SOME SUBMERSIONS OF CR-HYPERSURFACES OF KÄHLER-EINSTEIN MANIFOLD

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The Riemannian submersions of a CR-hypersurface \( M \) of a Kaehler-Einstein manifold \( \tilde{M} \) are studied. If \( M \) is an extrinsic CR-hypersurface of \( \tilde{M} \), then it is shown that the base space of the submersion is also a Kaehler-Einstein manifold.


1. Introduction. The study of the Riemannian submersions \( \pi : M \rightarrow B \) was initiated by O'Neill [14] and Gray [9]. This theory was very much developed in the last thirty five years. Besse’s book [3, Chapter 9] is a reference work. Bejancu introduced a remarkable class of submanifolds of a Kaehler manifold that are known as CR-submanifolds (see [1, 2]). On a CR-submanifold, there are two complementary distributions \( D \) and \( D^\perp \), such that \( D \) is \( J \)-invariant and \( D^\perp \) is \( J \)-anti-invariant with respect to the complex structure \( J \) of the Kaehler manifold. The integrability of the anti-invariant distribution \( D \) was proved by Blair and Chen [4].

Recently, Kobayashi [10] considered the similarity between the total space of a Riemannian submersion and a CR-submanifold of a Kaehler manifold in terms of the distribution. He studied the case of a generic CR-submanifolds in a Kaehler manifold and proved that the base space is a Kaehler manifold.

In Section 3, we extend the result of Kobayashi to the general case of a CR-submanifold.

In Section 4, we study a Riemannian submersion from an extrinsic hypersurface \( M \) of a Kaehler-Einstein manifold \( \tilde{M} \) onto an almost-Hermitian manifold \( B \). In this case, we prove that the basic manifold is a Kaehler-Einstein manifold. If \( \tilde{M} \) is \( C^{n+1} \), a standard example is the Hopf fibration \( S^{2n+1} \rightarrow CP^n \) equipped with the canonical metrics.

For the basic formulas of Riemannian geometry, we use [11, 12].

2. Preliminaries. Let \( \tilde{M} \) be a complex \( m \)-dimensional Kaehler manifold with complex structure \( J \) and Hermitian metric \( \langle \cdot, \cdot \rangle \). Bejancu [2] introduced the concept of a CR-submanifold of \( \tilde{M} \) as follows: a real Riemannian manifold \( M \), isometrically immersed in a Kaehler manifold \( \tilde{M} \), is called a CR-submanifold of \( \tilde{M} \) if there exists on \( M \) a differentiable holomorphic distribution \( D \) and its
orthogonal complement $D^\perp$ on $M$ is a totally real distribution, that is, $JD_x^\perp \subseteq T_x^\perp M$, where $T_x^\perp M$ is the normal space to $M$ at $x \in M$ for any $x \in M$. It is easily seen that each real orientable hypersurface of $M$ is a CR-submanifold. The Riemannian metric induced on $M$ will be denoted by the same symbol $\langle \cdot, \cdot \rangle$.

Let $\tilde{\nabla}$ (resp., $\nabla$) be the operator of covariant differentiation with respect to the Levi-Civita connection on $\tilde{M}$ (resp., $M$). The second-fundamental form $B$ is given by

$$B(E,F) = \tilde{\nabla}_EF - \nabla_{\tilde{E}}F$$

(2.1)

for all $E,F \in \Gamma(TM)$, where $\Gamma(TM)$ is the space of differentiable vector field on $M$. We denote everywhere by $\Gamma(\tau)$ the space of differentiable sections of a vector bundle $\tau$.

For a normal vector field $N$, that is, $N \in \Gamma(T_x^\perp M)$, we write

$$\tilde{\nabla}_EN = -L_NE + \nabla_x^\perp N,$$

(2.2)

where $-L_NE$ (resp., $\nabla_x^\perp N$) denotes the tangential (resp., normal) component of $\tilde{\nabla}_EN$.

Let $\mu$ be the orthogonal complementary vector bundle of $JD_x^\perp$ in $T_x^\perp M$, that is, $T_x^\perp M = JD_x^\perp \oplus \mu$.

It is clear that $\mu$ is a holomorphic subbundle of $T_x^\perp M$, that is, $J\mu = \mu$.

**Definition 2.1** (Kobayashi [10]). Let $M$ be a CR-submanifold of a Kaehler manifold $\tilde{M}$. A submersion from a CR-manifold $M$ onto an almost-Hermitian manifold is a Riemannian submersion $\pi : M \to M'$ with the following conditions:

(i) $D^\perp$ is the kernel of $\pi_*$,

(ii) $\pi_* : D_x \to T_{\pi(x)} M'$ is a complex isometry for every $x \in M$.

