A NEW CHARACTERIZATION OF SOME ALTERNATING AND SYMMETRIC GROUPS

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We suppose that \( p = 2^\alpha 3^\beta + 1 \), where \( \alpha \geq 1 \), \( \beta \geq 0 \), and \( p \geq 7 \) is a prime number. Then we prove that the simple groups \( A_n \), where \( n = p, p + 1, \) or \( p + 2 \), and finite groups \( S_n \), where \( n = p, p + 1, \) are also uniquely determined by their order components. As corollaries of these results, the validity of a conjecture of J. G. Thompson and a conjecture of Shi and Bi (1990) both on \( A_n \), where \( n = p, p + 1, \) or \( p + 2 \), is obtained. Also we generalize these conjectures for the groups \( S_n \), where \( n = p, p + 1. \)

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1. Introduction. Let \( G \) be a finite group. We denote by \( \pi(G) \) the set of all prime divisors of \( |G| \). We construct the prime graph of \( G \) as follows. The prime graph \( \Gamma(G) \) of a group \( G \) is the graph whose vertex set is \( \pi(G) \), and two distinct primes \( p \) and \( q \) are joined by an edge (we write \( p \sim q \)) if and only if \( G \) contains an element of order \( pq \). Let \( t(G) \) be the number of connected components of \( \Gamma(G) \) and let \( \pi_1, \pi_2, \ldots, \pi_{t(G)} \) be the connected components of \( \Gamma(G) \). If \( 2 \in \pi(G) \), then we always suppose that \( 2 \in \pi_1 \).

Now \( |G| \) can be expressed as a product of coprime positive integers \( m_i \), \( i = 1, 2, \ldots, t(G) \), where \( \pi(m_i) = \pi_i \). These integers are called the order components of \( G \). The set of order components of \( G \) will be denoted by \( \text{OC}(G) \). Also we call \( m_2, \ldots, m_{t(G)} \) the odd-order components of \( G \). The order components of non-abelian simple groups having at least three prime graph components are obtained by Chen [7, Tables 1, 2, 3]. Similarly, the order components of non-abelian simple groups with two-order components can be obtained by using the tables in [18, 28].

The following groups are uniquely determined by their order components: Suzuki-Ree groups [6], Sporadic simple groups [4], \( \text{PSL}_2(q) \) [7], \( E_6(q) \) [2], \( G_2(q) \), where \( q \equiv 0 \pmod{3} \) [3], \( F_4(q) \), where \( q \) is even [15], \( \text{PSL}_3(q) \), where \( q \) is an odd prime power [14], \( \text{PSL}_3(q) \), where \( q = 2^n \) [13], \( \text{PSU}_3(q) \), where \( q > 5 \) [16], and \( A_p \), where \( p \) and \( p - 2 \) are primes [12].

It was proved by Oyama [20] that a finite group which has the same table of characters as an alternating group \( A_n \) is isomorphic to \( A_n \). It was also proved by Koike [17] that a finite group which has the isomorphic subgroup-lattice as an alternating group \( A_n \) is isomorphic to \( A_n \).
Let $\pi_e(G)$ denote the set of orders of elements in $G$. Shi and Bi [27] proved that if $\pi_e(G) = \pi_e(A_n)$ and $|G| = |A_n|$, then $G \cong A_n$. Iranmanesh and Alavi [12] proved that if $p$ and $p-2$ are primes and $OC(G) = OC(A_p)$, then $G \cong A_p$. Praeger and Shi [21] and Shi and Bi [26] proved that $A_8, A_9, A_{11}, A_{13}, S_7,$ and $S_8$ are characterizable by their element orders. Also recently, Kondrat’ev and Mazurov [19] and Zavarnitsin [29] proved that if $\pi_e(G) = \pi_e(A_n)$, where $n = s, s+1, s+2$ and $s$ is a prime number, then $G \cong A_n$.

Now we prove the following theorems.

