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A FIXED POINT THEOREM FOR A NONLINEAR TYPE CONTRACTION

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ABSTRACT. A well-known result of Boyd and Wong [1] on nonlinear contractions is extended. Several other known results are obtained as special cases.

INTRODUCTION.

In this paper, we extend a well-known result of Boyd and Wong [1] and obtain as consequences several other known results (see [2], [3], [4], [5]).

Throughout this paper, let (X,d) be a complete metric space, R^+ the nonnegative reals and $\phi = \phi(t_1,t_2,t_3,t_4,t_5):(R^+)^5 \to R^+$ a function which is (a) continuous from right in each coordinate variable (b) nondecreasing in t_2, t_3, t_4, t_5, and satisfies the inequality (c) $\phi(t,s,s,as,bs) < \mathrm{Max}\{t,s\}$ if $\mathrm{Max}\{t,s\} \neq 0$ where $\{a,b\} \subseteq \{0,1,2\}$ with $a + b = 2$. Note that (c) implies that $\phi(t,t,t,t,t) < t$ for any $t > 0$.

2. MAIN RESULTS.

The following is the main result of this paper.

THEOREM 1. Let $f,g:X \to X$ be two commutative mappings such that

(i) $fX \subseteq gX$,

(ii) g is continuous,

(iii) $d(fx,fy) \leq \phi(d(gx,gy),\ d(fx,gx),\ d(fy,gy),\ d(fx,gy),\ d(fy,gx))$,

for each $x, y \in X$. Then, there exists a unique $u \in X$ with $fu = gu = u$.

We first prove the following lemma which simplifies the proof of the above theorem.

LEMMA. Under the conditions of Theorem 1, if there exists a $v \in X$ such that $fv = gv$, then there exists a unique $u \in X$ with $fu = gu = u$.

PROOF. We show that for any $w \in X$

$$f(w) = g(w) \text{ implies } f(v) = f(w) \tag{2.1}$$

Suppose $t = d(fv,fw) > 0$. Then it follows by (iii) that

$$t \leq \phi(t,0,0,t,t) \leq \phi(t,t,t,t,t) < t,$$

a contradiction. Thus $fv = fw$. Now, since $fw = gw$, therefore, $f(fw) = g(fw)$ and consequently by (2.1)

$$f(w) = f(fw) = g(fw).$$

Thus, if we set $u = f(w)$, then $fu = gu = u$. The uniqueness of u now follows from (2.1).

PROOF OF THEOREM 1. Let x_o be an arbitrary point in X. Construct a sequence $\{y_n\}$ in X as follows. Let $y_o = fx_o$. By (i) there exists a $x_1 \in X$ such that $y_o = gx_1$. Set $y_1 = fx_1$. Thus, if $y_o, y_1,..y_n$ are obtained with $y_n = fx_n$, there exists by (i) a $x_{n+1} \in X$ such that $y_n = gx_{n+1}$. Let $y_{n+1} = fx_{n+1}$. Thus, for each $n \in I$ (nonnegative Integers),

$$y_n = fx_n = gx_{n+1}. \tag{2.2}$$

We shall show that $\{y_n\}$ is a Cauchy sequence in X. For this, let for each $n \in I$, $d_n = d(y_n,y_{n+1})$. Then by (i) and (b),

$$d_{n+1} = d(fx_{n+1},fx_{n+2}) \leq \phi(d_n,d_n,d_{n+1},0,d_n + d_{n+1}). \tag{2.3}$$

