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Research Article
On the Existence of Positive Solutions of a Nonlinear Differential Equation

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We study some existence results for the nonlinear equation $(1/A)(Au')' = u\psi(x,u)$ for $x \in (0,\omega)$ with different boundary conditions, where $\omega \in (0,\infty]$, A is a continuous function on $[0,\omega)$, positive and differentiable on $(0,\omega)$, and ψ is a nonnegative function on $(0,\omega)\times[0,\infty)$ such that $t \mapsto t\psi(x,t)$ is continuous on $[0,\infty)$ for each $x \in (0,\omega)$. We give asymptotic behavior for positive solutions using a potential theory approach.

1. Introduction

In this paper, we study the following nonlinear equation:

$$\frac{1}{A}(Au')' = f(x,u), \quad \text{in } (0,\omega), \tag{1.1}$$

where $\omega \in (0,\infty]$ and A is a continuous function on $[0,\omega)$, which is positive and differentiable on $(0,\omega)$.

Several results have been obtained for (1.1) with different boundary conditions (see [1–7] and references therein).

In [5], Mâagli and Zeddini generalize the result of Taliaferro [7] who took $A(t) = 1$. Indeed, they studied (1.1) with the following boundary conditions $u(0) = u(1) = 0$ and a nonlinear term $f(x,u) = -\varphi(x,u)$, where φ is a nonnegative continuous function on $(0,1)\times(0,\infty)$, nonincreasing with respect to the second variable and the function A satisfies $\int_0^1 (1/A(t))dt < \infty$.

Usually $A(t) = t^{n-1}$, $n \geq 2$, so the integral $\int_0^1 (1/A(t))dt$ diverges. The condition $\int_0^1 (1/A(t))dt < \infty$ seems to be too restrictive from an application view point.

Our aim in this paper is to study (1.1) with a nonlinear term $f(x,u) = u\psi(x,u)$ and two boundary conditions. More precisely, we assume that $x \mapsto 1/A(x)$ is integrable in the neighborhood of ω and the integral $\int_0^\omega (1/A(t))dt$ may diverges and we search a positive continuous solution u of (1.1).

Our paper is organized as follows. In Section 2, we give some properties of the Green's function $G(x,y)$ of the operator $u \mapsto -(1/A)(Au')'$ with $Au'(0) = 0$ and $u(\omega) = 0$, which will be used later. We recall (see [4]) that for x, y in $[0,\omega)$, we have

$$G(x,y) = A(y)\int_{x\vee y}^{\omega} \frac{1}{A(t)}dt. \tag{1.2}$$

We refer in this paper to Vf, the potential of a measurable nonnegative function f defined on $(0,\omega)$ by

$$Vf(x) = \int_0^\omega G(x,y)f(y)dy. \tag{1.3}$$

Note that Vf is a lower semicontinuous function on $(0,\omega)$. Moreover, for two nonnegative measurable functions f and g with $f \le g$ and Vg is continuous, we have Vf is continuous.

In Section 3, we are interested to the following problem:

$$\begin{gathered} \frac{1}{A}(Au')' = u\psi(x,u), \quad \text{a.e in } (0,\omega), \\ u > 0, \\ \lim_{x\to 0}\frac{u(x)}{\rho(x)} = c > 0, \\ u(\omega) := \lim_{x\to\omega} u(x) = 0, \end{gathered} \tag{P_1}$$

where $\rho(x) = \int_x^\omega (1/A(t))dt$.

We assume that ρ and ψ satisfy the following conditions.

(H_0) The function $t \mapsto t\psi(x,t)$ is continuous on $[0,\infty)$ for each $x \in (0,\omega)$.

(H_1) The integral $\int_0^\omega (1/A(t))dt$ diverges.

(H_2) For each $a > 0$, there exists $q_a = q \in K$ such that for $0 \le s \le t \le a$ and $x \in (0,\omega)$, we have

$$t\psi(x,t\rho(x)) - s\psi(x,s\rho(x)) \le q(x)(t-s), \tag{1.4}$$

where K is the set of nonnegative Borel measurable functions q on $(0,\omega)$ satisfying $\int_0^\omega G(0,y)q(y)dy < \infty$.

Under these hypotheses, we prove the following result.