**Theorem 1.1.** Let $p = 2^{a}3^{b} + 1$, where $\alpha \geq 1$, $\beta \geq 0$, and $p \geq 7$ is a prime number. Let $M = A_n$, where $n = p, p + 1, p + 2$. Then $OC(G) = OC(M)$ if and only if $G \cong M$.

**Theorem 1.2.** Let $p = 2^{a}3^{b} + 1$, where $\alpha \geq 1$, $\beta \geq 0$, and $p \geq 7$ is a prime number. Let $M = S_n$, where $n = p, p + 1$. Then $OC(G) = OC(M)$ if and only if $G \cong M$.

In this paper, all groups are finite and by simple groups we mean non-abelian simple groups. All further unexplained notations are standard and we refer, for example, to [10]. Also frequently we use the results of Williams [28] and Kondrat’ev [18] about the prime graph of simple groups.

2. Preliminary results

**Remark 2.1.** Let $N$ be a normal subgroup of $G$ and $p \sim q$ in $\Gamma(G/N)$. Then $p \sim q$ in $\Gamma(G)$. In fact if $xN \in G/N$ has order $pq$, then there is a power of $x$ which has order $pq$.

**Definition 2.2** (see [11]). A finite group $G$ is called a 2-Frobenius group if it has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where $K$ and $G/H$ are Frobenius groups with kernels $H$ and $K/H$, respectively.

**Lemma 2.3** (see [28, Theorem A]). If $G$ is a finite group with its prime graph having more than one component, then $G$ is one of the following groups:
(a) a Frobenius or 2-Frobenius group;
(b) a simple group;
(c) an extension of a $\pi_1$-group by a simple group;
(d) an extension of a simple group by a $\pi_1$-solvable group;
(e) an extension of a $\pi_1$-group by a simple group by a $\pi_1$-group.

**Lemma 2.4** (see [28, Lemma 3]). If $G$ is a finite group with more than one prime graph component and has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $H$ and $G/K$ are $\pi_1$-groups and $K/H$ is a simple group, then $H$ is a nilpotent group.

The next lemma follows from [1, Theorem 2].

**Lemma 2.5.** Let $G$ be a Frobenius group of even order and let $H$, $K$ be Frobenius complement and Frobenius kernel of $G$, respectively. Then $t(\Gamma(G)) = 2$, where $t(\Gamma(G))$ is the number of connected components of $\Gamma(G)$. 


and the prime graph components of $G$ are $\pi(H)$, $\pi(K)$ and $G$ has one of the following structures:
(a) $2 \in \pi(K)$ and all Sylow subgroups of $H$ are cyclic;
(b) $2 \in \pi(H)$, $K$ is an abelian group, $H$ is a solvable group, the Sylow subgroups of odd order of $H$ are cyclic groups, and the 2-Sylow subgroups of $H$ are cyclic or generalized quaternion groups;
(c) $2 \in \pi(H)$, $K$ is an abelian group, and there exists $H_0 \leq H$ such that $|H : H_0| \leq 2$, $H_0 = Z \times \text{SL}(2,5)$, $(|Z|, 2, 3, 5) = 1$, and the Sylow subgroups of $Z$ are cyclic.

The next lemma follows from [1, Theorem 2] and Lemma 2.4.

**Lemma 2.6.** Let $G$ be a 2-Frobenius group of even order. Then $t(\Gamma(G)) = 2$ and $G$ has a normal series $1 \leq H \leq K \leq G$ such that
(a) $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi(K/H) = \pi_2$;
(b) $G/K$ and $K/H$ are cyclic, $|G/K|$ divides $|\text{Aut}(K/H)|$, $(|G/K|, |K/H|) = 1$, and $|G/K| < |K/H|$;
(c) $H$ is nilpotent and $G$ is a solvable group.

**Lemma 2.7** (see [8, Lemma 8]). Let $G$ be a finite group with $t(\Gamma(G)) \geq 2$ and let $N$ be a normal subgroup of $G$. If $N$ is a $\pi_i$-group for some prime graph component of $G$ and $m_1, m_2, \ldots, m_r$ are some order components of $G$ but not a $\pi_i$-number, then $m_1 m_2 \cdots m_r$ is a divisor of $|N| - 1$.

