NONLINEAR FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS IN HILBERT SPACE

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Abstract. Let $X$ be a Hilbert space and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial \Omega$. We establish the existence and norm estimation of solutions for the parabolic partial functional integro-differential equation in $X$ by using the fundamental solution.

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1. Introduction. Let $X$ be a Hilbert space and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial \Omega$. We consider the following parabolic partial functional integro-differential equation.

$$
\frac{\partial u}{\partial t} = A_0 u(t, x) + A_1 u(t - h, x) + \int_{-h}^{0} a(s) A_2 u(t + s, x) \, ds$$

$$+ \int_{0}^{t} \left\{ k(t, s) G(s, u(s - h), x) + H(t, s, u(s - h, x)) \right\} \, ds$$

$$+ F(t, u(t - h, x)) + f(t, x), \quad 0 < t \leq T, \quad x \in \Omega,$$

where $A_i (i = 0, 1, 2)$ are elliptic differential operators, $f$ is a forcing function, $h > 0$ is a delay time, $a(s)$ is a real scalar function on $[-h, 0], G, H,$ and $F$ are nonlinear functions, and $k$ is a kernel. The boundary condition attached to (1.1) is, e.g., given by the Dirichlet boundary condition

$$u|_{\partial \Omega} = 0, \quad 0 < t \leq T,$$

and the initial condition is given by

$$u(\theta, x) = g(\theta, x), \quad \theta \in [-h, 0], \quad x \in \Omega.$$  (1.3)

From [4], the above mixed problems (1.1), (1.2), and (1.3) can be formulated abstractly as

$$
\frac{du(t)}{dt} = A_0 u(t) + A_1 u(t - h) + \int_{-h}^{0} a(s) A_2 u(t + s) \, ds$$

$$+ \int_{0}^{t} \left\{ k(t, s) G(s, u_s) + H(t, s, u_s) \right\} \, ds$$

$$+ F(t, u_t) + f(t), \quad 0 < t \leq T,$$

$$u(\theta) = g(\theta), \quad \theta \in [-h, 0].$$  (1.5)
where the state $u(x)$ of the system (1.5) lies in an appropriate Hilbert space and $A_i(i = 0, 1, 2)$ are unbounded operators associated with $s_i(i = 0, 1, 2)$, respectively. Next, we explain the notation $u_1$ in (1.5). Let $I = [-h, 0]$. If a function $u(t)$ is continuous from $I \cup [0, T]$ into a Hilbert space $X$, then $u_t$ is an element in $C = C([-h, 0]; X)$, which has the point-wise definition
\[
 u_t(\theta) = u(t + \theta) \quad \text{for } \theta \in I. \tag{1.6}
\]

Let $\Delta_T = \{(s, t); 0 \leq s \leq t \leq T\}$. We assume in (1.5) that $G : [0, T] \times C \rightarrow X$, $H : \Delta_T \times C \rightarrow X$, $F : [0, T] \times C \rightarrow X$ and the kernel $k : \Delta_T \rightarrow R$ ($R$ denotes the set of real numbers) are continuous, $f : [0, T] \rightarrow V^*$ with some enlarged space $V^* \supset H$ and $g : [-h, 0] \rightarrow V$ with some dense subspace $V \subset H$. It is assumed that the inclusions $V \subset H \subset V^*$ are continuous and $V^*$ is the dual space of $V$.

Many authors [2, 8] studied the following delay differential equation:
\[
 \frac{du(t)}{dt} = A_0 u(t) + A_1 u(t-h) + \int_{-h}^{0} a(s) A_2 u(t+s) \, ds + f(t), \quad \text{a.e. } t \geq 0,
\]
\[
 u(\theta) = g(\theta), \quad \theta \in [-h, 0].
\]

The fundamental solution is constructed in Tanabe [8]. In this paper, we establish the existence and norm estimation of solutions for the equation (1.5) by using the fundamental solution.