This definition is given by Kobayashi for the case where $\mu$ is a null subbundle of $T_x^\perp M$ (see [10]). If $JD_x^\perp = T_x^\perp M$ for any $x \in M$, we say that $M$ is a *generic CR-submanifold* of $\tilde{M}$ (Yano and Kon [15]). For example, any real orientable hypersurface of $\tilde{M}$ is a generic CR-submanifold of $\tilde{M}$.

Concerning the basic notions on the Riemannian submersions, see O'Neill [14] and Gray [9].

The vertical distribution of a Riemannian submersion is an integrable distribution. In our case, the distribution vertical is $D^\perp$, which is integrable according to a theorem by Blair and Chen [4].

The sections of $D^\perp$ (resp., $D$) are called the *vertical vector fields* (resp., the *horizontal vector fields*) of the Riemannian submersion $\pi : M \to M'$. The letters $U$, $V$, $W$, and $W'$ will always denote vertical vector fields, and the letters $X$, $Y$, $Z$, and $Z'$ denote horizontal vector fields. For any $E \in \mathfrak{X}(M)$, $vE$ and $hE$ denote the vertical and horizontal components of $E$, respectively. A horizontal vector field $X$ on $M$ is said to be basic if $X$ is $\pi$-related to a vector field $X'$ on $M'$. 

It is easy to see that every vector field \( X' \) on \( M' \) has a unique horizontal lift \( X \) to \( M \), and \( X \) is basic.

Conversely, let \( X \) be a horizontal vector field and suppose that \( \langle X,Y \rangle_x = \langle X,Y \rangle_y \) for all \( Y \) basic vector fields on \( M \), for all \( x,y \in \pi^{-1}(x') \), and for all \( x' \in M' \). Then, the vector field \( X \) is basic. We have the following O'Neill’s lemma (see [8, 14]).

**Lemma 2.2.** Let \( X \) and \( Y \) be basic vector fields on \( M \). Then, they are satisfying the following:

(i) the horizontal component \( h[X,Y] \) of \( [X,Y] \) is a basic vector field and \( \pi_* h[X,Y] = [X',Y'] \circ \pi \),

(ii) \( h(\nabla_X Y) \) is a basic vector field corresponding to \( \nabla'_X Y' \), where \( \nabla' \) is the Levi-Civita connection on \( (M', \langle \cdot, \cdot \rangle') \),

(iii) \([X,U] \in \Gamma(D^\perp) \) for any vertical field \( U \in \Gamma(D^\perp) \).

We recall that a Riemannian submersion \( \pi : (M,g) \to (M',g') \) determines the fundamental tensor field \( T \) and \( A \) by the formulas

\[
T_E F = h \nabla_{vE} vF + v \nabla_{vE} hF, \\
A_E F = v \nabla_{hE} hF + h \nabla_{hE} vF, 
\]

for all \( E,F \in \Gamma(TM) \) (cf. O’Neill [14] and Besse [3]).

It is easy to prove that \( T \) and \( A \) satisfy

\[
T_U V = T_U U, \\
A_X Y = \frac{1}{2} v[X,Y],
\]

for any \( U,V \in \Gamma(D^\perp) \) and \( X,Y \in \Gamma(D) \).

Formula (2.4) means that the restriction of \( T \) to the integrable distribution \( D^\perp \) is the second-fundamental form of the fiber submanifolds in \( M \), and (2.5) measures the integrability of the distribution \( D \).

We have the following properties:

\[
\nabla_U X = T_U X + h \nabla_U X, \\
\nabla_X U = v \nabla_X U + A_X U, \\
\nabla_X Y = h \nabla_X Y + A_X Y,
\]

for any \( X,Y \in \Gamma(\mathcal{H}) \) and \( U \in \Gamma(\mathcal{V}) \).

**3. Kaehler structure on the basic space \( M' \).** From (2.1), we have

\[
\tilde{\nabla}_X Y = h \nabla_X Y + v \nabla_X Y + \overline{h}B(X,Y) + \overline{v}B(X,Y)
\]

for any \( X,Y \in \Gamma(D) \).
Here, we denote by \( h \) and \( v \) (resp., \( \overline{h} \) and \( \overline{v} \)) the canonical projections on \( D \) and \( D^\perp \) (resp., \( \mu \) and \( JD^\perp \)). Define a tensor field \( C \) on \( M \) as the vertical component \( v(\nabla_X Y) \) of \( \nabla_X Y \) (cf. Kobayashi [10]). The tensor field \( C \) is known to be a skew-symmetric tensor field defined by Kobayashi such that

\[
C(X,Y) = \frac{1}{2} v[X,Y]
\]

for all \( X,Y \in \Gamma(D) \).

Note that the tensor field \( C \) is the restriction of \( A \) to \( \Gamma(\mathfrak{H}) \times \Gamma(\mathfrak{H}) \).

