TAG-MODULES WITH COMPLEMENT SUBMODULES H-PURE

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ABSTRACT

The concept of a QTAG-module MR was given by Singh[8]. The structure theory of such modules has been developed on similar lines as that of torsion abelian groups. If a module M is such that M@M is a QTAG-module, it is called a strongly TAG-module. This in turn leads to the concept of a primary TAG-module and its periodicity. In the present paper some decomposition theorems for those primary TAG-modules in which all h-neat submodules are h-pure are proved. Unlike torsion abelian groups, there exist primary TAG-modules of infinite periodicities. Such modules are studied in the last section. The results proved in this paper indicate that the structure theory of primary TAG-modules of infinite periodicity is not very similar to that of torsion abelian groups.

KEY WORDS AND PHRASES: QTAG-modules, complement submodules, h-pure submodules, h-neat submodules, and basic submodules.

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1 INTRODUCTION

A module MR satisfying the following two conditions is called a TAG-module [2].
(I) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

(II) Given any two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U, any homomorphism f: W → V can be extended to a homomorphism g: U → V provided the composition length d(U/W) ≤ d(V/f(W)).

If a module satisfies condition (I), it is called a QTAG-module [8]. The main purpose of this paper is to prove some decomposition theorems for a module M, such that M@M is a QTAG-module and that is to prove some decomposition theorems for a module M, such that M@M is a QTAG-module and that every h-neat (complement) submodule of M is h-pure. An example of such an h-reduced primary TAG -
module, which is not decomposable, is given at the end of the paper. However, it follows from the results in this paper that any torsion reduced module over a bounded (hnp)-ring, with every complement submodule pure, is decomposable. The main results are given in Theorems (5.5), (5.12) and (5.14). In section 3, a necessary and sufficient condition for a QTAG-module to admit only one basic submodule is given. In section 4 the concept of neat height of a uniform element in a QTAG-module is discussed. The concept of neat height is used to give, in Theorems (4.6) and (4.7), some criterions for a QTAG-module, such that every h-neat module is $I$-embedded in the sense of Moore[5]. The results in sections 3 and 4 can be of independent interest.

2 PRELIMINARIES

A module in which the lattice of its submodules is linearly ordered under inclusion is called a serial module; in addition if it has finite composition length, it is called a uniserial module. Let $M_R$ be a QTAG-module. An $x \in M$ is called a uniform element, if $xR$ is a non-zero uniform (hence uniserial) submodule of $M$. For any module $A_R$ with a composition series, $d(A)$ denotes its composition length. Let $x \in M$ be uniform. Then $e(x) = d(xR)$ is called the exponent of $x$. The equation $[x, y] = n$, will give that $y$ is a uniform element of $M$, such that $x \leq yR$ and $d(yR/xR) = n$. For basic definitions of height of an element of $M$, the submodule $H_k(M)$ for $k \geq 0$, one may refer to [6] or [8]. For any submodule $N$ of $M$, and any $y \in N$, $h_N(y)$ will denote the height of $y$ in $N$; however we write $h(y)$ for $h_N(y)$. A submodule $N$ of $M$ is said to be h-pure in $M$, if $H_k(M) \wedge N = H_k(N)$ for every $k \geq 0$. For any module $K$, $\text{soc}(K)$ denotes the socle of $K$. $M_R$ is said to be decomposable, if it is a direct sum of uniserial modules.

By using [8, Lemma(2.3)], one can prove the following:

Proposition(2.1). A submodule $N$ of a QTAG-module $M$ is h-pure in $M$ if and only if for any uniform $x \in \text{soc}(N)$, $h_N(x) = h(x)$.

The following is of frequent use in this paper.

Proposition(2.2) [8, Lemma(3.9)]. Let $N$ be any h-pure submodule of a QTAG-module $M$. Then for any uniform $x \in M$, there exists a uniform $x' \in M$, such that for $x = x + M \in M/N$, $e(x) = e(x')$, $\bar{x} = \bar{x'}$ and $M \cap x'R = 0$.

By using the above proposition, we get that if $M/N$ is decomposable for some h-pure submodule $N$, then $M = T \oplus N$, for some decomposable submodule $T$ of $M$. Let $K_R$ be any module. For the definitions of $K$-injective modules and $K$-projective modules one may refer to [1]. Lemmas (2.2) and (2.4) in [8] give the following:

Proposition(2.3). Let $A$ and $B$ be two uniserial submodules of a QTAG-modules $M$, such that $A \cap B = 0$.

(i) If $d(A) \leq d(B)$, then $B$ is $A$-injective.

(ii) If $d(A) \geq d(B)$, then $B$ is $A$-projective.

(iii) If $d(A) = d(B)$, then $A \equiv B$ if and only if either $\text{soc}(A) \equiv \text{soc}(B)$, or $A/H_1(A) \equiv B/H_1(B)$.

$M$ is said to be bounded, if for some $k$, $H_k(M) = 0$. Any h-pure bounded submodule of $M$ is a summand of $M$ [8, Remark(3.8)]. $M$ is said to be h-divisible, if $h(x) = \infty$ for every $x \in M$. If a uniform
element \( x \in \text{soc}(M) \) has finite height, then for any uniform \( y \in M \), with \([x, y] = h(x), yR \) being an h-pure submodule of \( M \), is a summand of \( M \). For general properties of rings and modules one may refer to [3].

\( \xi 3 \) BASIC SUBMODULES

Throughout \( M_k \) is a QTAG-module. A submodule \( B \) of \( M \) is called a basic submodule of \( M \), if \( B \) is a decomposable h-pure submodule of \( M \), such that \( M/B \) is h-divisible [7]. As pointed out in [8, Remark(3.12)], \( M \) has a basic submodule and any two basic submodules of \( M \) are isomorphic.

Lemma(3.1). Let \( A_1, A_2, \ldots, A_k \) be any finitely many uniserial summands of \( M \), such that \( d(A_i) < d(A_{i+1}) \) and \( N = \sum_{i=1}^{i=k} A_i \). Then \( N \) is an h-pure submodule of \( M \).

Proof. Consider a uniform element \( x \in \text{soc}(N) \). Then \( x = \sum_{i=1}^{i=k} x_i \in A_i \). If for any \( i < j \), \( x_i \neq 0 \neq x_j \), then by the hypothesis \( h(x_i) < h(x_j) \). Thus \( h(x) = \{h(x_i) : x_i \neq 0\} \). As each \( A_i \) is h-pure, \( h(x) = h(A_i(x_i)) = h_0(x_i) \). This gives \( h(x) = h_0(x) \). Hence \( N \) is -pure.

Lemma(3.2). Let \( M \) be such that \( \cap_k H_k(M) = 0 \) and let \( M \) have a basic submodule \( B \neq M \). Then for some simple submodule \( S \) of \( \text{soc}(M) \), there exists an h-pure submodule \( N \subseteq M \) such that every \( yR \) is uniserial, \( d(yR) < d(yR) \) and \( S \equiv \text{soc}(yR) \). The heights of the (non-zero) elements of the homogeneous components of \( \text{soc}(M) \), determined by \( S \), do not have an upper bound.