Theorem 1.1. *Assume (H_0)–(H_2), then the problem (P_1) has a positive solution $u \in C^1((0,\omega))$ satisfying*

$$c_1\rho(x) \le u(x) \le c\rho(x), \tag{1.5}$$

where c_1 is a positive constant.

If we replace hypothesis (H_2) by the following condition:

(H_3) for each $a > 0$, there exists $q_a = q \in K$ such that for $0 \le s \le t \le a$ and $x \in (0,\omega)$, we have

$$t\psi(x,t) - s\psi(x,s) \le q(x)(t-s), \tag{1.6}$$

we obtain the following result.

Theorem 1.2. *Under hypotheses (H_0) and (H_3), the problem*

$$\begin{gathered} \frac{1}{A}(Au')' = u\psi(x,u), \quad \text{a.e in } (0,\omega), \\ u > 0, \\ Au'(0) := \lim_{x\to 0} Au'(x) = 0, \\ u(\omega) := \lim_{x\to\omega} u(x) = c > 0 \end{gathered} \tag{P_2}$$

has a positive bounded solution $u \in C([0,\omega]) \cap C^1((0,\omega))$ satisfying

$$c_1 \le u(x) \le c, \quad \forall x \in (0,\omega), \tag{1.7}$$

where c_1 is a positive constant.

In order to simplify our statements, we adopt the following notation.

Notation.

(i) $\mathcal{B}((0,\omega))$ denotes the set of Borel measurable functions on $(0,\omega)$.

(ii) $\mathcal{B}^+((0,\omega))$ is the set of nonnegative Borel measurable functions on $(0,\omega)$.

(iii) We denote by $C([0,\omega]) := \{u \in C((0,\omega)),\ \lim_{x\to 0} u(x), \text{ and } \lim_{x\to\omega} u(x) \text{ exist}\}$, and by $C^+([0,\omega])$ the set of nonnegative ones.

(iv) Let f and g be two positive functions defined on a set S.

(a) We call $f \preceq g$, if there exists a constant $c > 0$, such that

$$f(x) \le cg(x), \quad \forall x \in S. \tag{1.8}$$

(b) We call $f \sim g$, if there exists a constant $c > 0$ such that

$$\frac{1}{c}g(x) \le f(x) \le cg(x), \quad \forall x \in S. \tag{1.9}$$

(v) For $s,t \in [0,\omega)$, we denote $s \vee t = \max(s,t)$.

(vi) We denote

$$K = \left\{q \in \mathcal{B}^+((0,\omega)),\ \int_0^\omega G(0,y)q(y)\,dy < \infty\right\}. \tag{1.10}$$

2. Properties of Green's function

In the sequel, we denote

$$\Gamma(x,y)=\int_{x\vee y}^{\omega}\frac{1}{A(t)}dt,\quad \text{for } x,y\in[0,\omega),$$
$$\delta(x)=\min\left(1,\int_{x}^{\omega}\frac{1}{A(t)}dt\right),\quad \text{for } x\in(0,\omega). \tag{2.1}$$

Let $a\in(0,\omega)$, then for each $x\in(0,\omega)$, we have

$$\Gamma(x,a)\sim\delta(x). \tag{2.2}$$

Indeed, the result follows from the following inequalities:

$$\min(\alpha,1)\min(1,\beta)\le\min(\alpha,\beta)\le\max(\alpha,1)\min(1,\beta),\quad \text{for } \alpha,\beta>0. \tag{2.3}$$

First, we give the following version of a comparison principle.

Proposition 2.1. *The following properties hold.*

(1) *Let* $f\in\mathcal{B}^+((0,\omega))$, *then for a fixed* $a\in(0,\omega)$,

$$Vf(x)\ge Vf(a)\frac{\Gamma(x,a)}{\Gamma(a,a)},\quad \forall x\in[0,\omega). \tag{2.4}$$

(2) *The function* $x\mapsto\Gamma(x,0)/\delta(x)$, *is nonincreasing on* $(0,\omega)$.

(3) *For each* $x,y\in(0,\omega)$,

$$\frac{\delta(y)}{\delta(x)}G(x,y)\le G(0,y). \tag{2.5}$$

Proof. (1) Let $x,y\in[0,\omega)$ and $a\in(0,\omega)$, then we have

$$\Gamma(x,y)\Gamma(a,a)\ge\Gamma(x,a)\Gamma(a,y), \tag{2.6}$$

which implies the result.