From Definition 2.1 and Lemma 2.2, we obtain that \( Jh\nabla_X Y \) (resp., \( h\nabla_X JY \)) is a basic vector field and corresponds to \( J'\nabla'_{X'} Y' \) (resp., \( \nabla'_{X'} J'Y' \)) for any basic vector fields \( X \) and \( Y \) on \( M \).

On the Kaehler manifold \( \tilde{M} \), we have

\[
\tilde{\nabla}_E JF = J\tilde{\nabla}_E F.
\]  

From (3.1) and (3.3), we obtain the following proposition.

**Proposition 3.1.** For any basic vector fields \( X \) and \( Y \) on \( M \),

\[
\begin{align*}
Jh\nabla_X Y &= h\nabla_X JY, \\
JC(X,Y) &= \overline{\nabla}B(X,JY), \\
C(X,JY) &= J\overline{\nabla}B(X,Y), \\
J\overline{\nabla}B(X,Y) &= \overline{\nabla}B(X,JY).
\end{align*}
\]

**Theorem 3.2.** Let \( M \) be a CR-submanifold of a Kaehler manifold \( \tilde{M} \) and \( \pi : M \to M' \) be a CR-submersion of \( M \) on an almost-Hermitian manifold \( M' \). Then, \( M' \) is a Kaehler manifold.

**Proof.** From Lemma 2.2 and (3.4), we obtain that \( \nabla'_{X'} J'Y' = J'\nabla'_{X'} Y' \), so that \( M' \) is a Kaehler manifold.

**Remark 3.3.** Proposition 3.1 is proved for generic CR-submanifolds of \( \tilde{M} \) (i.e., \( \mu = 0 \)) in [10].

4. Riemannian submersions from extrinsic hyperspheres of Einstein-Kaehler manifolds. We recall that a totally umbilical submanifold \( M \) of a Riemannian manifold \( \tilde{M} \) is a submanifold whose first-fundamental form and second-fundamental form are proportional.

The extrinsic hyperspheres are defined to be totally umbilical hypersurfaces, having nonzero parallel mean-curvature vector field (cf. Nomizu and Yano [13]). Many of the basic results concerning extrinsic spheres in Riemannian and Kaehlerian geometry were obtained by Chen [5, 6, 7].
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Let $M$ be an orientable hypersurface in a Kaehler manifold $\tilde{M}$. Then, $M$ is an extrinsic hypersphere of $\tilde{M}$ if it satisfies

$$B(E,F) = \langle E, F \rangle_H$$  \hspace{1cm} (4.1)$$

for any vector fields $E$ and $F$ on $M$. Here, $H$ denote the mean-curvature vector field of $M$. If we put $k = \| H \|$ (where the norm $\| \cdot \|$ is, with respect to a scalar product, induced on every tangent space to $M$), then $k$ is a nonzero constant function on the extrinsic hypersphere $M$.

We denote by $N$ the global unit normal vector field to $M$. Then, $\xi = -JN$ is a global unit vector on $M$ such that $N = J\xi$. Let $D$ be the maximal $J$-invariant subspace (with respect to $J$) of the tangent space $T_pM$ for every $p \in M$. We see that $M$ is a CR-hypersurface of $M$ such that $TM = D \oplus D^\perp$, where $D^\perp$ is the one-dimensional anti-invariant distribution generated by the vector field $\xi$ on $M$.

The anti-invariant distribution $D^\perp$ is integrable, and its leaves are totally geodesic in $M$ (but not in $\tilde{M}$).

This is an easy consequence from Gauss and Weingarten’s formulas of the leaves of $D^\perp$ in $M$. This means that O'Neill’s tensor $T$ vanishes on the fibres of the Riemannian submersion $\pi : M \rightarrow B$.

The main result of this section is the following theorem.

**Theorem 4.1.** Let $M$ be an orientable extrinsic hypersphere of an Kaehler-Einstein manifold $\tilde{M}$. If $\pi : M \rightarrow B$ is a CR-submersion of $M$ on an almost-Hermitian manifold $B$, then $B$ is an Kaehler-Einstein manifold.

To prove **Theorem 4.1**, we need several lemmas.

**Lemma 4.2.** Following the assumptions of **Theorem 4.1**, then

$$\langle A_x\xi, A_y\xi \rangle = k^2\langle X, Y \rangle$$ \hspace{1cm} (4.2)$$

for any horizontal vector $X$ on $M$.

