CS-MODULES AND ANNIHILATOR CONDITIONS

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Received 4 June 2002

We study $S$-$R$-bimodules $SM_R$ with the annihilator condition $S = l_S(A) + l_S(B)$ for any closed submodule $A$, and a complement $B$ of $A$, in $M_R$. Such annihilator condition has a direct connection with the CS-condition for $M_R$. We make use of this to give a new characterization of CS-modules. Bimodules $SM_R$ for which $r_M l_S(A) = A$ (for every closed submodule $A$ of $M_R$) are also dealt with. Such modules are called $W^*$-modules. We give the extra added annihilator conditions to $W^*$-modules to be equivalent to the continuous (quasicontinuous) modules.

2000 Mathematics Subject Classification: 16D80.

1. Introduction. Let $R$ and $S$ be rings and let $SM_R$ be a bimodule. For any $X \leq M$ and $T \leq S$, write $l_S(X) = \{ s \in S : sX = 0 \}$ and $r_M(T) = \{ m \in M : Tm = 0 \}$. Let $\lambda : S \rightarrow \text{End}(M_R)$ be the canonical ring homomorphism. For each $s \in S$, we identify $\lambda(s)$ with $s$. A submodule $A$ is essential in $M$ (denoted by $A \leq^e M$) if $A \cap B \neq 0$ for every nonzero submodule $B$ of $M$. A submodule $A$ is closed in $M$ if it has no proper essential extensions in $M$. $A \leq^e M$ signifies that $A$ is a direct summand of $M$ (or simply a summand). A module $M$ is called a CS-module if every closed submodule of $M$ is a summand. The module $M$ is continuous if it is a CS-module and satisfies condition $(C_2)$: if $A \cong B \leq M$ with $A \leq^e M$, then $B \leq^e M$. A generalization of condition $(C_2)$ is $(GC_2)$ (see [4]): if $A$ is a submodule of $M$ with $A \cong M$, then $A \leq^e M$. The module $M$ is quasicontinuous if it is a CS-module and satisfies condition $(C_3)$: if $A, B \leq^e M$ with $A \cap B = 0$, then $A \oplus B \leq^q M$. It is known that $M$ is quasicontinuous if and only if $M = A \oplus B$ whenever $A$ and $B$ are complements of each other in $M$ (see [3, Theorem 2.8]).

Camillo et al. [1] have dealt with Ikeda-Nakayama rings that are related to continuous and quasicontinuous rings.

For a bimodule $SM_R$, Wisbauer et al. [4] have studied the annihilator condition $l_S(A \cap B) = l_S(A) + l_S(B)$ for any submodules $A$ and $B$ of $M_R$, and the condition $S = l_S(A) + l_S(B)$ for any submodules $A$ and $B$ of $M_R$ with $A \cap B = 0$. Consequently, they obtained new characterizations of quasicontinuous modules. We adapt their ideas here to study a variation of the above annihilator condition which is connected to CS-modules, and obtain a new characterization of CS-modules in Section 2.
In Section 3, we study the bimodules \( S M_R \) which satisfy the following condition:

\[ S = l_S(A) + l_S(B) \quad (1.1) \]

for any two relative complements \( A \) and \( B \) in \( M_R \). Such modules are clearly quasicontinuous modules, while there are quasicontinuous modules which do not satisfy condition (1.1). For example, consider \( R \) as a commutative integral domain with field of quotients \( Q \) and let \( M = Q \oplus Q \). In Lemma 3.2, we give a necessary and sufficient condition for quasicontinuous modules to satisfy condition (1.1). In the case of \( S = \text{End}(M_R) \), every quasicontinuous module must have condition (1.1). As a generalization of this condition, we introduce the concept of \( W^* \)-modules (bimodules \( S M_R \) for which \( A = r_M l_S(A) \) for every closed submodule \( A \) of \( M_R \)). It is clear that any bimodule with condition (1.1) is a \( W^* \)-module, while in general the converse is not true. Proposition 3.8 indicates when a \( W^* \)-module satisfies condition (1.1).