(2) It follows from the fact that $x\mapsto\Gamma(x,0)$ is nonincreasing and

$$\frac{\Gamma(x,0)}{\delta(x)}=\max\left(1,\Gamma(x,0)\right). \tag{2.7}$$

(3) For $x,y\in(0,\omega)$, we distinguish the following cases:

(i) if $y\le x$, then

$$\frac{\delta(y)}{\delta(x)}G(x,y)=\delta(y)A(y)\frac{\Gamma(x,y)}{\delta(x)}$$
$$\le\delta(y)A(y)\frac{\Gamma(x,0)}{\delta(x)}\le A(y)\Gamma(y,0)=G(0,y); \tag{2.8}$$

(ii) if $y \geq x$, then $\delta(y)/\delta(x) \leq 1$, which implies that

$$\frac{\delta(y)}{\delta(x)} G(x,y) \leq G(x,y) \leq G(0,y), \tag{2.9}$$

and this completes the proof. □

Next, we will give some inequalities satisfied by Green's function.

Theorem 2.2 (3G-theorem). *For each $x,y,z \in [0,\omega)$,*

$$\frac{G(x,z)G(z,y)}{G(x,y)} \leq \frac{\delta(z)}{\delta(x)} G(x,z) + \frac{\delta(z)}{\delta(y)} G(y,z). \tag{2.10}$$

Proof. We remark that assertion (2.10) is equivalent to

$$\frac{\Gamma(x,z)\Gamma(z,y)}{\Gamma(x,y)} \leq \frac{\delta(z)}{\delta(x)} \Gamma(x,z) + \frac{\delta(z)}{\delta(y)} \Gamma(z,y). \tag{2.11}$$

Since $\Gamma(x,y)$ is symmetric in x,y, we will discuss three cases.

(i) If $z \leq x \leq y$, then $\Gamma(x,z) = \int_x^\omega (1/A(t))dt$, $\Gamma(z,y) = \int_y^\omega (1/A(t))dt$, and $\Gamma(x,y) = \int_y^\omega (1/A(t))dt$. Since $\delta(z)/\delta(x) \geq 1$, then we have the result.

(ii) If $x \leq y \leq z$, then we obtain $\Gamma(x,z) = \Gamma(z,y) = \Gamma(z,0)$ and $\Gamma(x,y) = \Gamma(y,0)$. Hence, we have

$$(2.11) \Longleftrightarrow \frac{\Gamma(z,0)}{\delta(z)} \leq \frac{\Gamma(y,0)}{\delta(x)} + \frac{\Gamma(y,0)}{\delta(y)}. \tag{2.12}$$

Now, using the second assertion of Proposition 2.1, we obtain the result.

(iii) If $x \leq z \leq y$, then we obtain $\Gamma(x,z) = \int_z^\omega (1/A(t))dt$, $\Gamma(z,y) = \int_y^\omega (1/A(t))dt$, and $\Gamma(x,y) = \int_y^\omega (1/A(t))dt$. So if $\delta(z) = 1$, then $\delta(x) = 1$, and if $\delta(z) = \int_z^\omega (1/A(t))dt$, then $\delta(y) = \int_y^\omega (1/A(t))dt$.

This proves (2.11). □

In the sequel, for a fixed $q \in \mathcal{B}^+((0,\omega))$, we put

$$\begin{aligned} \|q\| &= \sup_{x \in (0,\omega)} \int_0^\omega \frac{\delta(y)}{\delta(x)} G(x,y) q(y) dy, \\ \alpha_q &= \sup_{x,y \in (0,\omega)} \int_0^\omega \frac{G(x,z)G(z,y)}{G(x,y)} q(z) dz. \end{aligned} \tag{2.13}$$

Then, we have the following result.

Proposition 2.3. *Let $q \in K$, then*

$$\|q\| \leq Vq(0) \leq \alpha_q \leq 2\|q\|. \tag{2.14}$$

Proof. By Proposition 2.1, we have $(\delta(y)/\delta(x))G(x,y) \leq G(0,y)$, which implies that $\|q\| \leq Vq(0)$.