Proof. Let \( M/B \). Consider a uniform \( z \in \text{soc}(M/B) \). By (2.2) there exists a uniform \( z_1 \in \text{soc}(M) \) such that \( z = z_1 \). As \( \cap_k H_k(M) = 0 \), \( h(z_1) \) is finite. Let \( h(z_1) = n_1 \). Then there exists \( y_1 \in M \), such that \([z_1, y_1] = n_1 \). Then \( y_1R \) is an h-pure submodule of \( M \) and \( B \cap y_1R = 0 \). However \( h(z) = \infty \). So there exists a uniform \( u_1 \in M \) with \( h(u_1) = zR \) and \( h(u_1) > n_1 \). By (2.2) we get uniform \( z_2 \in \text{soc}(M) \) with \( z = z_2 \), \( h(z_2) = n_2 > n_1 \). We get \( y_2 \in M \) such that \([z_2, y_2] = n_2 \). By continuing this process, we get an infinite sequence of uniform elements \( (y_i)_{i=1}^{i=\infty} \) of \( M \), such that each \( y_iR \) is an h-pure uniserial submodule, \( \text{soc}(y_iR) = zR \) for some \( z_i \in M \) satisfying \( z = z_i \), \([z_i, y_i] = n_i = h(z_i) \) and \( n_i < n_{i+1} \). If \( K = \sum_{i=1}^{i=k} y_iR \) is not a direct sum, we get a smallest \( i \geq 2 \), such that \( z_i \in \sum_{k=1}^{k=i} z_iR \). Then \( N = \sum_{k=1}^{k=i} y_iR \). By (3.1) \( N \) is an h-pure submodule of \( M \). For any uniform \( v \in N \), if \( v = \sum v_j \), with \( v_j \in y_iR \), then \( h(v) = \min(h(v_j)) \). This gives \( h(z_k) \leq \max\{h(z_k) : 1 \leq k \leq i-1\} \). This is a contradiction, as \( h(z_k) < h(z_i) \) for \( j < i \). Hence \( K = \Theta \Sigma y_iR \). By using (3.1) we get that \( K \) is an h-pure submodule. Clearly \( \text{soc}(K) \) is homogeneous. The last part is obvious.

Lemma(3.3). Let \( M \) be a QTAG-module such that \( M = \Theta \sum_{i=1}^{i=k} y_iR \) uniserial, \( \text{soc}(y_iR) \equiv \text{soc}(y_{i+1}R) \) and \( d(y_iR) < d(y_{i+1}R) \). Then \( M \) has a basic submodule \( B \neq M \).
Proof. By (2.3)(i) we get monomorphisms \( \sigma_i : y_iR \rightarrow y_{i+1}R \). Write \( \sigma_i(y_i) = w_i \). Then \( w_i \) is uniform and \( e(w_i) = e(y_i) \). Consider \( B = \sum w_i R \), and \( M = M/B \). Let \( z \in B \). Then \( z = \sum_{i=1}^{|z|} (y_i - \sigma_i(y_i)) r_i \), where \( r_i \in R \) and a positive integer \( s \). Here \( y_i r_i - \sigma_i(y_i) r_i = y_i R - \sigma_i(y_i) r_i R \). Using this, it can be easily proved that \( B = \sum \sigma_i R \) and \( y_i R \cap B = 0 \). Now \( y_i R \Rightarrow y_i R \Rightarrow \cdots \Rightarrow y_i R \Rightarrow \cdots \), gives \( M \) is a serial module of infinite length. It only remains to prove that \( B \) is h-pure. In view of (3.1) it is enough to prove that each \( w_i R \) is h-pure. Now \( y_i R \oplus y_{i+1} R \) being a summand of \( M \), is h-pure. But \( y_i R \oplus y_{i+1} R = w_i R \oplus y_{i+1} R \). So \( w_i R \) is h-pure in \( M \). This completes the proof.

Theorem (3.4). A QTAG-module \( M_R \) has no basic submodule other than \( M \) if and only if \( M \) is h-reduced and for each homogeneous component \( K \) of \( \text{soc}(M) \), there exists an upper bound on the heights of members of \( K \).

Proof. Let \( M \) be its only basic submodule. Then by definition \( M \) is decomposable and h-reduced. For a simple submodule \( S \) of \( M \), we get a summand \( M_S \) of \( M \), such that \( \text{soc}(M_S) \) is the homogeneous component of \( \text{soc}(M) \) determined by \( S \). If heights of members of \( \text{soc}(M_S) \) do not have an upper bound, we get a summand \( N = \sum y_i R \) of \( M_S \) such that each \( y_i R \) is uniserial and \( d(y_i R) < d(y_{i+1} R) \). By (3.3) \( N \) has a basic submodule \( B_1 \neq N \). As \( N \) is a summand of \( M \), we get a basic submodule \( B \) of \( M \) of which \( B_1 \) is a summand and \( B \neq M \). This is a contradiction. Conversely let the given conditions hold. Then \( \cap K \text{ h}(M) = 0 \). The rest follows from (3.3).

4. H-NEAT HEIGHT

Throughout \( M_R \) is a QTAG-module. A submodule \( N \) of \( M \) is called an h-neat submodule of \( M \) if \( H_1(M) \cap N = H_1(N) \). As observed in [8], any submodule \( N \) of \( M \) is h-neat if and only if it is a complement submodule of \( M \), any maximal essential extension \( K' \) of a submodule \( K \) of \( M \), is an h-neat submodule of \( M \). Any such \( K' \) is called an h-neat hull of \( K \). For any uniform \( x \in M \), the minimum of all \( d(K' / xR) \), where \( K' \) runs over all h-neat hulls of \( xR \), is called the h-neat height of \( x \) : it is denoted by \( h'(x) \). If \( x \in N \subset M \), then \( h'_N(x) \) will denote the neat height of \( x \) in \( N \). If \( N \) is an h-neat submodule of \( M \), then any h-neat submodule of \( N \) is h-neat in \( M \), so that for any uniform \( x \in N \), \( h'(x) \leq h'_N(x) \). We put \( h'(0) = \infty \). In an h-divisible QTAG-module \( M \), every uniform element is of infinite h-neat height.

For any two modules \( A_R \) and \( B_R \) any homomorphism from a submodule of \( A \) into \( B \) is called a subhomomorphism from \( A \) to \( B \); the set of all subhomomorphisms from \( A \) to \( B \) is denoted by \( \text{SH}(A,B) \). An \( f \in \text{SH}(A,B) \) is said to be maximal, if it has no extension in \( \text{SH}(A,B) \). Now (2.3) gives the following:

Lemma (4.1). Let \( xR \) and \( yR \) be any two uniserial submodules of \( M \), such that \( xR \cap yR = 0 \). Then

(a) For any maximal \( f \in \text{SH}(xR, yR) \), either \( \text{domain}(f) = xR \) or \( \text{range}(f) = yR \).

(b) Let \( z \in xR \oplus yR \) be uniform, \( z = x' + y', x' \in xR, y' \in yR \) and \( d(x'R) \geq d(y'R) \). The following
hold:

(i). $zR \equiv x'R$.

(ii) Given any $u = v+w, v \in xR, w \in yR$ such that $z \in uR,$

(a) if $y' \neq 0$, then $[x', v] = [y', w]$;

(b) if $y' = 0$, then $e(w) \leq [x', v].$

Lemma (4.2). Let $xR$ and $yR$ be two uniserial submodules of $M$ such that $xR \cap yR = 0$. Let $z = x' + y', x' \in xR, y' \in yR$, be uniform such that $d(y'R) \leq d(x'R)$. For $T = xR \oplus yR$, the following hold:

(i). For $y' \neq 0$, $h'_t(z)$ is the minimum of $[x', x]$ and $[y', y]$.