Now, if for some $n \in I$, $d_{n+1} > d_n$, then by (b) and (c)

$$d_{n+1} \leq \phi(d_n,d_{n+1},d_{n+1},0,2d_{n+1}) < d_{n+1},$$

a contradiction. Thus for each $n \in I$, $d_{n+1} \leq d_n$, that is $\{d_n\}$ is a nonincreasing sequence of nonnegative reals and consequently there exists a $d \in R^+$ such that $\{d_n\} \to d$. Clearly $d = 0$, for otherwise by (2.3) and (c),

$$d \leq \phi(d,d,d,0,2d) < d,$$

a contradiction. Thus,

$$d_n \to 0. \tag{2.4}$$

Suppose, now that $\{y_n\}$ is not a Cauchy sequence. Then there exists a $E > 0$ such that for each $k \in I$, there exist integers $n(k)$, $m(k)$ with $k \leq n(k) < m(k)$ satisying

$$E_k = d(y_{n(k)},y_{m(k)}) > E.$$

Let $m(k)$ be the least integer greater than $n(k)$ such (2.4) holds. This implies that for each $k \in I$, $d(y_{n(k)},y_{m(k)-1}) \leq E$. Consequently, for each $k \in I$,

$$E < E_k \leq d(y_{n(k)},Y_{m(k)-1}) + d(y_{m(k)-1},Y_{m(k)}) \leq E + d_k. \tag{2.5}$$

Hence, it follows by (2.4) that as $k \to \infty$, $E_k \to E$.

However, for each $k \in I$,

$$E_k \leq d_{n(k)} + d(fx_{n(k)+1},fx_{m(k)+1}) + d_{m(k)},$$

$$\leq 2d_k + \phi(E_k,d_k,d_k,E_k+d_k,E_k+d_k),$$

Therefore, as $k \to \infty$,

$$E \leq \phi(E,0,0,E,E) < E,$$

contradicting the existence of $E > 0$. Thus, $\{y_n\}$ is a Cauchy sequence in X. Consequently, there is a $v \in X$ such that $\{y_n\} \to v$, that is

$$fx_n = gx_{n+1} \to v. \tag{2.6}$$

We show that for this v,

$$\alpha = d(fv,gv) = 0.$$

Suppose $\alpha > 0$. Now by (ii) and (2.6) we have,

$$fgx_n = gfx_n \to gv \text{ and } g^2x_n \to gv.$$

Also, it follows by (b) and (iii) that,

$$d(f(gx_n),fv) \leq \phi(d(g^2x_n,gv),\ d(fgx_n,g^2x_n),\ \alpha,\ d(fgx_n,gv),\ \alpha + d(gv,g^2x_n)).$$

Therefore, as $n \to \infty$, the above inequality yields that

$$\alpha = d(gv,fv) \leq \phi(0,0,\alpha,0,\alpha) < \alpha,$$

a contradiction. Thus $fv = gv$ and hence by the above lemma, there is a unique $u \in X$ satisfying $fu = gu = u$.

In the special case when g is taken to be the identity map of x in Theorem 1, we have

COROLLARY 1. Let $f:X \to X$ satisfy either of the following conditions: for all $x,y \in X$,

(A). $d(fx,fy) \leq \phi(d(x,y),\ d(x,fx),\ d(y,fy),\ d(x,fy),\ d(y,fx))$.

(B). $d(fx,fy) \leq \alpha(d(x,fx) + d(y,fy)) + \beta(d(x,fy) + d(y,fx)) + \Psi(d(x,y))$

where $\alpha \geq 0$, $\beta \geq 0$ and $\Psi:R^+ \to R^+$ is a right continuous function satisfying $\Psi(t) < (1-2\alpha-2\beta)t$ if $t > 0$. Then f has a unique fixed point in X.

PROOF. The conclusion is an obvious consequence of Theorem 1 if (A) holds. In case of condition (B), let $\phi:(R^+)^5 \to R^+$ be defined by

$$\phi(t_1,t_2,t_3,t_4,t_5) = \Psi(t_1) + \alpha(t_2 + t_3) + \beta(t_4 + t_5).$$

then ϕ satisfies conditions (a), (b) and (c). Thus the conclusion again follows by Theorem 1.

It may be remarked that if $\alpha = \beta = 0$ in (B) then Corollary 1 yields a well-known result of Boyd and Wong [1]. If $\Psi(t) = \alpha t$, then Corollary 1 yields certain results of Hardy and Rogers [2], Kannan [3], Reich [4], Sehgal [5]. All these results are special cases of Theorem 1.

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