On the other hand, using Lebesgue's theorem and Fatou's lemma, we obtain that

$$\begin{aligned} Vq(0) &= \int_0^\omega G(0,z)q(z)dz = \sup_{x\in(0,\omega)} \int_0^\omega G(x,z)q(z)dz \\ &= \sup_{x\in(0,\omega)} \int_0^\omega \lim_{y\to\omega} \frac{G(x,z)G(z,y)}{G(x,y)} q(z)dz \le \sup_{x\in(0,\omega)} \liminf_{y\to\omega} \int_0^\omega \frac{G(x,z)G(z,y)}{G(x,y)} q(z)dz \\ &\le \sup_{x,y\in(0,\omega)} \int_0^\omega \frac{G(x,z)G(z,y)}{G(x,y)} q(z)dz = \alpha_q. \end{aligned} \tag{2.15}$$

Now, by (2.10) we deduce that

$$\alpha_q \le 2\|q\|. \tag{2.16}$$

This completes the proof. □

Remark 2.4. It is clear that if $q \in K$, then the function

$$x \longmapsto Vq(x) = \int_x^\omega \frac{1}{A(t)} \left(\int_0^t A(s)q(s)ds \right) dt \tag{2.17}$$

is continuous on $[0,\omega)$

In the next two propositions, we will give some estimates on the potential Vq, for a convenient function q.

Proposition 2.5. *Let $\lambda \ge 0$, $\alpha < \min(\lambda+1,2)$, and $\beta < 2$. Put $A(x) = x^\lambda$ and $q(x) = 1/x^\alpha(1-x)^\beta$, for $x \in (0,1)$. Then*

$$Vq(x) \sim \begin{cases} (1-x)^{2-\beta} & \text{if } 1 < \beta < 2, \\ (1-x)\log\left(\dfrac{2}{1-x}\right) & \text{if } \beta = 1, \\ (1-x) & \text{if } \beta < 1. \end{cases} \tag{2.18}$$

Proof. Since the function $x \mapsto Vq(x)$ is continuous and positive on $[0,1/2]$, then we deduce that $Vq(x) \sim 1$, for $x \in [0,1/2]$.

Now, assume that $x \in [1/2,1)$. Using the fact that for $t \in [x,1)$, we have $1/2 \le t \le 1$, then we obtain

$$Vq(x) \sim \int_x^1 \left(\int_0^t \frac{s^{\lambda-\alpha}}{(1-s)^\beta} ds \right) dt. \tag{2.19}$$

Since $\alpha < \min(\lambda+1,2)$ and $\beta < 2$, then for each $t \in [x,1)$, we have

$$\begin{aligned} \int_0^t \frac{s^{\lambda-\alpha}}{(1-s)^\beta} ds &\sim \left(\int_0^{1/2} s^{\lambda-\alpha} ds + \int_{1/2}^t (1-s)^{-\beta} ds \right) \\ &\sim \left(1 + \int_{1/2}^t (1-s)^{-\beta} ds \right). \end{aligned} \tag{2.20}$$

(i) If $\beta < 1$, then since $\int_{1/2}^{t}(1-s)^{-\beta}ds = (1/(1-\beta))((1/2)^{1-\beta} - (1-t)^{1-\beta})$, we deduce that $1 + \int_{1/2}^{t}(1-s)^{-\beta}ds \sim 1$. So $Vq(x) \sim 1 - x$.

(ii) If $\beta > 1$, then since $\int_{1/2}^{t}(1-s)^{-\beta}ds \sim (1-t)^{1-\beta} - 2^{\beta-1}$, we deduce that $1 + \int_{1/2}^{t}(1-s)^{-\beta}ds \sim (1-t)^{1-\beta}$. So $Vq(x) \sim \int_x^1 (1-t)^{1-\beta}dt \sim (1-x)^{2-\beta}$.

(iii) If $\beta = 1$, then since $\int_{1/2}^{t}(1-s)^{-1}ds = \log(1/2(1-t))$, we deduce that $\int_0^t (s^{\lambda-\alpha}/(1-s)^{\beta})ds \sim \log(e/2(1-t))$.

Now using the fact that for $\mu \in \mathbb{R}$ and $\sigma > 1$,

$$\int_x^{+\infty} \frac{(\log(t))^{\mu}}{t^{\sigma}}dt \sim \frac{(\log(x))^{\mu}}{(\sigma-1)x^{\sigma-1}}, \quad \text{as } x \longrightarrow \infty, \tag{2.21}$$

we deduce that $\int_x^1 \log(e/2(1-t))dt \sim (1-x)\log(e/2(1-x))$, as $x \to 1$.

So $Vq(x) \sim (1-x)\log(e/2(1-x))$.