In Section 4, we discuss the equivalence between \( W^* \)-modules and continuous (quasicontinuous) modules over an arbitrary ring \( S \). Then we draw the consequences when \( S \) is the endomorphism ring of \( M_R \).

### 2. CS-modules and annihilator conditions

The proofs of the lemmas and propositions, presented in this section, are adaptations of the arguments in [4].

**Lemma 2.1.** Let \( S M_R \) be a bimodule. If for every closed submodule \( A \) of \( M_R \) there exists a complement \( B \) of \( A \) in \( M_R \) such that \( S = l_S(A) + l_S(B) \), then \( M_R \) is a CS-module.

**Proof.** Let \( A \) be a closed submodule of \( M_R \). Then by assumption there exists a complement \( B \) of \( A \) in \( M_R \) such that \( S = l_S(A) + l_S(B) \). Write \( l_S = u + v \), where \( u \in l_S(A) \) and \( v \in l_S(B) \). It follows that \( a = va \) for all \( a \in A \), \( b = ub \) for all \( b \in B \), and \( vB = uA = 0 \). Thus \( B \subseteq r_M(v) \subseteq r_M(v^2) \) and \( r_M(v^2) \cap A = 0 \). Since \( B \) is a complement of \( A \) in \( M_R \), we have \( B = r_M(v) = r_M(v^2) \). Similarly, \( A = r_M(u) = r_M(u^2) \). Now we show that \( (vu)M = 0 \). Let \( vum = a + b \), where \( m \in M \), \( a \in A \), and \( b \in B \). Noting that \( vu = uv \), we have that \( (v^2u^2)m = (vu)(a + b) = 0 \). Hence \( u^2m \in r_M(v^2) = r_M(v) \), and this gives that \( u^2vm = vu^2m = 0 \). Then \( vm \in r_M(u^2) = r_M(u) \); and thus \( vum = uv = 0 \). So \( (vu)M \cap (A + B) = 0 \). Since \( A + B \) is essential in \( M_R \), \( (vu)M = 0 \). So \( uM \subseteq r_M(v) = B \) and \( vM \subseteq r_M(u) = A \) and hence \( M = vM + uM = A + B = A \oplus B \). Therefore \( A \) is a summand of \( M_R \).

**Remark 2.2.** The converse of Lemma 2.1 is not true. For example, there are torsion-free CS-modules over commutative integral domains, which do not satisfy the given condition in Lemma 2.1.

The next lemma follows from [4, Lemma 3].
Lemma 2.3. Let $SM_R$ be a bimodule, where $SM$ is faithful, and let $M_R = A \oplus B$. If the projection $f$ of $M$ onto $A$ along $B$ is given by $f(m) = sm$ for some $s \in S$, and all $m \in M$, then $S = l_S(A) + l_S(B)$.

For any submodules $A$ and $B$ of $M_R$ and any $t \in S$, define $\alpha_t : A + B \to M$, $a + b \to ta$ (see [4]).

Proposition 2.4. Let $SM_R$ be a bimodule such that $SM$ is faithful. The following are equivalent:

1. $M_R$ is CS and for any $f^2 = f \in \text{End}(M_R)$, there exists $s \in S$ such that $f(m) = sm$, for all $m \in M_R$;
2. for every closed submodule $A$ of $M_R$, there exists a complement $B$ of $A$ in $M_R$ such that $S = l_S(A) + l_S(B)$;
3. for every closed submodule $A$ of $M_R$, there exists a complement $B$ of $A$ in $M_R$ such that $S = l_S(A) \oplus l_S(B)$;
4. for every closed submodule $A$ of $M_R$, there exists a complement $B$ of $A$ in $M_R$ such that for every $t \in S$, the diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & A + B \\
& \downarrow{\alpha_t} & \longrightarrow \\
& M \\
\end{array}
$$

(2.1)

can be extended by $\lambda(s)$, for some $s \in S$.