(ii). For $y' = 0$, let $f \in SH(xR, yR)$ be maximal with $s = d(ker f)$, minimal under the condition that $xR \subseteq ker f$. If $domain(f) = uR$, then $h'_t(z) = [x', u]$ minimum of $[x', x]$ and $e(y) + s - e(x')$.

Proof. $g : x'R \rightarrow y'R$ such that $g(x'r) = y'r$ is an $R$-epimorphism. If $w = a+b, a \in xR, b \in yR$, is uniform and $z \in wR$, then $f : aR \rightarrow bR$ such that $f(ar) = br$, is an extension of $g$; further $[z, w] = [x', a]$. Any extension $h : aR \rightarrow yR, a' \in xR$, of $g$ gives uniform $w' = a' + h(a')$ such that $z \in w'R$. Consequently $wR$ is an $h$-neat hull of $zR$ if and only if $f$ is maximal. In that case by (4.1) either domain($f$) = $xR$ or range($f$) = $yR$. Thus for domain($f$) = $xR$, and $uR = ker f$, $e(a)$ is the minimum of $e(x)$ and $e(y)+e(u)$. To minimize $e(a)$, we need to minimize $s = e(u)$. So that for minimal $e(u)$, $h'_t(z) = [x', a] = e(a) - e(x') = min\{e(x), e(y)+e(u)\} - e(x') = min\{e(x') + e(y), e(y)+e(u) \cdot e(x')\}$, as $e(x) - e(x') = [x', x]$. If $y' \neq 0$, then $e(x') + e(y') - e(x') = e(y) - e(x') = [y', y]$. For $y' = 0$, it is obvious that $xR \subseteq ker f$. This proves the result.

Lemma (4.3). Let $M = A \oplus B$ and $x \in M$ be uniform. If $x = a+b, a \in A, b \in B$ and $d(aR) \geq d(bR)$, then the following hold:

(i). For $b \neq 0, h'_t(x) = min\{h'_t(a), h'_t(b)\}$.

(ii). If $b = 0$, and $B$ is $h$-divisible, then $h'_t(x) = h'_t(a)$

Proof. Now $g : aR \rightarrow bR$ given by $g(ar) = br$, is an epimorphism. Let $\pi_1$ and $\pi_2$ be the projections $A \oplus B \rightarrow A$, and $A \oplus B \rightarrow B$ respectively. Consider an $h$-neat hull $K$ of $xR$. Then $K$ is serial. Let $K_t = \pi_t(K)$. As $d(bR) \leq d(aR)$, we get an epimorphism $\sigma : K_t \rightarrow K_2$ such that for any $x_1 \in K_1, \sigma(x_1) = x_2$ if and only if $x_1 + x_2 \in K$. Further $aR \subseteq K_1, bR \subseteq K_2$ and $d(K/xR) = d(K_t/aR)$. By using (2.3) we get that either $K_1$ is $h$-neat or $K_2$ is $h$-neat in $M$.

Case I : $b \neq 0$. Then either $K_1$ is an $h$-neat hull of $aR$ or $K_2$ is an $h$-neat hull of $bR$. So that $h'_t(x) \geq min\{h'_t(a), h'_t(b)\}$. Let $t = min\{h'_t(a), h'_t(b)\} < h'_t(x)$. To be definite let $t = h'_t(a)$. Then we get an $h$-neat hull $aR$ of $aR$ with $[a, a] = t$, and a uniform $b_1$ in $M$ with $[b, b] \geq t$. By (2.3) $g$ extends to a homomorphism $f : aR \rightarrow b_1R$. Then $(a_1 + f(a_1))R$ is an $h$-neat hull of $xR$ with $[x, a_1 + f(a_1)] < h'_t(x)$ This is a contradiction. Similar arguments hold if $t = h'_t(b)$. This proves (i).

Case II : $b = 0$ and $B$ is $h$-divisible. Any $h$-neat serial submodule of $B$ is either zero or of infinite length. Thus for $K$ to be an $h$-neat hull of $xR$ it is necessary and sufficient that $K_1$ is an $h$-neat hull of $aR$.
Thus for $x = a$, $h'(x) = h'_a(a)$

Lemma (4.4). Let $K_R = \bigoplus_{i=1}^{k} x_i R$ be a QTAG-module with each $x_i R$ uniserial. Let $z = \sum z_i x_i \in x_i R$, be uniform. Let $z_i$ be such that $e(z) = e(z_i)$. Then $h'(z)$ is the minimum of the following numbers:

(i) All $[z_i, x_i]$ with $z_i \neq 0$.

(ii) The neat heights of $z_i$ in various $x_j R \oplus x_k R$, with $z_j = 0$.

Proof. The hypothesis on $z_i$ gives that for any $i$, $\sigma_i : z_i R \rightarrow z_i R$ such that $\sigma_i(z_i) = z_i R$ is an epimorphism. Let $y = \sum y_i$, $y_i \in x_i R$, be any uniform element in $K$ such that $z \in y R$. Then $\theta_i : y_i R \rightarrow y_i R$ given by $\theta_i(y_{ij}) = y_{ij}$ is an extension of $\sigma_i$. Clearly if $z_i \neq 0$, then $[z_i, y_i] = [z_i, y_i]$. So that $e(y)$ is not more than $s$, the minimum of all those $[z_i, x_i]$ for which $z_i \neq 0$. Thus $h'(z) \leq s$. However, if every $z_i \neq 0$, then by (2.3), it is immediate that for $y R$ to be an h-neat hull of $z R$, it is necessary that $[z, y] = s$, i.e., $h'(z) = s$. Suppose that for some $j$, $z_j = 0$ and that for $T = x_j R \oplus x_k R$, $h'_j(z_j) < s$. We have a maximal $f \in \text{SH}(x_i R, x_j R)$ with $\ker f$ of smallest length among those containing $z_i R$. Let $w_0 R = \text{domain}(f)$, then $s' = h'_j(z_j) = [z_j, w_0]$. By using (2.3), we obtain a uniform $y = \sum y_i$, with $z \in y R$, $y_i = w_0$ and $y_i = y_i$. Then $y R$ is an h-neat hull of $z R$ such that $[z, y] = s'$. Thus $h'(z) \leq s_0$, the minimum of the numbers listed in (i) and (ii). Suppose $h'(z) < s_0$. We get a uniform $w = \sum w_i$, $w_i \in x R$ such that $w R$ is an h-neat hull of $z R$ and $[z, w] = h'(z)$. Then for some $j$, $w_j R = x_j R$. For this $j$, $z_j = 0$ and $(w_j + w_j) R$ is an h-neat hull of $z_i R$. Consequently for $T = x_j R \oplus x_k R$, $h'_j(z_j) \leq h'(z)$. This is a contradiction. This completes the proof.

We now give a criterion in terms of h-neat heights, for a QTAG-module, in which every h-neat submodule is h-pure. We shall give a more general result. Analogous to the definition of an l-embedded subgroup of an abelian p-group given by Moore [5], we define an l-embedded submodule of a QTAG-module. Let $Z^*$ be the set of all non-negative integers and $l : Z^* \rightarrow Z^*$ be any function such that $n \leq l(n)$, $n \in Z^*$. A submodule $N$ of a QTAG-module $M$ is said to be l-embedded if $H_y(M) \cap N \subseteq H_y(N)$ for every $n \in Z^*$. Then if $I$ is the identity map on $Z^*$, a submodule $N$ of $M$ is h-pure if and only if $N$ is l-embedded. Given $l : Z^* \rightarrow Z^*$ satisfying $(l(n)) \geq n$, we define $l_1 : Z^* \rightarrow Z^*$ such that for any $n \in Z^*$, $l_1(n)$ is the minimum of all $l(k), k \geq n$. Then $l_1$ is monotonic. Further any submodule $N$ of $M$ is l-embedded if and only if it is $l_1$-embedded. So without loss of generality we assume that $l$ is monotonic. Further define $l(\infty) = \infty$.