Thus, by combination of the two cases we obtain the result. □

The following results will be used later.

Let $q \in K$ and $\varphi \in C^{+}([0,\omega]) \cap C^{1}((0,\omega))$ be the solution of the problem

$$\begin{aligned} &\frac{1}{A}(Au')' - qu = 0 \quad \text{in } (0,\omega), \\ &Au'(0) = 0, \qquad u(0) = 1. \end{aligned} \tag{Q}$$

Then we have the following.

Proposition 2.6 (see [4]). (i) *φ is nondecreasing on $[0,\omega)$.*

(ii) *For each $x \in [0,\omega)$,*

$$1 \leq \varphi(x) \leq e^{(Vq(0)-Vq(x))}. \tag{2.22}$$

In the sequel, we denote by

$$G_q(x,y) = A(y)\varphi(y)\varphi(x)\int_{x\vee y}^{\omega} \frac{1}{A(t)\varphi^2(t)}dt \tag{2.23}$$

the Green's function of the operator $u \mapsto -(1/A)(Au')' + qu$, with $Au'(0) = 0$ and $u(\omega) = 0$. Let $V_q f(x) = \int_0^{\omega} G_q(x,y)f(y)dy$, for $f \in \mathcal{B}^{+}((0,\omega))$. Then we have the following.

Proposition 2.7. *Let $q \in K$, then the following resolvent equation holds:*

$$V = V_q + V_q(qV) = V_q + V(qV_q). \tag{2.24}$$

Moreover, for each $u \in \mathcal{B}^{+}((0,\omega))$ such that $V(qu) < \infty$,

$$(I - V_q(q.))(I + V(q.))u = (I + V(q.))(I - V_q(q.))u = u. \tag{2.25}$$

For each $x, y \in [0,\omega)$,

$$e^{-Vq(0)}G(x,y) \le G_q(x,y) \le G(x,y), \quad \forall x,y \in (0,\omega), \tag{2.26}$$

$$1 - V_q(q)(x) \ge e^{-Vq(0)}, \quad \forall x \in [0,\omega), \tag{2.27}$$

$$(\rho - V_q(q\rho))(x) \ge \rho(x)e^{-Vq(0)}, \tag{2.28}$$

where $\rho(x) = \Gamma(x,0) = \int_x^{\omega}(1/A(t))dt$.

Proof. The proofs of (2.25) and (2.26) can be found in [4, Theorem 4].

For each $x \in (0,\omega)$, we obtain by Fubini-Tonelli's theorem that

$$\begin{aligned} V_q(q)(x) &= \int_0^{\omega} G_q(x,y)q(y)dy \\ &= \int_0^{\omega} A(y)\varphi(y)\varphi(x)\left(\int_{x\vee y}^{\omega}\frac{1}{A(t)\varphi^2(t)}dt\right)q(y)dy \\ &= \varphi(x)\int_x^{\omega}\frac{1}{A(t)\varphi^2(t)}\left(\int_0^t A(y)\varphi(y)q(y)dy\right)dt. \end{aligned} \tag{2.29}$$

Now using that φ is the solution of the problem (Q), and by integrating by parts, we have

$$V_q(q)(x) = 1 - \frac{\varphi(x)}{\varphi(\omega)}. \tag{2.30}$$

On the other hand, we deduce from (2.22) that

$$0 < \varphi(\omega) \le e^{Vq(0)}, \tag{2.31}$$

which proves (2.27).

For each $x \in (0,\omega)$, we obtain by Fubini-Tonelli's theorem that

$$\begin{aligned} V_q(q\rho)(x) &= \int_0^{\omega} G_q(x,y)q(y)\rho(y)dy \\ &= \int_0^{\omega} A(y)\varphi(y)\varphi(x)\left(\int_{x\vee y}^{\omega}\frac{1}{A(t)\varphi^2(t)}dt\right)q(y)\rho(y)dy \\ &= \varphi(x)\int_x^{\omega}\frac{1}{A(t)\varphi^2(t)}\left(\int_0^t A(y)\varphi(y)q(y)\rho(y)dy\right)dt. \end{aligned} \tag{2.32}$$

Now using that φ is the solution of the problem (Q) and ρ is differentiable on $(0,\omega)$, we obtain by integrating by parts that

$$\rho(x) - V_q(q\rho)(x) = \varphi(x)\int_x^{\omega}\frac{1}{A(t)\varphi^2(t)}dt = \frac{G_q(0,x)}{A(x)}. \tag{2.33}$$

Hence, from the lower inequality of (2.26), we deduce that

$$\rho(x) - V_q(q\rho)(x) \ge e^{-Vq(0)}\frac{G(0,x)}{A(x)} = e^{-Vq(0)}\Gamma(x,0) = e^{-Vq(0)}\rho(x). \tag{2.34}$$

This completes the proof of (2.28). □

3. Proofs of the main results

In this section, we aim at proving Theorems 1.1 and 1.2.

