 2, \ldots, m$). It follows that $M_i \not\subseteq N$ for some $i$ (otherwise $M = N$). If $M_j \not\subseteq N$ and $M_j \not\subseteq N$ for $i \neq j$, then $P_i = P_j$, a contradiction (for if $t \in P - P_i$ then $M_i = tM_i \subseteq tM \subseteq N$). Therefore, without loss of generality, we can assume that $M_1 \not\subseteq N$ and $M_1 \subseteq N$, so $P_1 = P$ and $P_i \not\subseteq P$ ($i = 2, 3, \ldots, m$). Then $M_2 + M_3 + \cdots + M_m \subseteq N$ and

\[ N = N \cap M = N \cap (M_1 + \cdots + M_m) = M_2 + \cdots + M_m + (N \cap M_1). \]  

(2.1)

Now the assertion follows from Lemma 2.9.

(ii) Since $M = M_1 + N$, we have $M/N = (M_1 + N)/N \cong M_1/(M_1 \cap N)$, as required. \hfill \Box

**Proposition 2.11.** Let $R$ be a Dedekind domain, and let $M$ be a $0 \neq P$-secondary $R$-module. Then $M$ is a $P$-primary module.

**Proof.** Let $r \in R$. If $r \in P$, then the homothety $M \overset{r}{\longrightarrow} M$ is nilpotent since $M$ is secondary. Suppose that $r \notin P$. If $ra = 0$ for some $0 \neq a \in M$, then by [6, Theorem 6.1], there exists $0 \neq b \in M$ and $Q \subseteq \text{Ass}(M)$ such that $r \in Q$ and $Q = (0 :_R b)$. As $(0 : M) \subseteq (0 : b) = Q$, we have $P = Q$, a contradiction. So the homothety $M \overset{r}{\longrightarrow} M$ is injective, as required. \hfill \Box

**Remarks.** (i) Let $R$ be a domain which is not a field. Then $R$ is a prime $R$-module (since $R$ is torsion-free) but it is not secondary (even it is not pure-injective).

(ii) Let $R$ be a local Dedekind domain with maximal ideal $P = Rp$. We show that the module $E(R/P)$ is not prime (but it is $(0)$-secondary). Set $E = E(R/P)$ and $A_n = (0 :_E P^n) (n \geq 1)$. Then by [2, Lemma 2.6], $PA_{n+1} = A_n$, $A_n \subseteq E$ is a cyclic $R$-module with $A_n = Ra_n$ such that $pa_{n+1} = a_n$, every nonzero proper submodule $L$ of $E$ is of the form $L = A_m$ for some $m$ and $E$ is Artinian module with a strictly increasing sequence of submodules

\[ A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq A_{n+1} \subseteq \cdots. \]  

(2.2)

We claim that $(A_n :_R E) = 0$ for every $n$. Suppose that $r \in (A_n :_R E)$ with $r \neq 0$. Then $rE \subseteq A_n$ and for all $a \in M$, we have $a = rb$ for some $b \in M$ since $E$ is injective (= divisible). Thus $a = rb \in A_n$, so $E = A_n$, a contradiction. Therefore $(A_n :_R E) = 0$ for
every integer $n \geq 1$. However no $A_n$ is a prime submodule of $E$, for if $m$ is any positive integer, then $p^m \notin (A_n :_RE) = 0$ and $a_{n+m} \notin A_n$, but $p^m a_{m+n} = a_n \in A_n$.

**Theorem 2.12.** Let $R$ be a Dedekind domain, and let $M$ be an $R$-module. Then $M$ is $0 \neq P$-second if and only if $M$ is $P$-prime.

**Proof.** By Proposition 2.11, it is enough to show that if $M$ is $P$-prime, then $M$ is $P$-second. Since $(0 : M) = P$ is a maximal ideal in $R$, so $M$ is a vector space over $R/P$, hence $M$ is $P$-second.

**Proposition 2.13.** Let $R$ be a Dedekind domain. Then any $0 \neq P$-prime $R$-module is a direct sum of copies of $R_P/PR_P \cong R/P$.

**Proof.** By the proof of Proposition 2.11, every element of $R – P$ acts invertibly on $M$, so the $R$-module structure of $M$ extends naturally to a structure of $M$ as a module over the localisation $R_P$ of $R$ at $P$. Therefore, we can assume that $R$ is a commutative local Dedekind domain with maximal ideal $P = Rp$. Let $M_j$ denote the indecomposable summand of $M$, so $M_j$ is $P$-prime. Let $m_j$ be a nonzero element of $M_j$, hence $(0 : m_j) = (0 : M) = P$. Then $Rm_j \cong R/P$ is pure in $M_j$ since $m_j$ is not divisible by $p$ in $M_j$, but by [1, Proposition 1.3], the module $R/P$ is itself pure-injective, so $Rm_j$ is a direct summand of $M_j$, and hence $M_j \cong Rm_j$, as required.