Proof. (1)⇒(2). Let $A$ be a closed submodule of $M_R$. Since $M_R$ is a CS-module, there exists $f^2 = f \in \text{End}(M_R)$ such that $A = fM$. By (1), there exists $s \in S$ such that $f(m) = sm$, for all $m \in M_R$. Hence $(s^2 - s)M = (f^2 - f)M = 0$. Since $SM$ is faithful, it follows that $s$ is an idempotent in $S$. Now we have

$$l_S(A) = l_S(fM) = l_S(sm) = l_S(s) = S(1 - S).$$

(2.2)

Similarly, $l_S(B) = S_S$, where $B = (1 - f)M$. Thus $S = l_S(A) + l_S(B)$.

(2)⇒(1). It is clear by Lemma 2.1 that $M_R$ is CS. Now let $f^2 = f \in \text{End}(M_R)$, and denote $A = f(M)$. By (2), there exists a complement $B$ of $A$ in $M_R$ such that $S = l_S(A) + l_S(B)$. The argument of the proof of Lemma 2.1 shows that $M = A \oplus B$. Let $\pi$ be the projection of $M$ onto $A$ along $B$. Then

$$l_S(A) = l_S(\pi M) = \{ s \in S : s \pi = 0 \}$$

(2.3)

(by considering $s$ the homomorphism given by left multiplication by $s$) and

$$l_S(B) = l_S((1 - \pi)M) = \{ s \in S : s(1 - \pi) = 0 \}.$$

(2.4)
Let $1 = s' + s$, where $s' \in I_S(A)$ and $s \in I_S(B)$. Thus $s'\pi = 0$ and $s(1-\pi) = 0$. It follows that $0 = s(1-\pi) = (1-s')(1-\pi) = 1-\pi-s'$. Therefore $f(m) = \pi(m) = sm$ for all $m \in M$.

(2)$\Rightarrow$(3). From the argument in the proof of Lemma 2.1, we have $M = A \oplus B$. Since $sM$ is faithful, we have $0 = I_S(M) = I_S(A + B) = I_S(A) \oplus I_S(B)$ and hence $S = I_S(A) \oplus I_S(B)$.

(3)$\Rightarrow$(4). Let $A$ be a closed submodule of $M_R$. By (3), there exists a complement $B$ of $A$ such that $S = I_S(A) \oplus I_S(B)$. Write $t = u + v$, where $u \in I_S(A)$ and $v \in I_S(B)$. Then $\alpha_t(a+b) = ta = (u + v)a = v(a+b) = \lambda(v)(a+b)$. This follows that $1-s)a + (-s)b = 0$, for all $a \in A$ and $b \in B$. So $1-s \in I_S(A)$ and $-s \in I_S(B)$ and hence $1 = (1-s) - (-s) \in I_S(A) + I_S(B)$. Therefore $S = I_S(A) + I_S(B)$.

**COROLLARY 2.5.** The following are equivalent for a bimodule $S M_R$ with $S = \text{End}(M_R)$:

1. $M_R$ is a CS-module;
2. for every closed submodule $A$ of $M_R$, there exists a complement $B$ of $A$ in $M_R$ such that $S = I_S(A) + I_S(B)$;
3. for every closed submodule $A$ of $M_R$, there exists a complement $B$ of $A$ in $M_R$ such that $S = I_S(A) \oplus I_S(B)$;
4. for every closed submodule $A$ of $M_R$, there exists a complement $B$ of $A$ in $M_R$ such that for every $t \in S$, diagram (2.1) can be extended by some $g : M \to M$.

**PROPOSITION 2.6.** Let $S$ be the center of $\text{End}(M_R)$. The following are equivalent:

1. for every closed submodule $A$ of $M_R$, there exists a complement $B$ of $A$ in $M_R$ such that $S = I_S(A) + I_S(B)$;
2. $M_R$ is CS and every idempotent of $\text{End}(M_R)$ is central;
3. $M_R$ is CS and every closed submodule of $M_R$ is fully invariant.