The next lemma follows from [5, Lemma 1.4].

**Lemma 2.8.** Suppose that $G$ and $M$ are two finite groups satisfying $t(\Gamma(M)) \geq 2$, $N(G) = N(M)$, where $N(G) = \{n \mid G$ has a conjugacy class of size $n\}$, and $Z(G) = 1$. Then $|G| = |M|$.

**Lemma 2.9** (see [5, Lemma 1.5]). Let $G_1$ and $G_2$ be finite groups satisfying $|G_1| = |G_2|$ and $N(G_1) = N(G_2)$. Then $t(\Gamma(G_1)) = t(\Gamma(G_2))$ and $\text{OC}(G_1) = \text{OC}(G_2)$.

**Lemma 2.10.** Let $G$ be a finite group and let $M$ be a non-abelian finite group with $t(M) = 2$ satisfying $\text{OC}(G) = \text{OC}(M)$.

1. Let $|M| = m_1 m_2$, $\text{OC}(M) = \{m_1, m_2\}$, and $\pi(m_i) = \pi_i$ for $i = 1, 2$. Then $|G| = m_1 m_2$ and one of the following holds:

   (a) $G$ is a Frobenius or 2-Frobenius group;

   (b) $G$ has a normal series $1 \leq H \leq K \leq G$ such that $G/K$ is a $\pi_1$-group, $K$ is a nilpotent $\pi_1$-group, and $K/H$ is a non-abelian simple group. Moreover, $\text{OC}(K/H) = \{m_1', m_2', \ldots, m_r', m_2\}$, $|K/H| = m_1' m_2' \cdots m_r' m_2$, and $m_1' m_2' \cdots m'_r | m_1$, where $\pi(m'_j) = \pi_j$, $1 \leq j \leq s$.

2. In case (b), $|G/K| | |\text{Out}(K/H)|$.

**Proof.** The proof of (1) follows from the above lemmas. Since $t(G) \geq 2$, we have $t(G/H) \geq 2$. Otherwise $t(G/H) = 1$, so that $t(G) = 1$. Since $H$ is a $\pi_i$-group,
we arrive at a contradiction. Moreover, we have $Z(G/H) = 1$. For any $xH \in G/H$ and $xH \notin K/H$, $xH$ induces an automorphism of $K/H$ and this automorphism

