A NEW PROOF OF MONOTONICITY FOR EXTENDED MEAN VALUES

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ABSTRACT. In this article, a new proof of monotonicity for extended mean values is given.

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1. Introduction. Stolarsky [14] first defined the extended mean values $E(r,s;x,y)$ and proved that it is continuous on the domain $\{(r,s;x,y) : r,s \in R, x,y > 0\}$ as follows

$$E(r,s;x,y) = \left( \frac{r \cdot y^s - x^s}{y^r - x^r} \right)^{1/(s-r)} , \quad rs(r-s)(x-y) \neq 0; \quad (1.1)$$

$$E(r,0;x,y) = \left( \frac{y^r - x^r}{\ln y - \ln x} \cdot \frac{1}{r} \right)^{1/r} , \quad (r(x-y)) \neq 0; \quad (1.2)$$

$$E(r,r;x,y) = e^{-1/r} \left( \frac{x^r}{y^r} \right)^{1/(x^r - y^r)} , \quad (r(x-y)) \neq 0; \quad (1.3)$$

$$E(0,0;x,y) = \sqrt{xy}, \quad x \neq y; \quad (1.4)$$

$$E(r,s;x,x) = x, \quad x = y. \quad (1.5)$$

It is convenient to write $E(r,s;x,y) = E(r,s) = E(x,y) = E$.

Several authors including Leach and Sholander [2, 3], Páles [6] and Yao and Cao [15] studied the basic properties, monotonicity and comparability of the mean values $E$. Feng Qi [9] and in collaboration with Qiu-mig Luo [7] further investigated monotonicity of $E$ from new viewpoints. Recently, Feng Qi [7] generalized the extended mean values and the weighted mean values [1, 4, 5] as a new concept of generalized weighted mean values with two parameters, and studied its monotonicity and other properties.

In this note, a new proof of monotonicity for extended mean values is given.

2. Lemmas. Let

$$g = g(t) - g(t;x,y) = y^t - x^t/t, t \neq 0; \quad (2.1)$$

$$g(0;x,y) = \ln y - \ln x.$$
It is easy to see that \( g \) can be expressed in integral form as

\[
g(t; x, y) = \int_x^y u^{t-1} \, du, \quad t \in \mathbb{R},
\]

and

\[
g^{(n)}(t) = \int_x^y (\ln u)^n u^{t-1} \, du, \quad t \in \mathbb{R}.
\]

Therefore, the extended mean values can be represented in terms of \( g \) by

\[
E(r, s; x, y) = \left( \frac{g(s; x, y)}{g(r; x, y)} \right)^{1/(s-r)}, \quad (r-s)(x-y) \neq 0;
\]

\[
E(r, r; x, y) = \exp \left( \frac{g'(r; x, y)}{g(r; x, y)} \right), \quad x-y \neq 0.
\]

Set \( F = F(r, s) = F(x, y) = F(r, s; x, y) = \ln E(r, s; x, y) \), then \( F \) also can be expressed as

\[
F(r, s; x, y) = \frac{1}{s-r} \int_r^s \frac{g'_t(t; x, y)}{g(t; x, y)} \, dt, \quad r-s \neq 0;
\]

\[
F(r, r; x, y) = \frac{g'_r(r; x, y)}{g(r; x, y)}.
\]

**Lemma 2.1.** Assume that the derivative \( f''(t) \) exists on an interval \( I \). If \( f(t) \) is an increasing or convex downward function respectively on \( I \), then the arithmetic mean of \( f(t) \),

\[
\phi(r, s) = \frac{1}{s-r} \int_r^s f(t) \, dt,
\]

\[
\phi(r, r) = f(r),
\]

is also increasing or convex downward respectively with \( r \) and \( s \) on \( I \).

**Proof.** Direct calculation yields

\[
\frac{\partial \phi(r, s)}{\partial s} = \frac{1}{(s-r)^2} \left[ (s-r)f(s) - \int_r^s f(t) \, dt \right],
\]

\[
\frac{\partial^2 \phi(r, s)}{\partial s^2} = \frac{(s-r)^2 f'(s) - 2(s-r)f(s) + 2 \int_r^s f(t) \, dt}{(s-r)^3} \equiv \frac{\phi(r, s)}{(s-r)^3},
\]

\[
\frac{\partial \phi(r, s)}{\partial s} = (s-r)^2 f''(s).
\]

In the case of \( f'(t) \geq 0 \), \( \partial \phi(r, s)/\partial s \geq 0 \), thus, \( \phi(r, s) \) increases with \( r \) and \( s \), since \( \phi(r, s) = \phi(s, r) \).

In the case of \( f''(t) \geq 0 \), \( \phi(r, s) \) increases with \( s \). Since \( \phi(r, r) = 0 \), it is easy to see that \( \partial^2 \phi(r, s)/\partial s^2 \geq 0 \) holds. Therefore, \( \phi(r, s) \) is convex downward with respect to either \( r \) or \( s \), since \( \phi(r, s) = \phi(s, r) \). \[\Box\]
**Lemma 2.2.** Let \( f, h : [a, b] \to \mathbb{R} \) be integrable functions, both increasing or both decreasing. Furthermore, let \( p : [a, b] \to \mathbb{R} \) be an integrable and nonnegative function. Then

\[
\int_a^b p(u) f(u) du \int_a^b p(u) h(u) du \leq \int_a^b p(u) du \int_a^b p(u) f(u) h(u) du. \tag{2.8}
\]

If one of the functions of \( f \) or \( h \) is nonincreasing and the other nondecreasing, then the inequality in (2.8) is reversed.