**PROOF.** (1)$\Leftrightarrow$(2) by Proposition 2.4.

(2)$\Rightarrow$(3). Let $A$ be a closed submodule of $M$. By CS, $A$ is a direct summand of $M_R$. Then $A = f(M)$ for some $f^2 = f \in \text{End}(M_R)$. For any $g \in \text{End}_R(M)$, since $f$ is central by (2), $g(A) = g(f(M)) = f(g(M)) \subseteq f(M) = A$. This shows that $A$ is a fully invariant submodule of $M$.

(3)$\Rightarrow$(2). Let $f, g \in \text{End}_R(M)$ with $f^2 = f$. Therefore $f(M)$ is a closed submodule of $M_R$. By (3), $g(f(M)) \subseteq f(M)$ and $g((1-f)(M)) \subseteq (1-f)(M)$. It follows that $fgf = gf$ and $(1-f)g(1-f) = g(1-f)$. Thus, $g - gf = g(1-f) = (1-f)g(1-f) = g - gf - fg + fgf = g - gf - fg + gf = g - fg$. This shows that $fg = gf$.

$\square$
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3. Condition (1.1) and its generalizations. The next lemma is clear.

**Lemma 3.1.** The following are equivalent for a bimodule \( SM_R \):

1. \( S = l_S(A) + l_S(B) \) for any two relative complements \( A \) and \( B \) of \( M_R \);
2. for any submodules \( A \) and \( B \) of \( M_R \) with \( A \cap B = 0 \), \( S = l_S(A) + l_S(B) \).

We say that a bimodule \( SM_R \) has condition (1.1) if it satisfies one of the equivalent conditions of **Lemma 3.1**.

The next lemma follows from [4, Lemma 3].

**Lemma 3.2.** Let \( SM_R \) be a bimodule such that \( S_M \) is faithful. Then the following are equivalent:

1. \( M \) has condition (1.1);
2. \( M \) is quasicontinuous and every idempotent in \( \text{End}(M_R) \) is a left multiplication by an element of \( S \).

**Remark 3.3** [4, Theorem 8]. In the case of \( S = \text{End}(M_R) \), it is clear from **Lemma 3.2** that an \( R \)-module \( M \) is quasicontinuous if and only if \( M \) has condition (1.1).

**Proposition 3.4.** Let \( SM_R \) be a bimodule which satisfies condition (1.1). Then \( A = r_M l_S(A) \) for all closed submodules \( A \) of \( M_R \).

**Proof.** Let \( A \) be a closed submodule of \( M_R \) and \( B \) a submodule of \( r_M l_S(A) \) such that \( A \cap B = 0 \). By Zorn’s lemma, there exists a complement \( C \) of \( A \) in \( M_R \) with \( B \subseteq C \). By condition (1.1), we have \( S = l_S(A) + l_S(C) \subseteq l_S(A) + l_S(B) \), so \( S = l_S(A) + l_S(B) \). Since \( l_S(A) = l_S r_M l_S(A) \leq l_S(B) \), it follows that \( S = l_S(B) \) and hence \( B = 0 \). This shows that \( A \leq^e r_M l_S(A) \). Since \( A \) is a closed submodule of \( M_R \), we have \( A = r_M l_S(A) \). \( \Box \)

A bimodule \( SM_R \) is called a \( W^* \)-module if \( A = r_M l_S(A) \) for every closed submodule \( A \) of \( M_R \). It is clear by **Proposition 3.4** that every bimodule \( SM_R \) with condition (1.1) is a \( W^* \)-module. But there are bimodules which are \( W^* \)-modules and do not satisfy condition (1.1). For example, let \( S = R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix} \), where \( F \) is any field and let \( M = r_R \). It is clear that \( M \) is \( W^* \)-module. But \( M_R \) is not quasicontinuous, and hence \( M \) does not satisfy condition (1.1).

**Lemma 3.5.** The following are equivalent for a bimodule \( SM_R \):

1. \( A \leq^e r_M l_S(A) \) for all submodules \( A \) of \( M_R \);
2. \( SM_R \) is a \( W^* \)-module.