<table>
<thead>
<tr>
<th>$p$</th>
<th>Finite simple $C_{pp}$ groups</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$A_5, A_6;$ $L_2(q)$, where $q$ is a Fermat prime, a Mersenne prime, or $q = 2^n$, $n \geq 3$, $L_3(2^2)$, $Sz(2^{2n+1})$, $n \geq 1$</td>
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<td>3</td>
<td>$A_5, A_6;$ $L_2(q)$, where $q = 2^3$, $3^{n+1}$, or $2.3^n \pm 1$ which is a prime, $n \geq 1$, $L_3(2^2)$</td>
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<td>5</td>
<td>$A_5, A_6, A_7; M_{11}, M_{22};$ $L_2(q)$, where $q = 7^2$, $5^n$, or $2.5^n \pm 1$ which is a prime, $n \geq 1$, $L_3(2^2)$, $S_4(q)$, $q = 3, 7$, $U_4(3)$; $Sz(q)$, $q = 2^3, 2^5$</td>
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<td>7</td>
<td>$A_7, A_8, A_9; M_{22}, J_1, J_2, HS;$ $L_2(q)$, $q = 2^3$, $7^n$, or $2.7^n - 1$ which is a prime, $n \geq 1$, $L_3(3)$, $L_4(3)$, $O_7(3)$, $S_4(3)$, $S_6(3)$, $O_9^+(3)$, $G_2(q)$, $q = 3^3, 3$; $F_4(2)$, $U_3(q)$, $q = 2^2$, $23$, $Sz(2^3)$, $3D_4(2)$, $2E_6(2)$, $2F_4(2)'$</td>
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<td>13</td>
<td>$A_{13}, A_{14}, A_{15}$; $Suz, Fi_{22};$ $L_2(q)$, $q = 3^3$, $5^2$, $13^n$, or $2.13^n - 1$ which is a prime, $n \geq 1$, $L_3(3)$, $L_4(3)$, $O_7(3)$, $S_4(3)$, $S_6(3)$, $O_9^+(3)$, $G_2(q)$, $q = 3^3, 3$; $F_4(2)$, $U_3(q)$, $q = 2^2$, $23$, $Sz(2^3)$, $3D_4(2)$, $2E_6(2)$, $2F_4(2)'$</td>
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<td>17</td>
<td>$A_{17}, A_{18}, A_{19}$; $J_3, He, Fi_{23}, Fi_{24}';$ $L_2(q)$, $q = 2^4$, $17^n$, $2.17^n \pm 1$ which is a prime, $n \geq 1$, $S_4(4)$, $S_6(2)$, $F_4(2)$, $O_8^+(2)$, $O_{10}^+(2)$, $2E_6(2)$</td>
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<td>19</td>
<td>$A_{19}, A_{20}, A_{21};$ $J_1, J_3, O'N, Th, HN; L_2(q)$, $q = 19^n$, $2.19^n - 1$ which is a prime, $n \geq 1$, $L_3(7)$, $U_3(2^3)$, $R(3^3)$, $2E_6(2)$</td>
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<td>37</td>
<td>$A_{37}, A_{38}, A_{39}$; $J_4, Ly'$; $L_2(q)$, $q = 37^n$, $2.37^n - 1$ which is a prime, $n \geq 1$, $U_3(11)$, $R(3^3)$, $2F_4(2^3)$</td>
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<td>73</td>
<td>$A_{73}, A_{74}, A_{75}$; $L_2(q)$, $q = 73^n$, $2.73^n - 1$ which is a prime, $n \geq 1$, $L_3(2^3)$, $S_6(2^3)$, $G_2(q)$, $q = 2^3$, $3^2$; $F_4(3)$, $E_6(2)$, $E_7(2)$, $U_3(3^2)$, $3D_4(3)$</td>
</tr>
<tr>
<td>109</td>
<td>$A_{109}, A_{110}, A_{111}$; $L_2(q)$, $q = 109^n$, $2.109^n - 1$ which is a prime, $n \geq 1$, $2F_4(2^3)$</td>
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<td>$p = 2^m + 1$, $m = 2^s$</td>
<td>$A_p, A_{p+1}, A_{p+2}$; $L_2(q)$, $q = 2^m p^k$, $2 \cdot p^k \pm 1$ which is a prime, $s \geq k \geq 1$, $S_6(2^b)$, $a = 2^c + 1$ and $b = 2^d$, $c \geq 1$, $c + d = s$, $F_4(2^e)$, $e \geq 1$, $4e = 2^f$, $O_2(m+1)(2^e)$, $s \geq 2$, $O_8^+(2^b)$, $c \geq 2$, $c + d = s$</td>
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<tr>
<td>Other</td>
<td>$A_p, A_{p+1}, A_{p+2}; L_2(q)$, $q = p^k$, $2 \cdot p^k - 1$ which is a prime, $k \geq 1$</td>
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is trivial if and only if \( xH \in Z(G/H) \). Therefore, \( G/K \leq \text{Out}(K/H) \) and since \( Z(G/H) = 1 \), (2) follows.

**Definition 2.11.** A group \( G \) is called a \( C_{pp} \) group if the centralizers of its elements of order \( p \) in \( G \) are \( p \)-groups.

**Lemma 2.12** (see [9]). Let \( p \) be a prime and \( p = 2^\alpha 3^\beta + 1 \), \( \alpha \geq 0 \) and \( \beta \geq 0 \). Then any finite simple \( C_{pp} \) group is given by Table 2.1.

3. Characterization of some alternating and symmetric groups. In the sequel, we suppose that \( p = 2^\alpha 3^\beta + 1 \), where \( \alpha \geq 1 \), \( \beta \geq 0 \), and \( p \geq 7 \) is a prime number.