We recall that $\rho(x) = \int_x^{\omega}(1/A(t))dt$.

Proof of Theorem 1.1. Let $c > 0$ and $q \in K$ satisfying (H_2). We denote by

$$\Lambda := \{u \in \mathscr{B}^+((0,\omega));\ c\rho e^{-Vq(0)} \leq u \leq c\rho\} \tag{3.1}$$

the nonempty convex set of $\mathscr{B}^+((0,\omega))$, and we define the operator T on Λ by

$$Tu(x) := c(\rho(x) - V_q(q\rho)(x)) + V_q(u(q - \psi(\cdot,u)))(x), \quad \forall x \in (0,\omega). \tag{3.2}$$

We claim that $T\Lambda \subset \Lambda$. Indeed, for $u \in \Lambda$ we have by (H_2),

$$Tu \leq c\rho - cV_q(q\rho) + cV_q(\rho(q - \psi(\cdot,c\rho))) \leq c\rho - cV_q(\psi(\cdot,c\rho)) \leq c\rho. \tag{3.3}$$

On the other hand, by using (2.28), we obtain that

$$Tu \geq c(\rho - V_q(q\rho)) \geq c\rho e^{-Vq(0)}. \tag{3.4}$$

Hence $T\Lambda \subset \Lambda$. Next, we prove that the operator T is nondecreasing on Λ. Let $u_1, u_2 \in \Lambda$ such that $u_1 \leq u_2$, then from (H_2), we have for each $x \in (0,\omega)$,

$$Tu_2(x) - Tu_1(x) = V_q[q(u_2 - u_1) + u_1\psi(\cdot,u_1) - u_2\psi(\cdot,u_2)](x) \geq 0. \tag{3.5}$$

Now, we consider the sequence $(u_j)_j$ defined by $u_0(x) = c(\rho(x) - V_q(q\rho)(x))$ and $u_{j+1}(x) = Tu_j(x)$, for $j \in \mathbb{N}$ and $x \in (0,\omega)$. Then since Λ is invariant under T, we have obviously $u_1 = Tu_0 \geq u_0$ and so from the monotonicity of T, we deduce that

$$u_0 \leq u_1 \leq \cdots \leq u_j \leq c\rho. \tag{3.6}$$

Hence, the sequence (u_j) converges on $(0,\omega)$ to a function $u \in \Lambda$. Now, using (H_0), (H_2), and the dominated convergence theorem, we deduce that $(Tu_j)_j$ converges to Tu on $(0,\omega)$. Consequently, we have

$$u(x) = c(\rho(x) - V_q(c\rho)(x)) + V_q(u(q - \psi(\cdot,u)))(x), \tag{3.7}$$

or equivalently

$$u(x) - V_q(qu)(x) = c\rho(x) - V_q(c\rho + u\psi(\cdot,u))(x). \tag{3.8}$$

Applying the operator $(I + V(q.))$ on both sides of the above equality and using (2.25), we deduce that u satisfies

$$u = c\rho - V(u\psi(\cdot,u)). \tag{3.9}$$

Finally, we need to verify that u is a positive continuous solution for the problem (P_1). Indeed, by (H_2) we have $u\psi(\cdot,u) \leq cq\rho$, then using the fact that $q \in K$ and ρ is bounded on each interval $[x_0,\omega)$ with $x_0>0$, we deduce the continuity of $V(cq\rho)$, which implies the continuity of u on $(0,\omega)$. Now since $q\in K$ and for each $x,y\in(0,\omega)$,we have

$$A(y)\frac{\rho(y)\rho(x\vee y)}{\rho(x)}\leq A(y)\rho(y)q(y)=G(0,y)q(y), \tag{3.10}$$