2. Preliminaries. Let $H$ be a pivot complex Hilbert space and $V$ be a complex Hilbert space such that $V$ is dense in $H$ and the inclusion map $i : V \rightarrow H$ is continuous. The norms of $H, V$, and the inner product of $H$ are denoted by $| \cdot |$, $\| \cdot \|$, and $\langle \cdot, \cdot \rangle$, respectively. Identifying the antidual of $H$ with $H$, we may consider that $V \subset H \subset V^*$. The norm of the dual space $V^*$ is denoted by $\| \cdot \|_*$. We consider the following linear functional differential equation on the Hilbert space $H$.
\[
 \frac{du(t)}{dt} = A_0 u(t) + A_1 u(t-h) + \int_{-h}^{0} a(s) A_2 u(t+s) \, ds + f(t), \quad \text{a.e. } t \geq 0,
\]
\[
 u(0) = g^0, \quad u(s) = g^1(s), \quad \text{a.e. } s \in [-h, 0].
\]

Let $a(u, v)$ be a bounded sesquilinear form defined in $V \times V$ satisfying Gårding’s inequality
\[
 \text{Re } a(u, u) \geq c_0 \|u\|^2 - c_1 |u|^2, \tag{2.2}
\]
where $c_0 > 0$ and $c_1 \geq 0$ are real constants. Let $A_0$ be the operator associated with this sesquilinear form
\[
 \langle v, A_0 u \rangle = -a(u, v), \quad u, v \in V, \tag{2.3}
\]
where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $V$ and $V^*$. The operator $A_0$ is bounded linear from $V$ into $V^*$. The realization of $A_0$ in $H$, which is the restriction of $A_0$ to the domain $D(A_0) = \{ u \in V : A_0 u \in H \}$, is also denoted by $A_0$. It is proved in Tanabe [6] that $A_0$ generates an analytic semigroup $e^{tA_0} = T(t)$ both in $H$ and $V^*$ and that $T(t) : V^* \rightarrow V$ for each $t > 0$. Throughout this paper, it is assumed that each $A_i(i = 1, 2)$ is bounded and linear from $V$ to $V^*$ (i.e., $A_i \in \mathcal{L}(V, V^*)$) such that $A_i$ maps $D(A_0)$ into $H$.  

The fundamental solution is constructed in Tanabe [8]. In this paper, we establish the existence and norm estimation of solutions for the equation (1.5) by using the fundamental solution.
endowed with the graph norm of $A_0$ to $H$ continuously. The real valued scalar function $a(s)$ is assumed to be Hölder continuous on $[-h,0]$. We introduce a Stieltjes measure $\eta$ given by

$$\eta(s) = -\chi_{(-\infty,-h]}(s)A_1 - \int_{s}^{0} a(\xi) d\xi A_2 : V \rightarrow V^*, \quad s \in [-h,0],$$

(2.4)

where $\chi_{(-\infty,-h]}$ denotes the characteristic function of $(-\infty,-h]$. Then the delay term in (2.1) is written simply as $\int_{-h}^{0} d\eta(s)u(t+s)$. The fundamental solution $W(t)$ of (2.1) is defined as a unique solution of

$$W(t) = \begin{cases} T(t) + \int_{0}^{t} T(t-s)\int_{-h}^{0} d\eta(s)W(s+\xi) ds, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

(2.5)

and $W(t)$ is constructed by Tanabe [7] under the Hölder continuity of $a(s)$.

**Theorem 2.1** [2]. The fundamental solution $W(t)$ is strongly continuous in $V,H,$ and $V^*$, and for each $t > 0$, $W(t) : V^* \rightarrow V$. Furthermore, $W(t)$ satisfies

$$\frac{d}{dt} W(t) = A_0 W(t) + \int_{-h}^{0} d\eta(s)W(t+s), \quad \text{a.e. } t > 0.$$ 

(2.6)

For each $t > 0$, we define the operator valued function $U_t(\cdot)$ by

$$U_t(s) = \int_{-h}^{s} W(t-s+\xi) d\eta(\xi) : V \rightarrow V, \quad \text{a.e. } s \in [-h,0].$$

(2.7)

Let $T > 0$ be fixed. Associated with $U_t(\cdot)$, we consider the operator $\mathcal{U} : L^2(-h,0;V) \rightarrow L^2(0,T;V)$ defined by

$$(\mathcal{U}g)(t) = \int_{-h}^{0} U_t(s)g^1(s) ds, \quad t \in [0,T]$$

(2.8)

for $g^1 \in L^2(-h,0;V)$.

