SOME INTERESTING SERIES ARISING FROM THE POWER SERIES EXPANSION OF $(\sin^{-1} x)^q$

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Starting from the power series expansions of $(\sin^{-1} x)^q$, for $1 \leq q \leq 4$, formulae are obtained for the sum of several infinite series. Some of these evaluations involve $\zeta(3)$.

1. Introduction

In [10], Choe deduced the formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad (1.1)$$

from the power series expansion of $\sin^{-1}(x)$ (see also [1, 16]). By applying a generalization of the procedure used by Choe to the power series expansions of $(\sin^{-1} x)^q$ for $1 \leq q \leq 4$, we obtain explicit formulae for the sum of several infinite series, see (2.1), (2.2), (2.3), (2.4), (2.5), and (2.6). For other applications based on the procedure used by Choe, see [11, 12, 17].

2. Main results

Let $m$ denote an integer. For $m \geq 0$, we have the following theorems.

Theorem 2.1.

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)(2k+2m+1)\binom{2k+2m}{k+m}} = 2^{-4m} \left( \sum_{r=1}^{m} \frac{\binom{2m}{m-r}}{r^2} + \frac{2m}{m} \frac{\pi^2}{8} \right). \quad (2.1)$$

Theorem 2.2.

$$\sum_{k=1}^{\infty} \frac{\binom{2k+2m}{k+m}}{k^2 \binom{2k}{k}} = \sum_{r=1}^{m} \frac{2\binom{2m}{m-r}}{r^2} + \frac{2m}{m} \frac{\pi^2}{6}. \quad (2.2)$$
Theorem 2.3.

\[
\sum_{k=1}^{\infty} \frac{(2k)}{(2k+1)(2k+2m+1)} \sum_{j=1}^{k} \frac{1}{(2j-1)^2} = 2^{-4m-1} \left( - \sum_{r=1}^{m} \frac{(2m-r)}{2r^4} + \pi^2 \sum_{r=1}^{m} \frac{(2m-r)}{8r^2} + \left( \frac{2m}{m} \right) \frac{\pi^4}{192} \right). \quad (2.3)
\]

Theorem 2.4.

\[
\sum_{k=1}^{\infty} \frac{(2k+2m+2)}{(k+1)(2k+1)} \sum_{j=1}^{k} \frac{1}{j^2} = -4 \sum_{r=1}^{m} \frac{(2m-r)}{r^4} + \frac{2\pi^2}{3} \sum_{r=1}^{m} \frac{(2m-r)}{r^2} + \left( \frac{2m}{m} \right) \frac{\pi^4}{60}. \quad (2.4)
\]

In addition, we have the following theorems.

Theorem 2.5.

\[
\sum_{k=1}^{\infty} \frac{1}{k(2k+1)} \sum_{j=1}^{k} \frac{1}{(2j-1)^2} = \frac{\pi^2}{4} \log 2 - \frac{7}{8} \zeta(3),
\]

\[
\sum_{k=1}^{\infty} \frac{1}{(k+1)(2k+1)} \sum_{j=1}^{k} \frac{1}{j^2} = \frac{\pi^2}{3} \log 2 - \frac{3}{2} \zeta(3). \quad (2.5)
\]

Theorem 2.6.

\[
\sum_{k=1}^{\infty} \frac{k}{(k+1)(2k+1)(2k-1)} \sum_{j=1}^{k} \frac{1}{j^2} = -\frac{\pi^2}{36} + \frac{2}{3} \log 2 + \frac{\pi^2}{9} \log 2 - \frac{1}{2} \zeta(3). \quad (2.6)
\]

In (2.5) and (2.6), \( \zeta \) represents the Riemann zeta function.

The following result in [14] \((m \geq 0)\) should be compared with (2.1):

\[
\sum_{k=0}^{\infty} \frac{(2k)}{(2k+2m+1)(2k+4m+1)} = \frac{\pi^2}{2^{8m+3}} \left( \frac{2m}{m} \right)^2. \quad (2.7)
\]

Also, the series appearing above in (2.3), (2.4), (2.5), and (2.6) bear some resemblance to Euler sums (see, e.g., [3, 4, 5, 9]). A very broad generalization which generalizes both Euler sums and polylogarithms is studied in [6]. For other interesting evaluations of series involving binomial coefficients, see, for example, [7, 8, 15, 18].
3. Proofs of Theorems 2.1, 2.2, 2.3, and 2.4