**Lemma 3.1.** Let \( G \) be a finite group and let \( M \) be \( A_n \), where \( n = p, p + 1 \), or \( p + 2 \), or \( S_n \), where \( n = p, p + 1 \). If \( OC(G) = OC(M) \), then \( G \) is neither a Frobenius group nor a 2-Frobenius group.

**Proof.** If \( G \) is a Frobenius group, then by Lemma 2.5, \( OC(G) = \{ |H|, |K| \} \), where \( K \) and \( H \) are Frobenius kernel and Frobenius complement of \( G \), respectively. Since \( |H| \mid |K| - 1 \), we have \( |H| < |K| \). Therefore, \( 2 \nmid |H| \), and hence \( 2 \mid |K| \). So, \( |H| = p, |K| = |G|/p \). We claim that there exists a prime \( p' \) such that \( 3n/4 < p' \). Note that \( p \leq n \), and hence \( p'^2 \nmid |A_n| \). Let \( \beta(n) \) be the number of prime numbers less than or equal to \( n \). In fact, by [22, Theorem 2] we have

\[
\frac{n}{\log n - 1/2} < \beta(n) < \frac{n}{\log n - 3/2},
\]

(3.1)

where \( n \geq 67 \). Thus

\[
\beta(n) - \beta\left(\frac{3n}{4}\right) > \frac{n}{\log n - 1/2} - \frac{3n/4}{\log(3n/4) - 3/2}.
\]

(3.2)

When \( n \geq 405 \), we get \( \beta(n) - \beta(3n/4) > 1 \), and for \( n < 405 \), we can immediately obtain the result by checking the table of prime numbers. Now let \( P' \) be the \( p' \)-Sylow subgroup of \( K \). Since \( K \) is nilpotent, \( P' \lneq G \). Then \( p \mid p' - 1 \), by Lemma 2.7, which is a contradiction since \( p' < p \). Therefore, \( G \) is not a Frobenius group.

Now let \( G \) be a 2-Frobenius group. By Lemma 2.6, there is a normal series \( 1 \leq H \leq K \leq G \) such that \( |K/H| = p \) and \( |G/K| < p \). So, \( |H| \neq 1 \) since \( |G| = |G/K| \cdot |K/H| \cdot |H| \). Since \( 2 \mid |H| \), let \( p' \) be as above and let \( P' \) be the \( p' \)-Sylow subgroup of \( H \). Now, \( p \mid p' - 1 \), which is impossible. Hence, \( G \) is not a 2-Frobenius group.

**Lemma 3.2.** Let \( G \) be a finite group and \( M = A_n \), where \( n = p, p + 1 \), or \( p + 2 \), or \( S_n \), where \( n = p, p + 1 \). If \( OC(G) = OC(M) \), then \( G \) has a normal series \( 1 \leq H \leq K \leq G \) such that \( H \) and \( G/K \) are \( \pi_1 \)-groups and \( K/H \) is a simple group. Moreover, the odd-order component of \( M \) is equal to an odd-order component of \( K/H \). In particular, \( t(\Gamma(K/H)) \geq 2 \). Also \( |G/H| \) divides \( |\text{Aut}(K/H)| \), and in fact \( G/H \leq \text{Aut}(K/H) \).
PROOF. The first part of the lemma follows from the above lemmas since the prime graph of $M$ has two prime graph components. For primes $p$ and $q$, if $K/H$ has an element of order $pq$, then $G$ has one. Hence, by the definition of prime graph component, the odd-order component of $G$ must be an odd-order component of $K/H$. Since $K/H \trianglelefteq G/H$ and $C_{G/H}(K/H) = 1$, we have

$$G/H = \frac{N_{G/H}(K/H)}{C_{G/H}(K/H)} \cong T, \quad T \leq \text{Aut}(K/H). \quad (3.3)$$

\square

THEOREM 3.3. Let $p = 2^\alpha 3^\beta + 1$, where $\alpha \geq 1$, $\beta \geq 0$, and $p \geq 7$ is a prime number. Let $M = A_n$, where $n = p, p + 1, p + 2$. Then $OC(G) = OC(M)$ if and only if $G \cong M$.