Proposition 4.5. Let $M$ be an h-reduced QTAG-module and $l : Z^* \rightarrow Z^*$ be a monotonic function such that $n \leq l(n)$, $n \in Z^*$. Then every h-neat submodule of $M$ is l-embedded if and only if $l(h'(y)) \leq l(h'(y) + 1) - 1$ for every uniform $y \in M$.

Proof. Let every h-neat submodule of $M$ be l-embedded. Consider a uniform $y \in M$. As $M$ is h-reduced, every h-neat hull of $y R$ is of finite length. Let $z R$ be an h-neat hull of $y R$ such that $[y, z] = h'(y)$ = $t$. Then $H_y(z R) = y R$ and $H_{t+1}(z R) < y R$. Then by the hypothesis, $H_{t+1}(M) \cap z R \subseteq H_{t+1}(z R) = y R$, but $H_{t+1}(M) \cap z R < y R$. Consequently $h(y) \leq l(t+1) - 1 = l(h'(y) + 1) - 1$. Conversely let the inequality hold. So every uniform $y \in M$ has finite height. Let there exist an h-neat submodule $N$ of $M$ that is not l-
embedded. We get smallest positive integer \( n \) such that \( H_{n_0}(M) \cap N \not\subseteq H_n(N) \). Then \( H_{n_{1}}(M) \cap N \subset H_{n_{-1}}(N) \). There exists a uniform \( y \in H_{n_0}(M) \cap N \) such that \( y \not\in H_n(N) \). As \( l(n) \geq l(n-1) \), \( y \in H_{n}(N) \). So that \( h_0(y) = n-1 \). Consequently \( h'(y) \leq n-1 \). By the hypothesis \( h(y) \leq l(h'(y)+1)-1 \leq l(n)-1 \). However as \( y \in H_{n_0}(M) \), \( h(y) \geq l(n) \). This is a contradiction. This proves the result.

**Theorem (4.6).** Let \( M \) be any QTAG-module and \( I : Z^{-} \to Z^{-} \) be a monotonic function such that \( n \leq l(n) \), \( n \in Z^{-} \). Then every \( h \)-neat submodule of \( M \) is \( I \)-embedded if and only if for any uniform \( y \in M \), \( h(y) \leq l(h'(y)+1)-1 \).

**Proof.** Let every \( h \)-neat submodule of \( M \) be \( I \)-embedded. Write \( M = L \oplus D \), where \( D \) is the largest \( h \)-divisible submodule of \( M \). Now \( L \) is \( h \)-reduced and every \( h \)-neat submodule of \( L \) is \( I \)-embedded in \( L \).

Consider a uniform \( y \in M \). Write \( y = y_1 + y_2 \), \( y_1 \in L \), \( y_2 \in D \). Suppose \( y_1 \neq 0 \). Then \( h(y) = h(y_1) \). By (4.3), \( h'(y_1) = h'_L(y_1) \). By using (4.5), we get \( h(y) = h(y_1) \leq l(h'(y_1)+1)-1 \). Suppose \( y_1 = 0 \). Then \( y = y_2 \in D \), hence \( h(y_2) = \infty \). Let \( K \) be any \( h \)-neat hull of \( y_R \). Consider any \( n \geq 0 \). Then \( H_{n_0}(M) = H_{n_0}(L) \oplus D \). As \( K \subseteq D \neq 0 \), \( H_{n_0}(M) \cap K \subset H_n(K) \), we get \( H_n(K) \neq 0 \). So that \( d(K) = \infty \), \( h'(y) = \infty = h(y) \). Once again \( h(y) = l(h'(y)+1)-1 \). Conversely let the given condition be satisfied. By essentially following the arguments in (4.5), we complete the proof.

**Theorem (4.7).** Let \( M = L \oplus D \) be a QTAG-module such that \( L \) is \( h \)-reduced and \( D \) is \( h \)-divisible.

For a monotonic function \( I : Z^{-} \to Z^{-} \) satisfying \( n \leq l(n) \), every \( h \)-neat submodule of \( M \) is \( I \)-embedded if and only if

(i) every \( h \)-neat submodule of \( L \) is \( I \)-embedded in \( L \); and

(ii) for any serial submodule \( W \) of \( D \), any non-zero homomorphism \( f : W \to L \) is a monomorphism.

**Proof.** Let every \( h \)-neat submodule of \( M \) be \( I \)-embedded. Then obviously (i) hold. Consider a non-zero homomorphism \( f : W \to L \) with \( \ker f \neq 0 \). Then \( bR = \soc(W) \subseteq \ker f \). Consider \( \soc(f(W)) = b_1 \). As \( h(b_1) < \infty \), by using (2.3) we can choose \( W \) to be such that \( f(W) \) is \( h \)-neat in \( L \). Then \( L = \{ x+f(x) : x \in W \} \) is an \( h \)-neat hull of \( bR \). So that \( h'(b) < \infty \). By (4.6) \( h'(b) = \infty \). This gives a contradiction.

Conversely, let the conditions be satisfied. Consider a uniform \( y \in M \). Let \( y = y_1 + y_2 \), \( y_1 \in L \), \( y_2 \in D \). Suppose \( y_1 \neq 0 \). Then by (4.3) \( h(y) = h_L(y_1) \leq l(h'_L(y_1)+1)-1 \). Suppose \( y_1 = 0 \). Then \( y = y_2 \in D \). Let \( K \) be any \( h \)-neat hull of \( y_R \). Let \( K_1 \) and \( K_2 \) be projections of \( K \) in \( L \) and \( D \) respectively. Then \( K \subseteq K_2 \) and we get an epimorphism \( f : K_2 \to K_1 \) with \( y \in \ker f \). By (ii), \( f = 0 \). Consequently \( K \subseteq D \) and hence \( d(K) = \infty \). So once again \( h(y) = l(h'(y)+1)-1 \). Hence (4.6) completes the proof.

By taking \( I = I \), we get the following:

**Corollary (4.8).** Let \( M = L \oplus D \) be a QTAG-module such that \( L \) is \( h \)-reduced and \( D \) is \( h \)-divisible.

Then the following are equivalent:

(i) Every \( h \)-neat submodule of \( M \) is \( h \)-pure in \( M \).

(ii) For any uniform \( y \in M \), \( h(y) = h'(y) \).
(iii) Every h-neat submodule of L is h-pure and for any uniserial submodule W of D any non-zero homomorphism \( f: W \to L \) is a monomorphism.

\[ \xi \ 5. \ HÆ\text{-NEAT SUBMODULES} \]

A module \( M_R \) is called a strongly TAG-module, if \( M \oplus M \) is a OTAG-module. We start with the following:

Lemma(5.1). Let \( M_R \) be a strongly TAG-module, A and B be two uniserial submodules of some homomorphic images of \( M \). Then the following hold:

(i) If \( d(A) < d(B) \), then B is A-injective.