3. Pure-injective modules

**Proposition 3.1.** Let $M$ be a $P$-secondary module over a commutative ring $R$. Then $H = H(M)$, the pure-injective hull, is $P$-secondary.

**Proof.** Let $r \in R$. If $r \notin P$, then $rM = M$, so $M$ satisfies the sentence for all $x$ there exists $y$ ($x = ry$), and hence so does $H$ (because any module and its pure-injective hull satisfy the same sentences [7, Chapter 4]). If $r \in R$, then $r^nM = 0$, so $M$ satisfies the sentence for all $x$ ($r^n x = 0$), hence so does in $H$, as required.

**Theorem 3.2.** The following conditions are equivalent for a Prufer domain $R$:

(i) the ring $R$ is a Dedekind domain;

(ii) every secondary $R$-module is pure-injective.

**Proof.** Let $R$ be a Dedekind domain and $M$ a secondary $R$-module. If $\text{Ann}(M) = 0$, then $M$ is divisible, hence injective. If $\text{Ann}(M) \neq 0$, then $M$ is a torsion $R$-module of bounded order, so that $M$ is $\Sigma$-pure-injective (see [15]). In both cases, $M$ is $\Sigma$-pure-injective (so pure-injective).

Conversely, let $R$ be a Prufer domain with the property that every secondary module is pure-injective. In order to prove that $R$ is Dedekind domain, it suffices to show that every divisible $R$-module is injective. Let $M$ be a divisible $R$-module. Then $M$ is secondary, Hence pure-injective. Since $R$ is Prufer, pure-injective modules are RD-injective (see [7]). The embedding of $M$ in its injective envelope $E(M)$ is an RD-pure monomorphism, because for every nonzero $r \in R$ we have that $M = rM$, so that $rE(M) \cap M \subseteq M \subseteq rM$. Since $M$ is the RD-injective, $M$ is a direct summand of $E(M)$. Thus $M$ is injective. This shows that $R$ is a Dedekind domain.
Remarks. (i) There is a module over a commutative regular ring which is injective but not secondary (see [9, Theorem 2.3]). The commutative regular ring $R = F \times F$, $F$ a field, is an Artinian Gorenstein, that is, $R$ is injective (so pure-injective) as an $R$-module. But $R$ is not secondary, because multiplication by $(1,0)$ is neither nilpotent nor surjective.

(ii) The above consideration thus leads us to the following question: are secondary modules pure-injective? The answer is yes because of the following reason. Every non-Noetherian Prufer domain has secondary modules that are not pure-injective. For instance, every non-Noetherian valuation domain has secondary modules that are not pure-injective.

**Proposition 3.3.** Let $M$ be an $R$-module.

(i) $M$ is $\Sigma$-secondary if and only if $M$ is secondary.

(ii) Let $M$ be a direct sum of modules $M_i$ ($i \in I$) where for each $i$, $M_i$ is secondary and $\text{Ann}(M_i) = \text{Ann}(M_j)$ for all $i, j \in I$. Then $M$ is secondary.

**Proof.** (i) The necessity is immediate by the definition. Conversely, suppose that $M$ is $P$-secondary. Given an index set $J$, and let $r \in R$. If $r \in P$, then $r^n M = 0$ for some $n$, so $r^n M(J) = 0$. If $r \notin P$ then $rM = M$, so $rM(J) = M(J)$, as required.

(ii) Since the annihilators of all direct summands coincide, we can assume that $M_i$ is $P$-secondary (say) for all $i \in I$. Now the proof of (ii) is similar to that (i) and we omit it.

**Corollary 3.4.** Let $M$ be an indecomposable $\Sigma$-pure-injective module over a commutative Prufer ring $R$. Then $M$ is secondary.

**Proof.** Set $P = \{r \in R : \text{Ann}_M r \neq 0\}$ and $P' = \cap_n P^n$. Then $P$ and $P'$ are prime ideals in $R$ by [8, Fact 3.1 and Lemma 2.1]. By [8, Fact 3.2], $M$ is either $P$-secondary or $P'$-secondary, as required.

**Corollary 3.5.** Every $\Sigma$-pure-injective module over a Prufer ring is representable.

**Proof.** Suppose $M$ is a $\Sigma$-pure-injective module over a commutative Prufer ring $R$. By [8, page 967], we can write $M = M_1 \oplus \cdots \oplus M_m$ where $M_i$ is secondary for all $i$ by Proposition 3.3 and Corollary 3.4, as required.

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This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

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