The power series expansions of \( (\sin^{-1} x)^q \) for \( 1 \leq q \leq 4 \) (valid for \( |x| \leq 1 \)) are given by (see [10], [2, pages 262-263])

\[
\sin^{-1} x = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{x^{2k+1}}{2k+1}, \quad \text{(3.1)}
\]

\[
(\sin^{-1} x)^2 = \sum_{k=1}^{\infty} \frac{2^{2k-1}}{k^2} x^{2k},
\]

\[
(\sin^{-1} x)^3 = 6 \sum_{k=1}^{\infty} \frac{(2k)}{2^{2k}} \left( \sum_{j=1}^{k} \frac{1}{(2j-1)^2} \right) x^{2k+1},
\]

\[
(\sin^{-1} x)^4 = 3 \sum_{k=1}^{\infty} \frac{2k}{(2k)} \left( \sum_{j=1}^{k} \frac{1}{j^2} \right) x^{2k+2} \frac{1}{(k+1)(2k+1)}.
\]

Multiplying each of (3.1) by \( x^{2m} \), where \( m \) is an integer, putting \( x = \sin \theta \) and integrating with respect to \( \theta \) from \( \theta = 0 \) to \( \theta = \pi/2 \), and using the well-known results (valid for nonnegative integers \( p \))

\[
\int_0^{\pi/2} \sin^{2p+1} \theta d\theta = \frac{2^p}{(2p+1) \binom{2p}{p}},
\]

\[
\int_0^{\pi/2} \sin^2 \theta d\theta = \frac{\pi}{2},
\]

we obtain

\[
\int_0^{\pi/2} \theta \sin^{2m} \theta d\theta = 2^{2m} \sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{(2k+1)(2k+2m+1) \binom{2k+2m}{k+m}}, \quad m \geq 0, \quad \text{(3.3)}
\]

\[
\int_0^{\pi/2} \theta^2 \sin^{2m} \theta d\theta = \frac{\pi}{2^{2m+2}} \sum_{k=1}^{\infty} \frac{\binom{2k+2m}{k+m}}{k^2 \binom{2k}{k}}, \quad m \geq -1, \quad \text{(3.4)}
\]

\[
\int_0^{\pi/2} \theta^3 \sin^{2m} \theta d\theta = 3(2^{2m+1}) \sum_{k=1}^{\infty} \frac{\binom{2k}{k}}{(2k+1)(2k+2m+1) \binom{2k+2m}{k+m}} \sum_{j=1}^{k} \frac{1}{(2j-1)^2}, \quad m \geq -1, \quad \text{(3.5)}
\]

\[
\int_0^{\pi/2} \theta^4 \sin^{2m} \theta d\theta = \frac{3\pi}{2^{2m+3}} \sum_{k=1}^{\infty} \frac{\binom{2k+2m+2}{k+m+1}}{(k+1)(2k+1) \binom{2k}{k}} \sum_{j=1}^{k} \frac{1}{j^2}, \quad m \geq -2. \quad \text{(3.6)}
\]
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For \(m \geq 0\), we evaluate the integrals on the left of (3.3), (3.4), (3.5), and (3.6) using the following formula valid for a nonnegative integer \(m\) (see [13, page 31]):

\[
\sin^{2m} \theta = 2^{-2m} \sum_{j=0}^{m-1} (-1)^{m+j} \binom{2m}{j} \cos (2(m-j)\theta) + \binom{2m}{m},
\]  

and the following easily checked formulae (valid for positive integers \(l\)):

\[
\int_0^{\pi/2} \theta \cos(2l\theta) d\theta = \frac{(-1)^l - 1}{4l^2},
\]

\[
\int_0^{\pi/2} \theta^2 \cos(2l\theta) d\theta = \frac{(-1)^l \pi}{4l^2},
\]

\[
\int_0^{\pi/2} \theta^3 \cos(2l\theta) d\theta = \frac{(-1)^l \pi^2}{16l^2} + \frac{1 - (-1)^l}{8l^4},
\]

\[
\int_0^{\pi/2} \theta^4 \cos(2l\theta) d\theta = (-1)^l \pi \left( \frac{\pi^2}{8l^2} - \frac{3}{4} \right).
\]

After some simplification, we obtain (2.1), (2.2), (2.3), and (2.4).

4. Special cases of Theorems 2.1, 2.2, 2.3, and 2.4

We record the special cases corresponding to \(0 \leq m \leq 2\).