then we obtain by (H_1) and the dominated convergence theorem that

$$\lim_{x\to 0}\frac{V(q\rho)(x)}{\rho(x)}=0, \tag{3.11}$$

which implies that $\lim_{x\to 0}(u(x)/\rho(x))=c$. This completes the proof. □

Example 3.1. Let $\gamma>1$ and let p be a nonnegative Borel measurable function on $(0,1)$ such that $\int_0^1 yp(y)(\log(1/y))^{\gamma}dy<\infty$. Then the problem

$$\begin{gathered}\frac{1}{x}(xu')'-p(x)u^{\gamma}=0,\quad \text{in } (0,1),\\ u>0,\\ \lim_{x\to 0}\frac{u(x)}{\log(1/x)}=c>0,\quad u(1)=0,\end{gathered} \tag{3.12}$$

has a positive solution $u\in C^2((0,1))$ satisfying

$$u(x)\sim\log\left(\frac{1}{x}\right). \tag{3.13}$$

Example 3.2. Let $\gamma>1$, $\lambda>1$, and put $A(x)=x^{\lambda}$. Let p be a nonnegative Borel measurable function on $(0,\infty)$ such that $\int_0^{\infty}(p(y)/y^{(\lambda-1)(\gamma-1)-1})dy<\infty$. Then the following problem:

$$\begin{gathered}\frac{1}{x^{\lambda}}(x^{\lambda}u')'-p(x)u^{\gamma}=0,\quad \text{in } (0,\infty),\\ u>0,\\ \lim_{x\to 0}\frac{u(x)}{x^{1-\lambda}}=c>0,\qquad \lim_{x\to\infty}u(x)=0,\end{gathered} \tag{3.14}$$

has a positive solution $u\in C((0,\infty))$ satisfying

$$u(x)\sim x^{1-\lambda}. \tag{3.15}$$

In the next, we will give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $c>0$, then by hypothesis (H_3), there exists $q\in K$ such that the function $t\mapsto t(\psi(x,t)-q(x))$ is nonincreasing on $[0,c]$. We consider the nonempty closed convex set Λ given by

$$\Lambda=\{u\in C([0,\omega]);\ ce^{-Vq(0)}\leq u(x)\leq c\}, \tag{3.16}$$

and we define the operator T on Λ by

$$Tu := c(1 - V_q(q)) + V_q((q - \psi(\cdot,u))u), \tag{3.17}$$

and $Tu(\omega) := \lim_{x\to\omega} Tu(x) = c$.

Now, by similar arguments as in the proof of Theorem 1.1, we obtain that $T\Lambda \subset \Lambda$ and T is an increasing operator on Λ. Let $(u_n)_n$ be the sequence of functions defined by

$$\begin{aligned} u_0 &= c(1 - V_q(q)), \\ u_{n+1} &= Tu_n, \quad \text{for } n \in \mathbb{N}. \end{aligned} \tag{3.18}$$

Then the sequence $(u_n)_n$ converges to a function $u = \sup_n u_n \in \Lambda$, satisfying

$$u = c - V(u\psi(\cdot,u)). \tag{3.19}$$

Since we have $\psi(\cdot,u) \le q$ and $Vq \in C^+([0,\omega))$, then $V(qu) \in C^+([0,\omega))$ and consequently $V(u\psi(\cdot,u)) \in C^+([0,\omega))$. Hence, u is a positive continuous solution of the problem (P_2). □

Example 3.3. Let $\gamma,\lambda \ge 0$, $\alpha < \min(\lambda+1,2)$, and $\beta < 2$. Put $A(x) = x^\lambda$, for $x \in (0,1)$. Then the problem

$$\begin{gathered} \frac{1}{A}(Au')' - \frac{u^{\gamma+1}(x)}{x^\alpha(1-x)^\beta} = 0, \quad \text{in } (0,1), \\ Au'(0) = 0, \quad u(1) = c > 0 \end{gathered} \tag{3.20}$$

has a positive solution $u \in C([0,1]) \cap C^1((0,1))$ satisfying for each x in $(0,1)$

$$0 \le c - u(x) \preceq \begin{cases} (1-x)^{2-\beta} & \text{if } 1 < \beta < 2, \\ (1-x)\log\left(\dfrac{2}{1-x}\right) & \text{if } \beta = 1, \\ (1-x) & \text{if } \beta < 1. \end{cases} \tag{3.21}$$

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