The inequality (2.8) is called Tchebycheff’s integral inequality; for details, see [1, 4].

**Lemma 2.3.** Let \( i, j, k \in \mathbb{N} \), we have

\[
g((2i+k+1) + 1) (t; x, y) g((2j+k+1) + 1) (t; x, y) \leq g((2k) + 1) (t; x, y) g((2i+j+k+1)) (t; x, y). \tag{2.9}
\]

If \( x, y \geq 1 \), then

\[
g((i+k)) (t; x, y) g((j+k)(t; x, y) \leq g((k)) (t; x, y) g((i+j+k)(t; x, y). \tag{2.10}
\]

If \( 0 < x, y \leq 1 \), then

\[
g((2i+k+1) + 1) (t; x, y) g((2j+k+1) + 1) (t; x, y) \leq g((2k) + 1) (t; x, y) g((2i+j+k+1)) (t; x, y); \tag{2.11}
\]

\[
g((2i+k+1) + 1) (t; x, y) g((2j+k+1) + 1) (t; x, y) \geq g((2k) + 1) (t; x, y) g((2i+j+k+1)) (t; x, y); \tag{2.12}
\]

\[
g((2i+k)) (t; x, y) g((2j+k)) (t; x, y) \leq g((2k)) (t; x, y) g((2i+j+k+1)) (t; x, y). \tag{2.13}
\]

**Proof.** By Tchebycheff’s integral inequality (2.8) applied to the functions \( p(u) = (\ln u)^2 u^{t-1}, f(u) = (\ln u)^{2i+1} \) and \( h(u) = (\ln u)^{2j+1} \) for \( i, j, k \in \mathbb{N} \), \( u \in [x, y] \), \( t \in \mathbb{R} \), inequality (2.9) follows easily.

By the same arguments, inequalities (2.10), (2.11), (2.12), and (2.13) also follow from Tchebycheff’s integral inequality.

**Lemma 2.4.** The functions \( g_{(2k+i+1)} (t; x, y) / g_{(2k)} (t; x, y) \) are increasing with respect to \( t, x, \) and \( y \) for \( i \) and \( k \) being nonnegative integers.

**Proof.** By simple computation, we have

\[
\left( \frac{g((2k+i+1)(t))}{g((2k)(t))} \right)' = \frac{g((2i+k+1))(t)g((2k)(t)) - g((2k+i+1))(t)g((2k+1)(t))}{[g((2k)(t))]^2}. \tag{2.14}
\]

Combining (2.9) and (2.14), we conclude that the derivative of \( g((2k+i+1)(t)) / g((2k)(t)) \) with respect to \( t \) is nonnegative, and \( g((2k+i+1)(t; x, y)) / g((2k)(t; x, y)) \) increases with \( t \).
\[
\frac{\partial}{\partial y} \left( \frac{g^{(2(k+i)+1)}_t(t;x,y)}{g^{(2k)}_t(t;x,y)} \right) = \frac{\partial}{\partial y} \left[ g^{(2(k+i)+1)}_t(t;x,y) g^{(2k)}_t(t;x,y) - g^{(2(k+i)+1)}_t(t;x,y) \frac{\partial}{\partial y} g^{(2k)}_t(t;x,y) \right] \\
= \frac{y^{t-1}(\ln y)^{2k}}{[g^{(2k)}_t(t;x,y)]^2} \left[ (\ln y)^{2i+1} \int_x^y (\ln u)^{2k} u^{t-1} du - \int_x^y (\ln u)^{2(i+k)+1} u^{t-1} du \right] \geq 0.
\]

(2.15)

Therefore, the desired monotonicity with respect to both \(x\) and \(y\) follows, for the involved functions are symmetric in \(x\) and \(y\). This completes the proof.

\[ \square \]

3. Proof of monotonicity

**Theorem 3.1.** The extended mean values \(E(r,s;x,y)\) are increasing with respect to both \(r\) and \(s\), or to both \(x\) and \(y\).

**Proof.** This is a simple consequence of Lemma 2.1 and Lemma 2.3 in combination with its integral forms (2.4) and (2.5) of \(E(r,s;x,y)\).

**Remark 1.** It may be pointed out that the method used in this paper could yield more general results (see [4, 12], and so on).

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**References**


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Space dynamics is a very general title that can accommodate a long list of activities. This kind of research started with the study of the motion of the stars and the planets back to the origin of astronomy, and nowadays it has a large list of topics. It is possible to make a division in two main categories: astronomy and astrodynamics. By astronomy, we can relate topics that deal with the motion of the planets, natural satellites, comets, and so forth. Many important topics of research nowadays are related to those subjects. By astrodynamics, we mean topics related to spaceflight dynamics.

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