NONAUTONOMOUS DIFFERENTIAL EQUATIONS
OF ALTERNATELY RETARDED
AND ADVANCED TYPE

QIONG MENG and JURANG YAN

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ABSTRACT. We obtain a solution formula of the differential equation \( \dot{x}(t) + a(t)x(t) + b(t)x(g(t)) = f(t) \). At the same time, we study its oscillation and asymptotic stability properties.

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1. Introduction and preliminary. In this paper, we investigate the global asymptotic behavior as well as oscillation of equations with piecewise constant argument

\[
\dot{x}(t) + a(t)x(t) + b(t)x(g(t)) = f(t) \quad \text{for } t > 0
\]

subject to the initial condition

\[
x(0) = x_0,
\]

where \( a(t), b(t), \) and \( f(t) \) are locally integrable functions on \([0, \infty)\), \( g(t) \) is a piecewise constant function defined by

\[
g(t) = np \quad \text{for } t \in [np - l, (n+1)p - l) \quad (n \in \mathbb{N}),
\]

where \( p \) and \( l \) are positive constants satisfying \( p > l \).

Since the argument deviation of (1.1), namely

\[
\tau(t) = t - g(t)
\]

is negative in \([np - l, np)\) and positive in \([np, (n+1)p - l)\), equation (1.1) is said to be of alternately advanced and retarded type.

Equations with piecewise constant argument (EPCA) deviation were investigated in many papers (see [1, 2, 3, 4, 5, 6, 7, 8, 9]). Since EPCA combine the features of both differential and difference equations, their asymptotic behavior as \( t \to \infty \) resembles in some cases the solution growth of differential equations, while in others it inherits the properties of difference equations. So this makes EPCA more interesting.

Definition 1.1. A function \( x : [0, \infty) \to \mathbb{R} \) is a solution of (1.1) and (1.2) if the following conditions hold:

(i) \( x \) is continuous on \([0, \infty)\).

(ii) \( x \) is differentiable in \([0, \infty)\), except possibly at the points \( t = np - l, n \in \{1, 2, \ldots\}, \) where one-sided derivatives exist.
(iii) \( x(0) = x_0 \) and \( x \) satisfies (1.1) in \((0, p - l)\) and in every interval of the form 
\([np - l, (n + 1)p - l)\) for \(n \in \{1, 2, \ldots\}\).

A solution of (1.1) and (1.2) is oscillatory if it has no last zero. Let \([\cdot]\) denote the greatest integer function. This paper was motivated by [7] in which the equation

\[
\dot{x}(t) + Ax(t) + Bx(g(t)) = f(t) \quad \text{for } t > 0
\]

(1.5)

was investigated, where \(A\) and \(B\) are \(r \times r\) matrices, \(x\) is an \(r\)-vector and \(f(t)\) is a locally integrable function on \([0, \infty)\).

2. The case \(a(t) \equiv 0\). In this case, (1.1) becomes

\[
\dot{x}(t) + b(t)x(np) = f(t) \quad \text{for } t > 0.
\]

(2.1)

To simplify the notation, define

\[
B(a,b) = 1 - \int_a^b b(s) \, ds, \quad B(0, -l) = 1, \quad x(np) = x_n, \quad I_n = [np - l, (n + 1)p - l) \quad \text{for } n = 1, 2, \ldots.
\]

(2.2)

**Theorem 2.1.** Let \(b(t)\) and \(f(t)\) be locally integrable on \([0, \infty)\). Then (1.2), (1.4), and (2.1) has a unique solution on \([0, \infty)\) given by

\[
x(t) = B(g(t), t) \left( \prod_{j=1}^{g(t)/p} \frac{B((j-1)p, jp-1)}{B(jp, jp-l)} \right)
\]

\[
\times \left[ x_0 + \sum_{j=1}^{g(t)/p} \left( \prod_{i=1}^{j} \frac{B((i-1)p, (i-1)p-1)}{B((i-1)p, ip-l)} \right) \int_{(j-1)p}^{jp} f(s) \, ds \right]
\]

\[
+ \int_{g(t)}^t f(s) \, ds,
\]

(2.3)

where \(B(a, b)\) is defined in (2.2).

In addition, if \(b(t)\) and \(f(t)\) are integrable on \((-\infty, 0]\), this solution can be continued backwards on \((-\infty, 0]\) and is given by

\[
x(t) = B(g(t), t) \left( \prod_{j=1}^{-g(t)/p} \frac{B((-j-1)p, -jp-1)}{B(-jp, -jp-l)} \right)
\]

\[
\times \left[ x_0 + \sum_{j=1}^{-g(t)/p} \left( \prod_{i=1}^{j} \frac{B((-i-1)p, (-i-1)p-1)}{B((-i-1)p, -ip-l)} \right) \int_{-(j-1)p}^{-jp} f(s) \, ds \right]
\]

\[
+ \int_{g(t)}^t f(s) \, ds
\]

(2.4)