**Theorem 2.2** [8]. Let $T > 0$ be fixed. Assume that $f \in L^2(0,T;V^*)$ and $g = (g^0,g^1) \in H \times L^2(-h,0;V)$. Then there exists a unique solution $u(t) = u(t;f,g)$ of (2.1) on $[0,T]$ satisfying

$$u \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H).$$

(2.9)

Further, for each $T > 0$, there is a constant $K_T$ such that

$$\int_{0}^{T} \|u(t)\|^2 dt + \int_{0}^{T} \|u(t)\|_{*}^2 dt \leq K_T \left( \|g^0\|^2 + \int_{-h}^{0} \|g^1(s)\|^2 ds + \int_{0}^{T} \|f(t)\|_{*}^2 dt \right).$$

(2.10)

This solution $u(t)$ is represented by

$$u(t;f,g) = W(t)g^0 + (\mathcal{U}g^1)(t) + \int_{0}^{t} W(t-s)f(s) ds.$$ 

(2.11)

In what follows, in order to consider the solutions in the state space $C = C([-h,0];H)$, we assume that $g = (g^0,g^1)$ is continuous in $H$, i.e.,

$$g(0) = g^0, \quad g(\cdot) = g^1(\cdot) \in C([-h,0];H).$$

(2.12)
Let
\[ \hat{u}(t; f, g) = \begin{cases} u(t; f, g), & t \in [0, T], \\ g(t), & t \in [-h, 0]. \end{cases} \] (2.13)

Then, by Theorem 2.2, we get
\[ \hat{u}(\cdot; f, g) \in C([-h, T]; H) \] (2.14)
if (2.12) is satisfied.

3. Existence and uniqueness of functional integro-differential equations. Using the fundamental solution \( W(t) \) in Section 2, we consider the following abstract functional integral equation.

\[
v(t) = u(t; f, g) + \int_0^t W(t-s) \left[ \int_0^s \{ k(s, \tau)G(\tau, v_{\tau}) \\ + H(s, \tau, v_{\tau}) \} d\tau + F(s, v_s) \right] ds, \quad 0 < t \leq T,
\]

\[
v(\theta) = g(\theta), \quad \theta \in [-h, 0],
\]
for \( t, s \in [0, T], \phi, \overline{\phi} \in C. \)

(A1) The nonlinear functions \( G : [0, T] \times C \to H, H : \Delta_T \times C \to H, F : [0, T] \times C \to H, \)

(A2) Let \( b_1, b_3 : [0, T] \to \mathbb{R}, b_2 : \Delta_T \to \mathbb{R}^+ \) be continuous functions such that
\[
|G(t, \phi) - G(t, \overline{\phi})|_X \leq b_1(t) |\phi - \overline{\phi}|_C;
\]
\[
|H(t, s, \phi) - H(t, s, \overline{\phi})|_X \leq b_2(t, s) |\phi - \overline{\phi}|_C;
\]
\[
|F(t, \phi) - F(t, \overline{\phi})|_X \leq b_3(t) |\phi - \overline{\phi}|_C
\]
(3.2)

(A3) The function \( k(t, s) \) is Hölder continuous with exponent \( \alpha \), i.e., there exists a positive constant \( a \) such that
\[
|k(t_1, s_1) - k(t_2, s_2)| \leq a (|t_1 - t_2|^\alpha + |s_1 - s_2|^\alpha)
\]
(3.3)

(A4) For all \( 0 \leq s \leq t \leq T, \)
\[
G(t, 0) = 0, \quad H(t, s, 0) = 0, \quad F(t, 0) = 0.
\]
(3.4)

**Theorem 3.1.** Let \( f \in L^2(0, T; V^*) \) and \( g = (g(0), g(\cdot)) \in H \times L^2(-h, 0; V) \) satisfy (2.12). Assume that the hypotheses (A1)-(A4) hold. Then there exists a time \( t_1 > 0 \) such that the functional integral equation (3.1) admits a unique solution \( v(t) \) on \([0, t_1] \).