Putting \(m = 0, 1, 2\) in (2.1), we get

\[
\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8},
\]

\[
\sum_{k=0}^{\infty} \frac{k+1}{(2k+1)^2(2k+3)} = \frac{1}{8} + \frac{\pi^2}{32},
\]

\[
\sum_{k=0}^{\infty} \frac{\binom{2k}{k} (2k+4)}{(2k+1)(2k+5)\binom{2k+4}{k+2}} = \frac{1}{64} + \frac{3\pi^2}{1024}.
\]

Putting \(m = 0, 1, 2\) in (2.2), we get

\[
\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6},
\]

\[
\sum_{k=1}^{\infty} \frac{(2k+2)}{k^2} \frac{\binom{2k}{k}}{k+1} = 2 + \frac{\pi^2}{3},
\]

\[
\sum_{k=1}^{\infty} \frac{(2k+4)}{k^2} \frac{\binom{2k}{k}}{2k+2} = \frac{17}{2} + \pi^2.
\]

The first results of (4.1) and (4.2) are of course well-known classical results.
Putting $m = 0, 1, 2$ in (2.3), we get

\[ \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} \sum_{j=1}^{k} \frac{1}{(2j-1)^2} = \frac{\pi^4}{384}, \]

\[ \sum_{k=1}^{\infty} \frac{\binom{2k}{k}}{(2k+1)(2k+3)} \sum_{j=1}^{k} \frac{1}{(2j-1)^2} = -\frac{1}{64} + \frac{\pi^2}{256} + \frac{\pi^4}{3072}, \quad (4.3) \]

\[ \sum_{k=1}^{\infty} \frac{\binom{2k}{k}}{(2k+1)(2k+5)} \sum_{j=1}^{k} \frac{1}{(2j-1)^2} = -\frac{1}{256} + \frac{17\pi^2}{16384} + \frac{\pi^4}{16384}. \]

Putting $m = 0, 1, 2$ in (2.4) gives

\[ \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} \sum_{j=1}^{k} \frac{1}{j^2} = \frac{\pi^4}{120}, \]

\[ \sum_{k=1}^{\infty} \frac{\binom{2k+4}{k+2}}{(k+1)(2k+1)} \sum_{j=1}^{k} \frac{1}{j^2} = -4 + \frac{2\pi^2}{3} + \frac{\pi^4}{30}, \quad (4.4) \]

\[ \sum_{k=1}^{\infty} \frac{\binom{2k+6}{k+3}}{(k+1)(2k+1)} \sum_{j=1}^{k} \frac{1}{j^2} = -\frac{65}{4} + \frac{17\pi^2}{6} + \frac{\pi^4}{10}. \]

We note that the first series evaluated in (4.4) is an Euler sum and the result is classical and was known to Euler (see, e.g., [5]).

**5. Proof of Theorem 2.5**

We consider the case $m = -1$ of (3.5), (3.6) (the case $m = -1$ of (3.4) gives a trivial result). We need the following result valid for a positive integer $n$ and $|x| < 2\pi$ (see [2, page 260]):

\[ \int_{0}^{x} \frac{u^n}{2} \cot \left( \frac{u}{2} \right) du = \cos \left( \frac{n\pi}{2} \right) n! \zeta(n+1) - \sum_{j=0}^{n} (-1)^{(j+1)/2} \frac{\Gamma(n+1)}{\Gamma(n+1-j)} x^{n-j} \text{Cl}_{j+1}(x), \]

where

\[ \text{Cl}_{2n}(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^{2n}}, \]

\[ \text{Cl}_{2n+1}(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^{2n+1}}, \quad (5.2) \]
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and \(\Gamma\) and \(\zeta\) represent the Gamma function and the Riemann zeta function respectively. We note that

\[
\begin{align*}
\text{Cl}_{2n}(\pi) &= 0, \\
\text{Cl}_{2n+1}(\pi) &= \left(\frac{1}{2^{2n}} - 1\right)\zeta(2n+1), \quad n \geq 1, \\
\text{Cl}_1(\pi) &= -\log 2.
\end{align*}
\] (5.3)

Putting \(x = \pi\) in (5.1), we obtain

\[
2^n \int_0^{\pi/2} \theta^n \cot \theta \, d\theta = n! \cos \left(\frac{n\pi}{2}\right)\zeta(n+1) - \sum_{j=0}^{n} (-1)^{j+1/2} \frac{\Gamma(n+1)}{\Gamma(n+1-j)} \pi^{n-j} \text{Cl}_{j+1}(\pi).
\] (5.4)