(ii) If \( d(A) > d(B) \), then B is A-projective.

(iii) If \( d(A) = d(B) \), then A \(_{-}\)\(-B\), whenever \( \text{soc}(A) \equiv \text{soc}(B) \) or \( A/H(A) \equiv B/H(B) \).

(iv) \( M \) is a TAG-module.

Proof. Now A and B are submodules of \( M/K \) and \( M/L \) for some submodules K and L of M. As N \( = M/K \oplus M/L \) is a homomorphic image of \( M \oplus M \), \( A \times 0, 0 \times B \) are submodules of N with zero intersection, (i), (ii), and (iii) follow from (2.3). Finally (iv) follows from (i).

Let \( M_R \) be a strongly TAG-module. Let \( \text{spec}(M) \) be the set of all simple \( R \)-modules which occur as composition factors of some finitely generated submodules of \( M \). Let \( S, S' \in \text{spec}(M) \). Then \( S' \) is called an immediate predecessor of \( S \) (and \( S \) is called an immediate successor of \( S' \)) if for some uniserial submodule \( A \) of \( M \), \( A/H(A) \equiv S' \) and \( H(A)/H_2(A) \equiv S \). By using (5.1) we get that any \( S \in \text{spec}(M) \) does not have more than one immediate successor and more than one immediate predecessor. (see also [9]).

Let \( S, S' \in \text{spec}(M) \). \( S' \) is called a k-th successor of \( S \), if there exists a sequence \( S = S_0, S_1, \ldots, S_k = S' \) of \( k+1 \) distinct members \( S_i \) of \( \text{spec}(M) \), such that for \( i < k \), \( S_i \) is an immediate successor of \( S_{i+1} \); in this situation \( S \) is called a k-th predecessor of \( S' \). \( S' \) is called a successor of \( S \), if \( S' \) is a k-th successor of \( S \) for some positive integer \( k \). Define \( S \sim S' \) if for some \( k \geq 0, S' \) is a k-th successor or k-th predecessor of \( S \). This is an equivalence relation. Any equivalence class \( C \) determined by this relation is called a primary class. For a torsion abelian group, each such \( C \) is a singleton. However for a torsion module over a bounded \((\text{hnp})\)-ring, each \( C \) is finite. For any primary class \( C \) in \( \text{spec}(M) \), the submodule \( M_{C} \) of all those \( x \in M \) such that every composition factor of \( xR \) is in \( C \), is called the C-primary submodule of \( M \). By using (5.1) one can easily see that \( M \) is a direct sum of its C-primary submodules. A module \( M \) is called a primary TAG-module if \( M \oplus M \) is a TAG-module such that \( \text{spec}(M) \) is a primary class. Consider a primary TAG-module \( M \). Let \( \text{spec}(M) \) have \( k \) members, then either \( k \) is finite or countable. This \( k \) is called the periodicity of \( M \). In this section we study primary TAG-modules.

Lemma(5.2). Let \( M_R \) be an h-reduced primary TAG-module of finite periodicity. If there exists a function \( f: Z^+ \to Z^+ \) such that for any uniform \( x \in M \), \( h(x) \leq f(h'(x)) \), then \( M \) is bounded.

Proof. Let \( M \) be of periodicity \( k \). For any uniform \( x \in M \), \( h'(x) < \infty \). This gives \( h(x) \leq f(h'(x)) < \infty \). Suppose \( M \) is not bounded. Then \( M \) has uniserial summands of arbitrarily large lengths. So we can...
write \( M = x_1 R \oplus x_2 R \oplus M' \), with \( x_i R \) non-zero uniserial, \( z R = \soc(x_i R) \), \( h(z) > \max(f(j) : 1 \leq j \leq k + d(x_i R)) \) and \( e(x_2) > k \). Now \( h(z) = [z_2, x_1] \). As \( M \) is of periodicity \( k \) and \( e(x_2) > k \), we get \( y_2 \in x_2 R \) such that \( [z_2, y_2] \leq k - 1 \) and \( \soc(x_2 R/y_2 R) \equiv \soc(x_2 R) \). This gives a maximal \( g \in \SH(x_2 R, x_1 R) \) with \( d(\ker g) = k \) and \( z_2 R \subseteq \ker g \). Consequently \( d(\text{domain}(g)) \leq k + d(x_1 R), h'(z_2) \leq k + d(x_1 R) \). As \( h(z_2) \leq f(h'(z_2)) \), we get \( h(z_2) \leq \max\{ f(j) : 0 \leq j \leq k + d(x_1 R) \} \). This is a contradiction. Hence \( M \) is bounded.

**Lemma (5.3).** Let \( M_R \) be any primary TAG-module of finite periodicity. If every h-neat submodule of \( M \) is h-pure, then either \( M \) is h-divisible or h-reduced.

**Proof.** Let \( M \) be neither h-reduced nor h-divisible. Then \( M = x R \oplus A \oplus M_1 \) for some uniform element \( x \) and a serial module \( A \) of infinite length. Let \( z R = \soc(A) \). Then \( h(z) = \infty \). If the periodicity of \( M \) is \( k \), then for some \( u, 1 \leq u \leq k \), we get a submodule of \( A \) of length \( u \) satisfying \( \soc(A/y R) \equiv \soc(x R) \). By (2.3), we get a maximal \( f \in \SH(A, x R) \) with \( d(\text{domain}(f)) \leq e(x) + u \). This gives an h-neat hull \( K \) of \( z R \) length \( e(x) + u \). As \( K \) is h-pure, we get \( h(z) = d(K) - 1 < \infty \). This is contradiction. Hence the result follows.

**Lemma (5.4).** Let \( M_R \) be a primary TAG-module of finite periodicity \( k \). Let \( T = x R \oplus A \) be a submodule of \( M \), with x R uniserial, such that every h-neat submodule of \( T \) is h-pure in \( T \). Then the following hold:

(i) If \( \soc(x R) \equiv \soc(A) \), then \( d(A) \leq d(x R) + k \).

(ii) If \( \soc(x R) \) is the \( u \)-th predecessor of \( \soc(A) \) for some \( u \geq 1 \), then \( d(A) \leq d(x R) + u \).

**Proof.** Let \( \soc(A) = z R \). Let \( \soc(x R) \equiv z R \). For a maximal \( f \neq 0 \) in \( \SH(A, x R) \) with \( z R \subseteq \ker f \) and \( d(\ker f) \) minimal, we have \( d(\ker f) = k \), \( \text{domain}(f) = y R \subseteq A \); further \( h'(z) = e(y) - 1 = [z, y] \leq e(x) + k - 1 \).

However by (4.8), \( h'(z) = h(z) \). So \( y R = A \). Consequently \( e(y) = d(A) \leq d(x R) + k \). Similarly (ii) follows.

We now prove the first decomposition theorem.

**Theorem (5.5).** Let \( M_R \) be a primary TAG-module of periodicity \( k < \infty \). Then every h-neat submodule of \( M \) is h-pure if and only if either \( M \) is h-divisible or \( M = \bigoplus_{\alpha \in \Lambda} x_{\alpha} R \) such that:

(i) each \( x_{\alpha} R \) is uniserial; and

(ii) for any two distinct \( \alpha, \beta \in \Lambda \) the following hold:

(\( a \)) if \( \soc(x_{\alpha} R) \equiv \soc(x_{\beta} R) \), then \( d(x_{\alpha} R) \leq d(x_{\beta} R) + k \),

(\( b \)) if \( \soc(x_{\alpha} R) \) is a \( u \)-th predecessor of \( \soc(x_{\beta} R) \), \( 1 \leq u \leq k - 1 \), then \( d(x_{\alpha} R) \leq d(x_{\beta} R) + u \).