**Proof.** We prove this theorem by using the method of successive approximations. Set \( v^0(t) = u(t; f, g), t \geq 0 \). Let \( \tilde{v}^0(t) \) be the extension of \( v^0(t) \) on \([-h, T] \) by (2.13). Then, by the assumptions on \( f \) and \( g \), we have \( \tilde{v}^0(t) \in C([-h, T]; H) \). By hypotheses (A1)-(A4), we define \( \{ \tilde{v}^n \}_{n=0}^\infty \subset C([-h, T]; H) \) successively by

\[ \tilde{v}^{n+1}(t) = u(t; f, g) + \int_0^t W(t-s) \left[ \int_0^s \{ k(s, \tau)G(\tau, \tilde{v}^n_{\tau}) \\ + H(s, \tau, \tilde{v}^n_{\tau}) \} d\tau + F(s, \tilde{v}^n_s) \right] ds, \quad 0 < t \leq T,
\]

\[
\tilde{v}^n(\theta) = g(\theta), \quad \theta \in [-h, 0],
\]
for \( t, s \in [0, T], \phi, \overline{\phi} \in C. \)
\[ \hat{v}^n(t) = u(t; f, g) \]
\[ + \int_0^t W(t-s) \left[ \int_0^s \{ k(s, \tau)G(\tau, \hat{v}_t^{n-1}) + H(s, \tau, \hat{v}_\tau^{n-1}) \} d\tau + F(s, v_s^{n-1}) \right] ds, \quad 0 < t \leq T, \]
\[ \hat{v}^n(\theta) = g(\theta), \quad \theta \in [-h, 0]. \]

It is obvious that \( M = \sup_{t \in [0, T]} \| W(t) \|_L(H) \) is finite and that
\[ \hat{v}^{n+1}(\theta) - \hat{v}^n(\theta) = 0, \quad \theta \in [-h, 0]. \]

For \( 0 \leq t \leq T \), we have, by (A1)-(A4) and the strong continuity of \( W(t) \) on \([0, T]\),
\[ |\hat{v}^{n+1}(t) - \hat{v}^n(t)| \]
\[ = \left| \int_0^t W(t-s) \left[ \int_0^s \{ k(s, \tau)G(\tau, \hat{v}_t^{n}) + H(s, \tau, \hat{v}_\tau^{n}) \} d\tau + F(s, v_s^{n}) \right] ds \right| \]
\[ - \int_0^t W(t-s) \left[ \int_0^s \{ k(s, \tau)G(\tau, \hat{v}_t^{n-1}) + H(s, \tau, \hat{v}_\tau^{n-1}) \} d\tau + F(s, v_s^{n-1}) \right] ds \]
\[ \leq M \int_0^t \left[ \int_0^s |k(s, \tau)| |G(\tau, \hat{v}_t^{n}) - G(\tau, \hat{v}_\tau^{n-1})| d\tau ds \right] \]
\[ + \int_0^t |H(s, \tau, \hat{v}_\tau^{n}) - H(s, \tau, \hat{v}_\tau^{n-1})| d\tau \]
\[ + M \int_0^t |F(s, v_s^{n}) - F(s, v_s^{n-1})| ds \]
\[ \leq M \int_0^t \left[ \int_0^s \left( |k(s, \tau)| |b_1(\tau)||\hat{v}_t^{n} - \hat{v}_\tau^{n-1}| + |b_2(s, \tau)| ||\hat{v}_t^{n} - \hat{v}_\tau^{n-1}| \right) d\tau ds \right] \]
\[ + M \int_0^t |b_3(s)| ||\hat{v}_t^{n} - \hat{v}_\tau^{n-1}| |ds \]
\[ \leq M \int_0^t KL_1 + L_2 ||\hat{v}_t^{n} - \hat{v}_\tau^{n-1}| |s ds + M \int_0^t L_3 ||\hat{v}_t^{n} - \hat{v}_\tau^{n-1}| |ds \]
\[ \leq \left[ M(KL_1 + L_2) + L_3 \right] ||\hat{v}^n - \hat{v}^{n-1}| |_{C([-h, T]; H)} \]
\[ = (c_1 t + c_2) \rightarrow \frac{1}{c_2} \rightarrow \frac{1}{c_1} < 1. \]

where \( c_1 = (1/2)M(KL_1 + L_2) \) and \( c_2 = ML_3 \). We now choose a sufficiently small constant \( t_1 > 0 \) such that
\[ L = (c_1 t_1 + c_2) t_1 < 1. \]