**Proof.** From Gauss’s formula (2.1) and the umbilicality of $M$, we get $\tilde{\nabla}_X\xi = \nabla_X\xi$ for any vector field $X$ on $M$. Then, we have

$$\langle \tilde{\nabla}_X JN, Y \rangle = \langle \nabla_X\xi, Y \rangle = \langle h\nabla_X\xi, Y \rangle = \langle A_x\xi, Y \rangle.$$ \hspace{1cm} (4.3)

On the other hand, $\tilde{M}$ is a Kaehler manifold, so that $\nabla$ commute with $J$:

$$\langle \tilde{\nabla}_X JN, Y \rangle = \langle J\tilde{\nabla}_X N, Y \rangle = -\langle \tilde{\nabla}_X N, JY \rangle = \langle B(X, JY), N \rangle$$

$$= \langle G(X, JY)H, N \rangle = k\langle X, JY \rangle.$$ \hspace{1cm} (4.4)
Consequently,
\[
\langle A_X \xi, A_Y \xi \rangle = k \langle X, J A_Y \xi \rangle = -k \langle J X, A_Y \xi \rangle = k^2 \langle X, Y \rangle. \tag{4.5}
\]

**Lemma 4.3.** Following the assumptions of Theorem 4.1, then
\[
\langle A_X Y, A_Z W \rangle = k^2 \langle X, J Y \rangle \langle Z, J W \rangle \tag{4.6}
\]
for any horizontal vector fields on \( M \).

**Proof.** We say that \( A_X Y \) is a vertical vector field, hence
\[
A_X Y = \langle A_X Y, \xi \rangle \xi. \tag{4.7}
\]
Then,
\[
\langle A_X Y, A_Z W \rangle = \langle A_X Y, \xi \rangle \langle A_Z W, \xi \rangle = k^2 \langle X, J Y \rangle \langle Z, J W \rangle. \tag{4.8}
\]

**Lemma 4.4.** Following the assumptions of Theorem 4.1, then
\[
\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + k^2 \{ \langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle \}, \tag{4.9}
\]
where \( \tilde{R} \) and \( R \) are the curvature tensor on \( \tilde{M} \) and \( M \), respectively.

**Proof.** We have the Gauss equation
\[
\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + \langle B(X, Z), B(Y, W) \rangle - \langle B(Y, Z), B(X, W) \rangle. \tag{4.10}
\]
Using the umbilicality condition, we get (4.9).

**Lemma 4.5.** For any horizontal vector fields \( X \) and \( Y \) on \( M \),
\[
\tilde{R}(\xi, X, Y, \xi) = 0, \quad \tilde{R}(\xi, J X, J Y, \xi) = 0. \tag{4.11}
\]

**Proof.** For a Riemannian submersion with totally geodesic fibres, the following formula is known:
\[
\tilde{R}(X, V, Y, U) = \langle (\nabla_X A)(Y, U), V \rangle + \langle A_X V, A_Y U \rangle. \tag{4.12}
\]
On the other hand, the first term on the right is skew-symmetric with respect to the vertical vector fields \( V \) and \( U \). From (4.12) and (4.9), we obtain (4.11).
Proof of Theorem 4.1. For the horizontal vector fields $X$, $Y$, $Z$, and $W$ on $M$, we have the following equation of O’Neill:

\[ R(X,Y,Z,W) = R'(X',Y',Z',W') - 2\langle A_X Y, A_Z W \rangle \]
\[ + \langle A_Y Z, A_X W \rangle - \langle A_X Z, A_Y W \rangle \]

(4.13)

(see [3, 14]).

By (4.9) and (4.11), we get the following formula that connects the curvature of $M'$ to the curvature of the Kähler manifold $\tilde{M}$:

\[ \tilde{R}(X,Y,Z,W) = R'(X',Y',Z',W') \]
\[ - k^2 \{ \langle X, JZ \rangle \langle Y, JW \rangle - \langle X, JW \rangle \langle Y, JZ \rangle \} \]
\[ + 2 \langle X, JY \rangle \langle Z, JW \rangle \}
\[ - k^2 \{ \langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle \}. \]

(4.14)

Let $(e_1, \ldots, e_p; Je_1, \ldots, Je_p)$ be a local $J$-frame of basic vector fields for the horizontal distribution $D$. Then, $(e_1', \ldots, e_p'; J'e_1, \ldots, J'e_p)$ is a local $J'$-frame if $\pi_{\text{star}} e_i = e'_i$ on the Kähler manifold $B$.

Using the above lemmas, from (4.14) by a straightforward calculation, we conclude that $B$ is a Kähler-Einstein manifold if $\tilde{M}$ is a Kähler-Einstein manifold.

\[ \square \]

Corollary 4.6. Let $\tilde{M}$ be a complex-form space and $M$ an orientable CR-hypersurface of $\tilde{M}$. Then, the base space of submersion $\pi : M \to B$ is also a complex-form space.

Proof. The corollary follows by straightforward calculation making use of (4.14).

\[ \square \]

Example 4.7. Let $S^{2n+1}$ be the standard hypersphere in $C^{n+1}$. Then, $S^{2n+1}$ is an extrinsic hypersphere in $C^{n+1}$, and we have the Hopf fibration $\pi : S^{2n+1} \to CP^n$ equipped with the canonical metrics.

References


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