**Proof.** Let every h-neat submodule of \( M \) be h-pure. By (5.2) \( M \) is either h-divisible or h-reduced. Let \( M \) be h-reduced. By (5.2) \( M \) is bounded. So that \( M = \bigoplus_{\alpha \in \Lambda} x_{\alpha} R \), for some uniserial submodules \( x_{\alpha} R \).

By applying (5.4) we complete the necessity. Conversely let the given conditions be satisfied. If \( M \) is h-divisible, then every h-neat submodule \( N \) of \( M \) being h-divisible, is a summand of \( M \), consequently \( N \) is h-pure. Let \( M \) be h-reduced. Consider a uniform \( z = \sum_{\alpha \in \Lambda} z_{\alpha} \in M \) with \( z_{\alpha} \in x_{\alpha} R \). Then \( h(z) = \min\{ h(z_{\alpha}) : z_{\alpha} \neq 0 \} = \min\{ [z_{\alpha}, x_{\beta}] : z_{\alpha} \neq 0 \} \). Consider \( T = x_R \oplus x_R \) with \( z_{\alpha} \neq 0, z_{\beta} \neq 0 \) and \( \alpha \neq \beta \). Let \( f \in \SH(x_{R}, x_R) \) be maximal with the property that \( z_R \subseteq \ker f \) and \( d(\ker f) \) is minimal. Either \( \text{domain}(f) = x_R \) or \( \text{range}(f) \)
= x_\lambda R. If f = 0, obviously domain(f) = x_\lambda R. Let f \neq 0. If soc(x_\lambda R) \equiv soc(x_\lambda R), then d(\ker f) = \lambda k for some \lambda > 0. If soc(x_\lambda R) \not\equiv soc(x_\lambda R), then for some u \geq 1 soc(x_\lambda R) is the u-th predecessor of soc(x_\lambda R) and d(\ker f) = u + \mu k for some \mu \geq 0. Thus (a) and (b) yield domain(f) = x_\lambda R. Consequently h_1'(z_\mu) = [z_\mu, x_\lambda] = h(z_\mu)

By (4.4), h'_1(z) = h(z). This proves the result.

The periodicity of a torsion abelian p-group is one. We get the following:

Corollary (5.6). Every neat subgroup of an abelian p-group G, p a prime number, is pure subgroup if and only if either G is a divisible group or G = A \oplus B, such that for some positive integer n, A is a direct sum of copies of Z/(p^n) and B is a direct sum of copies of Z/(p^{n+1}).

We now discuss the case of a primary TAG-module of infinite periodicity. Henceforth M will be a primary TAG-module of infinite periodicity.

Lemma (5.7). Let xR and yR be two h-neat uniserial submodules of M such that soc(xR) \neq soc(yR) and soc(yR) is a predecessor of soc(xR). Then:

(i) SH(yR, xR) = 0.
(ii) For any h-neat hull K of xR \oplus yR in M, yR is a summand of K; if in addition xR is h-pure in M, then K = xR \oplus yR.
(iii) If xR and yR both are h-pure, then xR \oplus yR is h-pure in M.

Proof. As M is of infinite periodicity and soc(yR) is a predecessor of soc(xR), soc(xR) is not a predecessor of soc(yR). Consequently SH(yR, xR) = 0. Let K be an h-neat hull of xR \oplus yR. As rank(K) = 2, K = A_1 \oplus A_2 with A_i serial. Consider the projections f_i : A_1 \oplus A_2 \to A_i. The restriction of one of f_i, say of f_1 to xR is a monomorphism. Then soc(xR) \equiv soc(A_1) and soc(yR) \equiv soc(A_2). Further f_2 embeds yR in A_2. By (i) SH(A_2, A_1) = 0. This yields yR \subseteq A_2. As yR \subseteq A_2 and yR is h-neat, we get yR = A_2. Let xR be h-pure in M. So that xR is h-pure in K. Consequently xR is a summand of K. As xR \neq yR, we get K = xR \oplus yR. This proves (ii). Finally let both xR and yR be h-pure in M. Then M = xR \oplus M_1 for some submodule M_1. Then K = xR \oplus (K \cap M_1). This gives yR = K \cap M_1. So K \cap M_1 is h-pure in M_1. Thus K \cap M_1 is a summand of M_1. Hence K is a summand of M. This gives (iii).

Lemma (5.8). Let K be any submodule of M with soc(K) homogeneous. Then:

(i) Given any two uniserial submodules A and B of M, either A \cap B = 0 or \ell_{h_{xy}} are comparable under inclusion,
(ii) for any uniform x \in K, h_K(x) = h_\mu'(x), and
(iii) any h-neat submodule of K is h-pure in K.

Proof. (i) Let A \cap B \neq 0 and d(A) \geq d(B). Then A+B = A \oplus C, with C a proper homomorphic image of B. Suppose C \neq 0, then soc(A) = soc(B) \equiv soc(C). This contradicts the hypothesis that soc(K) is homogeneous. Thus C = 0, so B \subseteq A. This proves (i). Consider a uniform x \in K. Then by (i) xR has unique h-neat hull D in K. Then h_\mu'(x) = d(D)-1. The uniqueness of D gives D is h-pure. Consequently h_K(x) = d(D)-1. This proves (ii). The last part is immediate from (ii).

Corollary (5.9). Let xR and yR be two uniserial submodules of M with xR \cap yR = 0. Every h-neat submodule of T = xR \oplus yR is h-pure in T if and only if
(i) $\text{soc}(x_R) \equiv \text{soc}(y_R)$, or
(ii) $\text{soc}(x_R) \not\equiv \text{soc}(y_R)$, one of them say $\text{soc}(x_R)$ is a $u$-th predecessor of $\text{soc}(y_R)$ for some $u$

$\geq 1$ and $d(y_R) \leq d(x_R) + u$.

Proof. Let every $h$-neat submodule of $T$ be $h$-pure. Let $\text{soc}(x_R) \equiv \text{soc}(y_R)$, and $\text{soc}(x_R)$ be a $u$-th predecessor of $\text{soc}(y_R)$. Then as in (5.4), we get $d(y_R) \leq d(x_R) + u$.

Conversely, let $T$ satisfy the given conditions. If $\text{soc}(x_R) \equiv \text{soc}(y_R)$, then $\text{soc}(T)$ is homogeneous, so by (5.8) every $h$-neat submodule of $T$ is $h$-pure in $T$. Let (ii) hold, $\text{soc}(y_R)$ be a $u$-th predecessor of $\text{soc}(x_R)$. As $M$ is of infinite periodicity, $\text{soc}(x_R)$ is not a predecessor of $y_R$. So $\text{SH}(y_R, x_R) \neq 0$, and for any $0 \neq f \in \text{SH}(x_R, y_R)$, $d(\ker f) = u$. Consider a uniform $z = x_1 + y_1, x_1 \in x_R, y_1 \in y_R$. Let $e(x_1) \geq e(y_1)$. If $y_1 \neq 0$, then $e(x_1) = e(y_1) + u$. So $[x_1, x] \leq [y_1, y]$, and hence $h_T^*(z) = [x_1, x] = h_T(z)$. Suppose $y_1 = 0$. Consider a maximal $f \in \text{SH}(x_R, y_R)$ with $z \in \ker f$. Then either $f = 0$ or $d(\ker f) = u$. As $d(x_R) \leq d(y_R) + u$, domain($f$) = $x_R$. Once again $h_T^*(z) = [x_1, x] = h_T(z)$. Let $e(x_1) > e(y_1)$, then $x_1 = 0$, as $\text{SH}(y_R, x_R) = 0$.