Then by (3.6), (3.8), and (3.9), we get
\[ ||\hat{v}^{n+1} - \hat{v}^n||_{C([-h, T]; H)} \leq L ||\hat{v}^n - \hat{v}^{n-1}| |_{C([-h, T]; H)} \]
\[ \leq L^n ||\hat{v}^1 - \hat{v}^0||_{C([-h, T]; H)}. \]

This implies that \( \{ \hat{v}^n \}_{n=0}^\infty \) converges uniformly to some \( \hat{v} \in C([ -h, 0]; H) \). Therefore,
\[ \lim_{n \to \infty} \sup_{t \in [0, t_1]} ||\hat{v}_t^n - \hat{v}_t||_{C([-h, 0]; H)} = 0. \]
Hence, by letting \( n \to \infty \) in (3.5), in view of (A1)–(A4) and (3.11), we get
\[
\hat{v}(t) = u(t; f, g)
+ \int_{0}^{t} \left[ W(t-s) \left\{ k(s, \tau) G(\tau, \hat{v}_{\tau}) + H(s, \tau, \hat{v}_{\tau}) \right\} d\tau \right] ds,
\quad 0 < t \leq t_{1},
\]
\[
\hat{v}(\theta) = g(\theta), \quad \theta \in [-h, 0].
\] (3.12)
This shows the local existence of a solution \( v(t) = \hat{v}(t) \mid_{[0,t_{1}]} \) of (3.1) on \([0,t_{1}]\). Let \( v_{1} \) and \( v_{2} \) be the solution of (3.1) on \([0,t_{1}]\). Then it is easy to see, similarly to the above, that
\[
\| \hat{v}^{1} - \hat{v}^{2} \|_{C([-h,t_{1}]; H)} \leq L \| \hat{v}^{1} - \hat{v}^{2} \|_{C([-h,t_{1}]; H)},
\] (3.13)
so that by \( L < 1, v_{1}(t) = v_{2}(t) \) on \([0,t_{1}]\). This proves the uniqueness. \( \Box \)

Since \( k(s, \tau) G(\tau, \hat{v}_{\tau}), H(s, \tau, \hat{v}_{\tau}), F(s, v_{s}) \in L^{2}(0,t_{1}; H) \subset L^{2}(0,t_{1}; V^{*}) \), by Theorem 2.1, we see that the solution \( v(t) \) of (3.1) satisfies
\[
\frac{d v(t)}{dt} = A_{0}v(t) + A_{1}v(t-h) + \int_{-h}^{t} a(s)A_{2}v(t+s)ds
+ \int_{0}^{t} \left\{ k(t,s) G(s,v_{s}) + H(t,s,v_{s}) \right\} ds
+ F(t,v_{t}) + f(t), \quad \text{a.e. } t \in [0,t_{1}],
\]
\[
v(\theta) = g(\theta), \quad \theta \in [-h, 0],
\] (3.14)
and \( v \in L^{2}(0,t_{1}; V) \cap W^{1,2}(0,t_{1}; V^{*}) \). In this sense, we call this \( v \) a mild solution of (1.5) on \([0,t_{1}]\). We give a norm estimation of the mild solution of (1.5) and establish the global existence of solutions with the aid of norm estimations. It is well known (cf. Lions and Magenes [3, Prop. 2.1, Thm. 3.1]) that the inclusion \( L^{2}(0,T; V) \cap W^{1,2}(0,T; V^{*}) \subset C([0,T]; H) \) is continuous, that is, there exists a constant \( c_{0} \) such that
\[
\| u \|_{C([0,T]; H)} \leq c_{0} \left( \| u \|_{L^{2}(0,T; V)} + \left\| \frac{d u}{dt} \right\|_{L^{2}(0,T; V^{*})} \right)
\] (3.15)
for all \( u \in L^{2}(0,T; V) \cap W^{1,2}(0,T; V^{*}) \).