PROOF. By Lemma 3.2, $G$ has a normal series $1 \leq H \leq K \leq G$ such that $\pi(H) \cup \pi(G/K) \subset \pi_1$, $K/H$ is a non-abelian simple group, $t(\Gamma(K/H)) \geq 2$, and the odd-order component of $M$ is an odd-order component of $K/H$. Therefore, $K/H$ is a finite simple $C_{pp}$ group. Now using Table 2.1, we consider each possibility of $K/H$ separately.

In the sequel, we frequently use the results of [28, Table I] and [18, Tables 2, 3].

STEP 1. Let $p = 7, 13, 17, 19, 37, 73$, or 109.

Since the proofs of these cases are similar, we state only one of them, say $p = 13$. Using Table 2.1, we have

1. $K/H \cong \text{Sz}$ or $F_{22}$. It is a contradiction since $3^7 | |\text{Sz}|$ and $3^9 | |F_{22}|$ but $3^7 \nmid |A_n|$, where $n = 13, 14, 15$;
2. $K/H \cong L_2(27), L_2(25), L_3(3), L_4(3), Sz(8), 2F_4(2)'$, or $U_3(4)$. If $K/H \cong L_2(27)$, then $|G|/|K/H| = |H| \cdot |G/K| \neq 1$. By Lemma 2.6, $|G/K| \mid |\text{Out}(K/H)| = 6$. So, $|H| \neq 1$. Let $P$ be the 5-Sylow subgroup of $H$. But since $H$ is nilpotent, $P < G$. Hence, $13 \nmid (|P| - 1)$, which is a contradiction. Other cases are similar;
3. $K/H \cong L_2(13r)$ or $L_2(2.13r - 1)$, where $2.13r - 1$ is a prime, $r \geq 1$. Note that $13^2 \nmid |G|$, hence $r = 1$. So, $K/H \cong L_2(13)$ or $L_2(25)$, and we can proceed similar to (2);
4. $K/H \cong O_7(3)$. It is a contradiction since $3^9 \mid |O_7(3)|$ but $3^9 \nmid |A_n|$;
5. $K/H \cong S_4(5)$ or $S_6(3)$. It is a contradiction since $5^4 \mid |S_4(5)|$ but $5^4 \nmid |A_n|$. Also $3^9 \mid |S_6(3)|$ but $3^9 \nmid |A_n|$;
6. $K/H \cong O_8^+(3)$. It is a contradiction since $3^{12} \mid |O_8^+(3)|$ but $3^{12} \nmid |A_n|$;
7. $K/H \cong G_2(3)$ or $G_2(8)$. If $K/H \cong G_2(3)$, then we get a contradiction since for $n = 13, 14$ we have $3^6 \mid |G_2(3)|$ but $3^6 \nmid |A_n|$. For $n = 15$, since $|\text{Out}(G_2(3))| = 2$, we have $|H| \neq 1$. Now we proceed similar to (2). If $K/H \cong G_2(8)$, then we get a contradiction since $2^{18} \mid |G_2(8)|$ but $2^{18} \nmid |A_n|$;
8. $K/H \cong F_4(2)$. It is a contradiction since $17 \mid |F_4(2)|$ but $17 \nmid |A_n|$;
9. $K/H \cong U_3(23)$. It is a contradiction since $23 \mid |U_3(23)|$ but $23 \nmid |A_n|$;
(10) $K/H \cong 3D_4(2)$ or $2E_6(2)$. It is a contradiction since $2^{12} \nmid |A_n|$. Also $19 \nmid |A_n|$;

(11) $K/H$ is an alternating group, namely $A_{13}, A_{14},$ or $A_{15}$.