**Proof.** (1) \( \Rightarrow \) (2). This implication is obvious.

(2) \( \Rightarrow \) (1). Let \( A \) be a submodule of \( M_R \) and \( C \) a maximal essential extension of \( A \) in \( M_R \). We have by (2) that \( A \leq^e C = r_M l_S(C) \). Since \( r_M l_S(A) \leq r_M l_S(C) \), we have \( A \leq^e r_M l_S(A) \). \( \Box \)

**Proposition 3.6.** If \( SM_R \) is a \( W^* \)-module, then \( r_M(T) = 0 \), or \( r_M(T) \) is uniform for every maximal left ideal \( T \) of \( S \).
Proof. Let $T$ be a maximal left ideal of $S$. Since $T \subseteq l_S r_M(T)$, we have either $l_S r_M(T) = T$ or $l_S r_M(T) = S$. If $l_S r_M(T) = S$, then $r_M(T) = 0$. If $l_S r_M(T) = T$, let $N$ be a nonzero submodule of $r_M(T)$. Then $T = l_S r_M(T) \subseteq l_S(N) \subseteq S$, and the maximality of $T$ yields $T = l_S(N)$. It follows that $r_M(T) = r_M l_S(N)$. Since $M$ is $W^*$-module, we have by Lemma 3.5 that $N \leq^e r_M(T)$. Therefore $r_M(T)$ is uniform.

Corollary 3.7. Let $sM_R$ be a $W^*$-module, where every maximal left ideal of $S$ is a left annihilator. Then $r_M(T)$ is uniform for every maximal left ideal $T$ of $S$.

Proof. Let $T$ be a maximal left ideal of $S$. From Proposition 3.6, it is enough to show that $r_M(T) \neq 0$. Let $r_M(T) = 0$. By assumption, $T = l_S r_M(T) = l_S(0) = S$, which contradicts the maximality of $T$.

Proposition 3.8. The following are equivalent for a bimodule $sM_R$:

1. $sM_R$ is a $W^*$-module and $l_S(A) + l_S(B)$ is a left annihilator for any two relative complements $A$ and $B$ in $M_R$;
2. $sM_R$ has condition (1.1).

Proof. (1)$\Rightarrow$(2). Let $A$ and $B$ be two relative complements in $M_R$. Then by (1), $S = l_S(0) = l_S(A \cap B) = l_S(r_M l_S(A) \cap r_M l_S(B)) = l_S r_M(l_S(A) + l_S(B)) = l_S(A) + l_S(B)$. Therefore $M$ has condition (1.1).

(2)$\Rightarrow$(1). This implication is obvious.

4. The relation between $W^*$-modules and (quasi-) continuous modules.

The following is an immediate consequence of Proposition 3.8.

Proposition 4.1. Let $sM_R$ be a bimodule with $S = \text{End}(M_R)$. Then the following are equivalent:

1. $sM_R$ is a $W^*$-module and $l_S(A) + l_S(B)$ is a left annihilator for any two relative complements $A$ and $B$ of $M_R$;
2. $M_R$ is quasicontinuous.

Proposition 4.2. Let $sM_R$ be a bimodule, where $sM$ is faithful. Then the following are equivalent:

1. $sM_R$ is a $W^*$-module, $l_S(A) + l_S(B)$ is an annihilator for any two relative complements $A$ and $B$ of $M_R$, and $M_R$ has $G_2$;
2. $M_R$ is a continuous module and every idempotent in $\text{End}(M_R)$ is a left multiplication by an element of $S$.

Proof. (1)$\Rightarrow$(2). We have by Proposition 3.8 that $M_R$ has condition (1.1). Therefore, by Lemma 3.2, $M_R$ is a quasicontinuous module. Let $s \in \text{End}(M_R)$ be a monomorphism, with $sM \leq^e M$. By $G_2$ it follows that $sM = M$. Then by [3, Lemma 3.14], $M_R$ is a continuous module. The rest of the proof of (2) follows from Lemma 3.2.