**Lemma 3.1** [5]. Let \( a(t), b(t), \) and \( c(t) \) be real valued nonnegative continuous functions defined on \( R^{+} \), for which the inequality
\[
c(t) \leq c_{0} + \int_{0}^{t} a(s)c(s)ds + \int_{0}^{t} \left\{ \int_{0}^{s} b(\tau)c(\tau) d\tau \right\} ds
\] (3.16)
holds for all \( t \in R^{+} \), where \( c_{0} \) is a nonnegative constant. Then
\[
c(t) \leq c_{0} \left( 1 + \int_{0}^{t} a(s) \exp \left\{ \int_{0}^{s} (a(\tau) + b(\tau)) d\tau \right\} ds \right) \quad \text{for all } t \in R^{+}.
\] (3.17)
By using Lemma 3.1, we get

\[ \|v(t)\|_{L^2([-h,0];V)} + \|f\|_{L^2(0,T;V^*)} \leq c \left( \|g(0)\| + \|g\|_{L^2([-h,0);V)} + \|f\|_{L^2(0,T;V^*)} \right) e^{Kt}, \quad (3.18) \]

where \( c \) is a positive constant which does not depend on \( v \).

**Proof.** From hypotheses (A1)-(A4), we have

\[ |v(t) + \theta; f, g)| \]

\[ \leq |u(t + \theta; f, g)| + \left| \int_{t}^{t+\theta} W(t + \theta - s) \times \left[ \int_{s}^{t} [k(s, \tau)G(\tau, v_{\tau}) + H(s, \tau, v_{\tau})] d\tau + F(s, v_{s}) \right] ds \right| \]

\[ \leq ||u(\cdot; f, g)||_{C([0,T]; H)} + M \left( \int_{0}^{t} \left[ \int_{0}^{s} K|b_{1}(\tau)| v_{\tau} || + |b_{2}(s, \tau)|| ||v_{\tau}|| ] d\tau + |b_{3}(s)|| ||v_{s}|| \right] ds. \]

Hence, by (2.10) and (3.15),

\[ \|v_{t}(\cdot; f, g)\| = \sup_{\theta \in [-h,0]} |v(t + \theta; f, g)| \]

\[ \leq K_{T}c_{0} \left( \|g(0)\| + \|g\|_{L^2([-h,0]; V)} + \|f\|_{L^2(0,T;V^*)} \right) \]

\[ + \int_{0}^{t} c_{1} ||v_{s}(\cdot; f, g)\|| ds + \int_{0}^{t} \int_{0}^{s} c_{2} ||v_{\tau}(\cdot; f, g)|| d\tau ds \]

\[ \leq c' \left( \|g(0)\| + \|g\|_{L^2([-h,0]; V)} + \|f\|_{L^2(0,T;V^*)} \right) \]

\[ + M \left( \int_{0}^{t} ||v_{s}(\cdot; f, g)|| ds + \int_{0}^{t} \int_{0}^{s} ||v_{\tau}(\cdot; f, g)|| d\tau ds \right). \]

By using Lemma 3.1, we get

\[ \|v_{t}(\cdot; f, g)\|_{C([0,T]; H)} \leq c \left( \|g(0)\| + \|g\|_{L^2([-h,0]; V)} + \|f\|_{L^2(0,T; V^*)} \right) \]

\[ \times \left( 1 + M \int_{0}^{t} \exp \left( \int_{0}^{s}(M + 1) d\tau \right) ds \right) \]

\[ \leq c' \left( \|g(0)\| + \|g\|_{L^2([-h,0]; V)} + \|f\|_{L^2(0,T; V^*)} \right) \]

\[ \times \left( 1 + M \exp \left( (M + 1)T \right) t \right) \]

\[ \leq c \left( \|g(0)\| + \|g\|_{L^2([-h,0]; V)} + \|f\|_{L^2(0,T; V^*)} \right) e^{Kt} \]

for some constants \( c \) and \( K \). This completes the proof. \( \square \)

**Theorem 3.3.** Assume that the conditions in Theorem 3.1 hold. Then there exists a unique solution \( v(t) \) on \([0,T]\) of (3.1) which satisfies the estimate

\[ \|v(\cdot; f, g)\|_{C([0,T]; H)} \leq c \left( \|g(0)\| + \|g\|_{L^2([-h,0]; V)} + \|f\|_{L^2(0,T; V^*)} \right) e^{Kt} \]

(3.22)

for some constants \( c \) and \( K \).
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References


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<td>First Round of Reviews</td>
<td>August 1, 2009</td>
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