ON PTOLEMY'S THEOREM

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The above theorem is:

Let a, b, c, d be (distinct, finite) points in the complex plane, lying, in this order, on a circle or a straight line. Then

$$|a-b| \cdot |c-d| + |a-d| \cdot |b-c| = |a-c| \cdot |b-d|. \tag{1}$$

(In the case of a circle, the theorem states that in a quadrilateral inscribed in a circle, the sum of the products of lengths of opposite sides equals the product of lengths of the diagonals.) We note here that (1) follows very easily from the elementary theory of fractional linear transformations.

Indeed let f be such a transformation, mapping the circle or straight line onto the real axis, so that

$$f(a) = \infty, f(b) = 0, f(c) = 1.$$

The $f(d) > 1$. Hence for the cross ratios below we have:

$$|(a,d,b,c)| = |(\infty, f(d), 0, 1)| = [f(d) - 1]/f(d),$$

$$|(a,b,d,c)| = |(\infty, 0, f(d), 1)| = 1/f(d),$$

$$|(a-b)(c-d)(a-c)^{-1}(b-d)^{-1}| + |(a-d)(b-c)(a-c)^{-1}(b-d)^{-1}|$$

$$= |(a,d,b,c)| + |(a,b,d,c) = 1$$

which implies (1).