(2)$\Rightarrow$(1). This implication is obvious.
Corollary 4.3. Let $S_MR$ be a bimodule with $S = \text{End}(M_R)$. Then the following are equivalent:

1. $S_MR$ is a $W^{*}$-module, $l_{S}(A) + l_{S}(B)$ is an annihilator for any two relative complements $A$ and $B$ of $M_R$, and $M_R$ has $GC_2$;
2. $M_R$ is a continuous module.

In particular, if $M_R$ is of finite uniform dimension, then $S$ is semiperfect.

Proof. It is clear that every monomorphism $f \in \text{End}(M_R)$ is an isomorphism (due to $GC_2$ and $M$ of finite uniform dimension). Hence, $M$ satisfies the assumptions in Camps and Dicks [2, Theorem 5], and so $\text{End}(M_R)$ is semilocal. Therefore by using [3, Proposition 3.5 and Lemma 3.7], idempotents of $S/J(S)$ lift to idempotents of $S$, and thus $S$ is semiperfect.

Lemma 4.4. Let $S_MR$ be a bimodule such that every finitely generated left ideal of $S$ is a left annihilator of a subset of $M_R$, and every closed submodule of $M_R$ is a right annihilator of a finite subset of $S$. Then $M$ has condition (1.1).

Proof. Let $A_1$ and $A_2$ be complements of each other in $M_R$. Then by assumption, we have $A_i = r_M(Y_i)$ for some finite subsets $Y_i$ of $S$. Again by assumption, $SY_i = l_{S}(K_i)$ for some subsets $K_i$ in $M_R$, where $i = 1, 2$. Now $S = l_{S}(A_1 \cap A_2) = l_{S}(r_M(Y_1) \cap r_M(Y_2)) = l_{S}r_M(SY_1 + SY_2) = SY_1 + SY_2$ (due to the assumption and since $SY_1 + SY_2$ is finitely generated). Hence $S = l_{S}(K_1) + l_{S}(K_2) = l_{S}r_Ml_{S}(K_1) + l_{S}r_Ml_{S}(K_2) = l_{S}r_M(Y_1) + l_{S}r_M(Y_2) = l_{S}(A_1) + l_{S}(A_2)$. Therefore $M$ satisfies condition (1.1).

Lemma 4.5. Let $S_MR$ be a bimodule and let every idempotent in $\text{End}(M_R)$ be a left multiplication by an element of $S$. If $M_R$ is a CS-module, then every closed submodule of $M_R$ is a right annihilator of a finite subset of $S$.

Proof. Let $A$ be a closed submodule of $M_R$. Then by CS, there exists $f^2 = f \in \text{End}(M_R)$ such that $A = r_M(1 - f) = \{m \in M : (1 - s)m = 0\} = r_M(1 - s)$, where $(1 - s) \in S$.

The following corollary is an immediate consequence of Lemmas 4.4 and 4.5.

Corollary 4.6. Let $S_MR$ be a bimodule, where $S = \text{End}(M_R)$. Let every finitely generated left ideal of $S$ be a left annihilator of a subset of $M$. Then the following are equivalent:

1. every closed submodule of $M$ is a right annihilator of a finite subset of $M$;
2. $M$ is a CS-module.

Theorem 4.7. Let $S_MR$ be a bimodule, where $S = \text{End}(M_R)$. Let every finitely generated left ideal of $S$ be a left annihilator of a subset of $M$. Then the following are equivalent:

1. $M$ is a CS-module;
2. $M$ is continuous.
Proof. By Lemmas 4.4 and 4.5, we have that $M$ has condition (1.1). By Remark 3.3, $M$ is quasicontinuous. To show that $M$ is continuous, by [3, Lemma 3.14], it is enough to show that every essential monomorphism $s \in S$ is an isomorphism. Let $s \in S$ be a monomorphism, with $sM \leq^e M$. By assumption, $S_S = l_S(X)$ for some subset $X$ of $M$. It follows that $X = 0$ and hence $S_S = s$. Then $s$ is a split monomorphism, and therefore $sM = M$. \qed

References


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