First suppose that $n = 13$. Since $|K/H| \leq |A_{13}|$, $K/H \cong A_{13}$. But $|G| = |A_{13}|$, and hence $H = 1$ and $K = G \cong A_{13}$. If $n = 14$, then $K/H \cong A_{13}$ or $A_{14}$. But if $r \neq 6$, then $\text{Aut}(A_r) = S_r$, and hence $|\text{Out}(A_r)| = 2$. If $K/H \cong A_{13}$, then $|G/K| \mid 2$, and hence $|H| \neq 1$. Now we get a contradiction similar to (2). Therefore, $K/H \cong A_{14}$, and hence $G \cong A_{14}$. If $n = 15$, we do similarly.

**STEP 2.** Let $p = 2^m + 1$, where $m = 2^s$.

Using Table 2.1, we have

(i) $K/H \cong L_2(2^m)$. Note that for every $m$ we have $|L_2(2^m)| \mid |G|$. Using Lemma 2.6, $|G/K| \mid |\text{Out}(K/H)|$. Also $|\text{Out}(L_2(2^m))| = m$. Hence, $|H| \neq 1$. Now let $p'$ be a prime number less than $p$ such that

$$p'||\frac{|A_n|}{m|K/H|}.$$ (3.4)

Let $P'$ be the $p'$-Sylow subgroup of $H$. Since $H$ is nilpotent, $P' \triangleleft G$. Hence, $p \mid (|P'|-1)$, which is a contradiction;

(ii) $K/H \cong L_2(2^k)$ or $L_2(2p^k \pm 1)$, where $2p^k \pm 1$ is a prime and $1 \leq k \leq s$.

We know that $p \mid |A_n|$, hence $k = 1$. Now we proceed similar to (i);

(iii) $K/H \cong S_a(2^b)$, where $a = 2^{c+1}$ and $b = 2^d$, $c \geq 1, c + d = s$. Let $q = 2^b$ and $f = 2^c$, Then $p = q^f + 1$ and we have

$$|S_a(2^b)| = q^{f^2}(q^f - 1)(q^f + 1)\prod_{i=1}^{f-1}(q^i - 1)(q^i + 1).$$ (3.5)

Each factor of the form $(q^i \pm 1)$ is less than or equal to $p$ and therefore divides $|A_n|$. Also $q^{f^2} = (2^m)^f \leq 2^{m^2} \leq 2^m$. Hence, $|S_a(2^b)| \mid |A_n|$. But $|\text{Out}(S_a(2^b))| = b$. Then $|H| \neq 1$ and we can proceed similar to (i);

(iv) $K/H \cong F_4(2^e)$, where $e \geq 1, 4e = 2^s$, or $O_2(2^{m+1})(2)$, where $s \geq 2$, or $O_4(2^b)$, where $c \geq 2, c + d = s$. Again this part is similar to (iii);

(v) $K/H \cong A_{p_1}, A_{p_1}, A_{p_2}$.

First suppose that $n = p$. Since $|K/H| \leq |A_p|$, $K/H \cong A_p$. But $|G| = |A_p|$, and hence $H = 1$ and $K = G \cong A_p$. If $n = p + 1$, then $K/H \cong A_p$ or $A_{p+1}$. But if $r \neq 6$, then $\text{Aut}(A_r) = S_r$, and hence $|\text{Out}(A_r)| = 2$. If $K/H \cong A_p$, then $|G/K| \mid 2$, and hence $|H| \neq 1$. Now we get a contradiction similar to (i). Therefore, $K/H \cong A_{p+1}$, and hence $G \cong A_{p+1}$. If $n = p + 2$, we do similarly.

**STEP 3.** For other primes $p$, we have $K/H \cong A_p, A_{p+1}, A_{p+2}; L_2(q)$, where $q = p^k, 2p^k - 1$ which is a prime, $k \geq 1$.

In fact the proof of this step is exactly similar to that of **Step 2** and we omit it for convenience. \qed
Theorem 3.4. If \( G \) is a non-abelian finite group with connected prime graph, then \( G \) is not characterizable with its order component.