Then for any uniform $v \in T$, with $z \in z_R$, we have $z_i \in y_R$. So $y_R$ is the only $h$-neat hull of $z_R$ in $T$. Thus in all cases $h_T^*(z) = h_T(z)$. By (4.8), the result follows.

Lemma (5.10). Let $N$ be the submodule of $M$ generated by those uniform elements $x \in M$ such that $\text{soc}(x_R)$ has no predecessor in $\text{soc}(M)$. Then:

(i) $\text{Soc}(N)$ is homogeneous.
(ii) $N$ is an $h$-pure submodule of $M$.
(iii) Any $h$-neat submodule of $N$ is $h$-pure in $M$.

Proof. Let $A$ be the set of those uniform $x \in M$ such that $\text{soc}(x_R)$ has no predecessor in $\text{soc}(M)$. For any $x, y \in A$, if $\text{soc}(x_R) \not\equiv \text{soc}(y_R)$, then one of them being a successor of the other, contradicts the hypothesis. So that $\text{soc}(x_R) \equiv \text{soc}(y_R)$ for all $x, y \in A$. Consider a uniform $z = x_1 + y_1, x_1 \in x_R, y_1 \in y_R$. For some $y_i \in A$, $z \in \sum y_i R = \oplus \sum B_j, B_j$'s uniserial. For some $j, z_R \equiv \text{soc}(B_j)$. But $B_j$ is a homomorphic image of some $y_i R$. As $\text{soc}(y_i R)$ has no predecessor in $\text{soc}(M)$, we get $y_i R \equiv B_j$. Hence $\text{soc}(N)$ is homogeneous. It is now immediate that if for any uniform $x \in M, \text{soc}(x_R) \subseteq N$, then $x \in N$. This fact gives (ii). By using (4.8) we get (iii).

The submodule $N$ of $M$ generated by those uniform elements $x \in M$, such that $\text{soc}(x_R)$ has no predecessor in $\text{soc}(M)$ is called a terminal submodule of $M$. We denote this submodule by $\text{Ter}(M)$.

Proposition (5.11). Let $M_\kappa$ be a primary TAG-module of infinite periodicity and $N = \text{Ter}(M)$. Then:

(i) Any submodule $K$ of $N$ has unique $h$-neat hull in $M$.
(ii) for any uniform $x \in N, h(x) = h'_t(x)$; and
(iii) for any decomposition $M = \oplus \sum_{x \in A} A_x, N = \sum (A_x \cap N)$.

Proof. By (5.10) $\text{soc}(N)$ is homogeneous. So given a uniform $x \in N$, by (5.8) any two uniform submodules of $N$ containing $x$ are comparable under inclusion. Thus there is unique $h$-neat hull $A_x$ of $x_R$ in $M$, and $A_x \subseteq N$. For $K$, the sum $L$ of those $A_x$ for which $x \in K$, is the unique $h$-neat hull of $K$. (ii) is immediate from (i). Consider any uniform $x \in N$. Then $x = \sum x_i, x_i \in A_i$. If some $x_i \neq 0$, and the
mapping $xR \to xR$, $xR \to xR$ is not one-to-one, then $soc(xR)$ will have a predecessor in $soc(M)$. This gives a contradiction. Hence $xR \equiv xR$, whenever $x_i \neq 0$. Thus $x_i \in \mathbb{N}$ and (iii) follows.

Theorem(5.12). Let $M_R$ be an $h$-reduced primary TAG-module of infinite periodicity. Then every $h$-neat submodule of $M$ is h-pure if and only if $M = \bigoplus \sum_{i \in \mathbb{N}} K_i \oplus Ter(M)$ satisfying the following conditions:

(i) for each $j$, $K_j$ is decomposable and $soc(K_j)$ is homogeneous,

(ii) for $j < j'$, with $K_j \neq 0 \neq K_{j'}$, if $z_j R$ and $z_{j'} R$ are uniserial summands of $K_j$ and $K_{j'}$ respectively, then $soc(z_{j'} R)$ is a $u$-th predecessor of $soc(z_j R)$ for some positive integer $u$ depending upon $j$ and $j'$, and $d(z_{j'} R) < d(z_j R) + u$; and

(iii) if $t$ is the length of a smallest length uniserial summand of $N$, and $S$ is the simple module determining $soc(N)$, then for any $K_j \neq 0$, if $S$ is a $v_j$-th predecessor of the simple module $S_j$ determining $soc(K_j)$, we have $d(z_{j'} R) < t + v_j$ for any uniserial summand $z_{j'} R$ of $K_j$.

Proof. Let every $h$-neat submodule of $M$ be $h$-pure. Now $N \cong Ter(M)$. As $N$ is $h$-reduced, it has a uniserial summand $xR$ of smallest length, say $t$. Consider $\overline{M} = M/N$. Let $S$ be a simple submodule of $\overline{M}$. Consider any uniform $y \in M$ such that $S = soc(yR)$. By (2.2), we choose $y$ to be uniform such that $yR \cap N = 0$. Then $soc(xR) \cong soc(yR)$. As $soc(xR)$ has no predecessor in $soc(M)$, it is a $v$-th predecessor of $soc(yR) = zR$ for some $v \geq 1$. Now $h(z) = h'(z) < \infty$. We get $y_1 \in M$ such that $[z, y_1] = h(z)$. Then in $xR \oplus y_1 R$, both the summands are $h$-pure in $M$. By (5.7) $xR \oplus y_1 R$ is $h$-pure. By (5.9), $d(y_1 R) \leq d(xR) + v$. So there is an upper bound on the heights of elements of a particular homogeneous component of $soc(\overline{M})$. Hence by (3.4) $\overline{M}$ is its only basic submodule, so it is decomposable. As $N$ is $h$-pure, by the observation following (2.2), we get $M = K \oplus N$, with $K$ its only basic submodule. As $M$ is primary, $spec(M)$ is countable. We get $K = \bigoplus \sum_{i \in \mathbb{N}} K_i$ satisfying (i). Finally (ii) and (iii) follow from (5.9).

Conversely, let $M$ satisfy the given conditions. Then $K$ satisfies conditions analogous to those given in (5.5). So on the similar lines as in (5.5), every $h$-neat submodule of $K$ is $h$-pure. Consider any uniform $y \in M$. Now $y = y_1 + y_2$, $y_1 \in K$, $y_2 \in N$. If $y_1 = 0$, $y \in N$ and by (5.11), $yR$ has unique $h$-neat hull in $M$; obviously then $h(y) = h'(y)$. Let $y_1 \neq 0$. Suppose $y_2 \neq 0$. then by using (4.3) $h'(y) = h(y)$. Suppose $y_2 = 0$ and $h'(y) < h(y)$. We get an $h$-neat hull $zR$ of $yR$ with $[y, z] = h'(y)$. Let $z = z_1 + z_2, z_1 \in K$, $z_2 \in N$. As $h(z_1) = h(z), z_2 \neq 0$. One of $z_1 R$ and $z_2 R$ is $h$-neat. As $yR \cong z_1 R$ and $[y, z_1] < h(y) = h(z)$, $z_1 R$ is not $h$-neat in $K$ and so in $M$. Consequently $z_2 R$ is $h$-neat in $N$, and by (5.10) it is $h$-pure. For some $v$, $soc(z_2 R)$ is a $v$-th predecessor of $soc(z_2 R)$. So that $d(z_1 R) = d(z_2 R) + v$. Write $soc(z_2 R) = gR$. Then by using condition (iii), we get $[g, z_1] \leq d(g) \leq d(z_2 R) + v - 1 = [g, z_1]$. Consequently $d(z_1 R) - 1 = h(g)$. So $z_1 R$ is $h$-pure. This is a contradiction. This completes the proof.