Using

\[
\int_0^{\pi/2} \theta^n \cot \theta \, d\theta = \frac{1}{n+1} \int_0^{\pi/2} \theta^{n+1} \csc^2 \theta \, d\theta, \quad n \geq 1,
\] (5.5)

in (5.4), we get

\[
\frac{2^n}{n+1} \int_0^{\pi/2} \theta^{n+1} \csc^2 \theta \, d\theta
\]

\[
= n! \cos \left(\frac{n\pi}{2}\right)\zeta(n+1) - \sum_{j=0}^{n} (-1)^{j+1/2} \frac{\Gamma(n+1)}{\Gamma(n+1-j)} \pi^{n-j} \text{Cl}_{j+1}(\pi).
\] (5.6)

From (5.6) and (5.3) we obtain

\[
\int_0^{\pi/2} \theta^2 \csc^2 \theta \, d\theta = \pi \log 2,
\] (5.7)

\[
\int_0^{\pi/2} \theta^3 \csc^2 \theta \, d\theta = \frac{3}{4} \pi^2 \log 2 - \frac{21}{8} \zeta(3),
\] (5.8)

\[
\int_0^{\pi/2} \theta^4 \csc^2 \theta \, d\theta = \frac{\pi^3}{2} \log 2 - \frac{9}{4} \pi \zeta(3).
\] (5.9)

Putting \(m = -1\) in (3.5) and (3.6) and using (5.8) and (5.9) give (2.5).

6. Proof of Theorem 2.6

We consider the case \(m = -2\) of (3.6). We need to evaluate \(\int_0^{\pi/2} \theta^4 \csc^4 \theta \, d\theta\). We have

\[
\int_0^{\pi/2} \theta^4 \csc^4 \theta \, d\theta = \theta^4 \csc^2 \theta (-\cot \theta) \bigg|_0^{\pi/2} + \int_0^{\pi/2} \cot \theta \frac{d}{d\theta} (\theta^4 \csc^2 \theta) \, d\theta
\]

\[
= 4 \int_0^{\pi/2} \theta^3 \cot \theta \csc^2 \theta \cot \theta \, d\theta - 2 \int_0^{\pi/2} \theta^4 \csc^2 \theta \cot^2 \theta \, d\theta.
\] (6.1)
Using \( \cot^2 \theta = \csc^2 \theta - 1 \) in the second integral on the right gives

\[
\int_0^{\pi/2} \theta^4 \csc^4 \theta \, d\theta = \frac{4}{3} \int_0^{\pi/2} \theta^3 \cot \theta \csc^2 \theta \, d\theta + \frac{2}{3} \int_0^{\pi/2} \theta^4 \csc^2 \theta \, d\theta.
\]  
(6.2)

Also,

\[
\int_0^{\pi/2} \theta^3 \cot \theta \csc^2 \theta \, d\theta = \theta^3 \csc \theta (- \csc \theta) \bigg|_0^{\pi/2} + \int_0^{\pi/2} \csc \theta \frac{d}{d\theta} (\theta^3 \csc \theta) \, d\theta
\]

\[
= -\frac{\pi^3}{8} + 3 \int_0^{\pi/2} \theta^2 \csc^2 \theta \, d\theta - \int_0^{\pi/2} \theta^3 \cot \theta \csc^2 \theta \, d\theta,
\]  
(6.3)

so that

\[
\int_0^{\pi/2} \theta^3 \cot \theta \csc^2 \theta \, d\theta = -\frac{\pi^3}{16} + \frac{3}{2} \int_0^{\pi/2} \theta^2 \csc^2 \theta \, d\theta.
\]  
(6.4)

From (6.2), (6.4), (5.7), and (5.9), we obtain

\[
\int_0^{\pi/2} \theta^4 \csc^4 \theta \, d\theta = -\frac{\pi^3}{12} + 2\pi \log 2 + \frac{\pi^3}{3} \log 2 - \frac{3}{2} \pi \zeta(3).
\]  
(6.5)

Putting \( m = -2 \) in (3.6) and using (6.5), we obtain (2.6).

7. Final remarks

In a future paper, we plan to investigate what happens when we multiply (3.1) by \( x^{2m+1} \) and carry out the same steps as we did here.

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References

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The study of dynamic equations on a time scale goes back to its founder Stefan Hilger (1988), and is a new area of still fairly theoretical exploration in mathematics. Motivating the subject is the notion that dynamic equations on time scales can build bridges between continuous and discrete mathematics; moreover, it often reveals the reasons for the discrepancies between two theories.

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