Proof. Clearly, \( OC(G) = OC(Z_{|G|}) \), but \( G \notin Z_{|G|} \). \( \square \)

Corollary 3.5. Every simple group with one component (see [28, Table I]) is not characterizable with this method.

Theorem 3.6. Let \( n \) be a positive integer. If there exist at least two non-isomorphic abelian groups of order \( n \), then abelian groups of order \( n \) are not characterizable with their order component.

Proof. The proof is obvious. \( \square \)

Remark 3.7. It was a conjecture that every finite simple group \( M \), where \( \Gamma(M) \) is not connected, is characterizable with its order components. But the following example is a counterexample.

Example 3.8. If \( q \) is an odd-prime power and \( n = 2^k \geq 4 \), then \( OC(S_{2n}(q)) = OC(O_{2n+1}(q)) \), but obviously \( S_{2n}(q) \notin O_{2n+1}(q) \).

Theorem 3.9. Let \( p = 2^\alpha 3^\beta + 1 \), where \( \alpha \geq 1 \), \( \beta \geq 0 \), and \( p \geq 7 \) is a prime number. Let \( M = S_n \), where \( n = p, p + 1 \). Then \( OC(G) = OC(M) \) if and only if \( G \cong M \).

Proof. Similar to the proof of Theorem 3.3, since \( G \) is a \( C_{pp} \) group, we have \( K/H \cong A_n \). Now using Lemma 3.2, we have

\[
A_n \leq \frac{G}{H} \leq \text{Aut}(A_n) = S_n. \tag{3.6}
\]

Therefore, \( G/H \cong A_n \) or \( \text{Aut}(A_n) = S_n \). If \( G/H \cong A_n \), then \( |H| = 2 \) and \( H < G \). Hence, \( H \leq Z(G) = 1 \), which is a contradiction. Therefore, \( G/H \cong S_n \), and since \( |G| = |S_n| \), we have \( G \cong S_n \). \( \square \)

4. Some related results

Remark 4.1. It is a well known conjecture of J. G. Thompson that if \( G \) is a finite group with \( Z(G) = 1 \) and \( M \) is a non-abelian simple group satisfying \( N(G) = N(M) \), then \( G \cong M \).

We can generalize this conjecture for the groups under discussion by our characterization of these groups.

Corollary 4.2. Let \( G \) be a finite group with \( Z(G) = 1 \) and let \( M \) be \( A_p \), \( A_{p+1}, A_{p+2}, S_p, \) or \( S_{p+1} \). If \( N(G) = N(M) \), then \( G \cong M \).

Proof. By Lemmas 2.8 and 2.9, if \( G \) and \( M \) are two finite groups satisfying the conditions of Corollary 4.2, then \( OC(G) = OC(M) \). So, Theorems 3.3 and 3.9 imply this corollary. \( \square \)
**Remark 4.3.** Shi and Bi in [26] put forward the following conjecture.

**Shi’s conjecture.** Let $G$ be a group and $M$ a finite simple group. Then $G \cong M$ if and only if

(i) $|G| = |M|$,  
(ii) $\pi_e(G) = \pi_e(M)$, where $\pi_e(G)$ denotes the set of orders of elements in $G$.

This conjecture is valid for sporadic simple groups [24], groups of alternating type [27], and some simple groups of Lie type [23, 25, 26]. As a consequence of Theorems 3.3 and 3.9, we prove a generalization of this conjecture for the groups under discussion.

**Corollary 4.4.** Let $G$ be a finite group and let $M$ be $A_p$, $A_{p+1}$, $A_{p+2}$, $S_p$, or $S_{p+1}$. If $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$, then $G \cong M$.

**Proof.** By assumption, we must have $OC(G) = OC(M)$. Thus the corollary follows by Theorems 3.3 and 3.9. □

**References**


[3] ———, *A new characterization of $G_2(q)$, $[q \equiv 0(\text{mod} 3)]*, J. Southwest China Normal Univ. (1996), 47-51.


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Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

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<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>February 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>May 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>August 1, 2009</td>
</tr>
</tbody>
</table>

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