We now discuss the case of $M$ being not necessarily $h$-reduced. Write $M = M_1 \oplus D$, where $D$ is the largest $h$-divisible submodule of $M$.

Lemma(5.13). If every $h$-neat submodule of $M$ is $h$-pure and $D \neq 0$, then $Ter(M) \subset D$; further $Ter(M)$ is $h$-divisible.
Proof. Suppose \( N = \text{Ter}(M) \subseteq D \). We get a uniform \( x \in \text{soc}(\mathfrak{m}) \), such that \( h(x) < \infty \). Then \( x = y + z \). Now \( 0 \neq y \in M_1, z \in D \). By (5.11) \( y \in N \). Consider any simple submodule \( S \) of \( D \). By the definition of \( S \), it is not a predecessor of \( \text{soc}(yR) \). So \( \text{soc}(yR) \) is a predecessor of \( S \). As \( D \) is \( h \)-divisible, there exists a uniserial submodule \( A \) of \( D \) and a homomorphism \( f : A \to M_1 \), with \( \text{range}(f) = yR \) and \( S \subseteq \text{ker}f \). This contradicts (4.8). Hence \( N \subseteq D \). As \( N \) is \( h \)-pure, it must be \( h \)-divisible.

Theorem (5.14) Let \( M_b \) be a primary TAG-module of infinite periodicity such that \( M \) is not \( h \)-divisible and let \( N = \text{Ter}(M) \). Then every \( h \)-neat submodule of \( M \) is \( h \)-pure if and only if the following hold:

(a) \( N \) is \( h \)-divisible, and

(b) \( M = N \oplus \sum_{i=1}^{n} K_i \), where \( K_i \) satisfy the following conditions:

(i) if \( K_i \neq 0 \), then \( \text{soc}(K_i) \) is a homogeneous component of \( \text{soc}(M) \),

(ii) each \( K_i \) is a direct sum of serial modules,

(iii) if \( K_i \neq 0 \), then the simple submodule \( S_i \) determining \( \text{soc}(K_i) \) is a \( v \)-th predecessor of the simple submodule determining \( \text{soc}(K_j) \) for some positive integer \( v \) depending on \( i \) and \( j \), and for any uniserial submodule \( A \) of \( K_j \), \( d(A) \leq t + v \), where \( t \) is the length of the smallest length uniserial summand of \( K_j \), and

(iv) if \( K_j \neq 0 \) and is not \( h \)-reduced, then for any \( i < j \), \( K_i \) is \( h \)-divisible.

Proof. Let \( D \) be the largest \( h \)-divisible submodule of \( M \). Then \( D \neq 0 \). Let every \( h \)-neat submodule of \( M \) be \( h \)-pure. By (5.13) \( N \) is \( h \)-divisible. Thus \( M = N \oplus M_1 \oplus M_2 \) such that \( D = N \oplus M_1 \) and \( M_2 \) is \( h \)-reduced. By applying (5.12) to \( M_2 \) and using the fact that \( M_1 \) is a direct sum of serial modules, we get \( M_1 \oplus M_2 = \oplus \sum_{i=1}^{n} K_i \) satisfying (i), (ii), and (iii). Finally (iv) is an immediate consequence of (iii).

Conversely let the given conditions be satisfied. By comparing these conditions with those in (5.12), we get \( M = D \oplus L \) such that \( N \subseteq D \). Then \( \text{SH}(N, L) = 0 \). Consider a uniserial submodule \( W \) of \( D \) and let \( f : W \to L \) be a non-zero homomorphism. Then \( W \subseteq N \). For some \( j \), \( W \) is isomorphic to a submodule of \( K_j \). This \( K_j \) is not \( h \)-reduced, \( f(W) \subseteq L \), and for some \( t \), \( f(W) \) is isomorphic to a submodule of \( K_t \). If \( j = t \), obviously \( f \) is a monomorphism. Suppose \( t < j \) Then \( K_j \) is \( h \)-divisible. Let \( xR = \text{soc}(f(W)) \).

As \( xR \subseteq L \), \( h(x) < \infty \). On the other hand \( x \in \text{soc}(K_t) \) yields \( h(x) = \infty \). This is a contradiction. Hence \( j < t \).

So \( \text{SH}(K_t, K_j) = 0 \). This once again contradicts the fact that \( f \neq 0 \). Thus \( j = t \). Hence \( f \) is a monomorphism. So by (4.8) the result follows.

We end this paper by giving an example of an \( h \)-reduced primary TAG-module \( M \) of which every \( h \)-neat submodule is \( h \)-pure, but it is not decomposable. Such a module has to be of infinite periodicity.

Example. Let \( F \) be a Galois field and \( R \) be the ring of infinite lower triangular matrices \( [a_{ij}] \) over \( F \), where \( i, j \) are indexed over the set \( P \) of all positive integers. Let \( \{ e_i : i, j \in P \} \) be the usual set of matrix units in \( R \). Then \( M_k = e_k R \) is a uniserial \( R \)-module with \( d(M_k) = k \); it is annihilated by the ideal \( A_k \) of \( R \) consisting of those \( [a_{ij}] \in R \), such that \( a_{ij} = 0 \) for \( i \leq k \). Observe that each \( R/A_k \) is isomorphic to the ring of \( k \times k \) lower triangular matrices over \( F \). So that any \( R/A_k \)-module is a TAG-module. Each \( M_k \) embeds in
$M_{k+1}$ under the mapping $e_k \times r \rightarrow e_{k+1} \times r$, $r \in R$. Let $T = \Pi_k M_k$. Then $M_k = \{x \in T : xA_k = 0$ for some $k\}$ is a primary TAG-module of infinite periodicity. Its socle is homogeneous. By (58) every $h$-neat submodule of $M$ is $h$-pure. Let $u$ be the smallest integer such that $x_u \neq 0$. Then $xR \equiv x_uR$. As $d(M_u) = u$, by using (2.3) it can be easily seen that $h(x) = u-1$. So that for any $i > 1$, $\text{soc}(H_{i+1}(M))/\text{soc}(H_i) \equiv \text{soc}(M_i)$. Suppose that $M$ is decomposable, then $M = \bigoplus_{j=1}^{\infty} N_j$, where $N_j$ is a direct sum of uniserial modules of length $j$. Then

$\text{soc}(H_{i+1}(M))/\text{soc}(H_i(M)) \equiv \text{soc}(N_i)$. Thus $\text{soc}(N_i) \equiv \text{soc}(M_i)$, a simple module. Consequently each $N_i$ is a uniserial module. As $F$ is finite, $N_i$ is a finite set. Consequently $M$ is countable. But by construction $M$ is uncountable. This is a contradiction. Hence $M$ is not decomposable.

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Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

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