**Proof.** We use the notation given in (2.2).

In each interval of the type \(I_n\), (2.1) becomes

\[
\dot{x}(t) + b(t)x(np) = f(t)
\]

(2.5)
which has a unique solution whenever a preassigned value for \( x(np) \) is given. The solution of (2.1), with \( x(np) = x_n \), is

\[
x(t) = B(np, t)x_n + \int_{np}^{t} f(s) \, ds \quad \text{for } t \in I_n,
\]

and with \( x((n+1)p) = x_{n+1} \) is

\[
x(t) = B((n+1)p, t)x_{n+1} + \int_{(n+1)p}^{t} f(s) \, ds \quad \text{for } t \in I_{n+1}.
\]

Continuity of the solution at \( t = (n+1)p - l \) requires

\[
B(np, (n+1)p - l)x_n + \int_{np}^{(n+1)p-l} f(s) \, ds.
\]

(2.8)

so that

\[
x_{n+1} = \frac{B(np, (n+1)p - l)}{B((n+1)p, (n+1)p - l)} x_n + \frac{1}{B((n+1)p, (n+1)p - l)} \int_{np}^{(n+1)p-l} f(s) \, ds,
\]

(2.9)

from which it follows that

\[
x_n = \left( \prod_{j=1}^{n} \frac{B((j-1)p, jp - l)}{B(jp, jp - l)} \right) \left[ x_0 + \sum_{j=1}^{n} \left( \prod_{i=1}^{j} \frac{B((i-1)p, (i-1)p - l)}{B((i-1)p, ip - l)} \right) \int_{(j-1)p}^{jp} f(s) \, ds \right].
\]

(2.10)

Substituting (2.10) into (2.6) yields (2.3). The continuation of (2.3) on \((-\infty, 0]\) is obtained in a similar way. This completes the proof.

**Theorem 2.2.** Let \( b(t) \) be locally integrable on \([0, \infty)\). Assume that \(|b(t)| < B_1 \) (\( B_1 > 0 \)) for \( t \in [0, \infty) \) and

\[
\left| \frac{B((n-1)p, np - l)}{B(np, np - l)} \right| < \alpha < 1 \quad \text{for } n \in \{1, 2, \ldots\}.
\]

(2.11)

(a) If \( f(t) \equiv 0 \) then the trivial solution of (2.1) is globally asymptotically stable.

(b) If \( \lim_{t \to -\infty} f(t) = 0 \) then every solution of (2.1) tends to zero as \( t \to \infty \).

**Proof.** (a) Note that for \( t \in I_n \)

\[
\left| B(g(t), t) \left( \prod_{j=1}^{a(t)/p} \frac{B((j-1)p, jp - l)}{B(jp, jp - l)} \right) x_0 \right| < B_2 \alpha^n |x_0|,
\]

(2.12)

where \( B_2 = 1 + B_1 \max\{l, p - l\} \). Therefore (a) is proved.

(b) We observe that the remaining term in (2.3) tends to zero as \( t \to \infty \). For \( t \in I_n \)

\[
\left| \int_{g(t)}^{t} f(s) \, ds \right| < \max\{l, p - l\} \max \{|f(t)| : t \in I_n\}.
\]

(2.13)
Similarly, \( F_j = \int_{(j-1)p}^{jp} f(s) \, ds \to 0 \) as \( j \to \infty \). Hence, given \( \varepsilon > 0 \), choose \( P_1 \) such that \( |F_j| < K \) if \( j < P_1 \) and \( |F_j| < \varepsilon(1 - \alpha)B_3/2B_2 \) for \( j \geq P_1 \), choose \( P_2 \) so that if \( n > P_2 \) then \( \alpha^n < \varepsilon B_3/2KB_2P_1 \), where \( B_3 = 1/|1 - B_1| \). If \( n > \max\{P_1, P_2\} \), then

\[
\left| \prod_{j=1}^{n} \frac{B((j-1)p, jp-l)}{B(jp, jp-l)} \left( \sum_{i=1}^{j} \frac{B((i-1)p, (i-1)p-l)}{B((i-1)p, ip-l)} \right) \int_{(j-1)p}^{jp} f(s) \, ds \right| 
\leq \sum_{j=1}^{P_1} \left( \left( \prod_{i=j+1}^{n} \frac{B((i-1)p, ip-l)}{B(ip, ip-l)} \right) \frac{1}{B(jp, jp-l)} |F_j| \right) + \frac{1}{B_3} \sum_{j=P_2+1}^{nP} \left( \left( \prod_{i=j+1}^{P_1} \alpha^i \right) |F_j| \right) \leq \frac{\varepsilon}{B_2},
\]

where we define \( \prod_{i=n+1}^{n} B((i-1)p, ip-l)/B(ip, ip-l) = 1 \). This completes the proof.

**Theorem 2.3.** Let \( b(t) \) be locally integrable on \([0, \infty)\). Every solution of the equation

\[
\dot{x}(t) + b(t)x(g(t)) = 0, \quad x(0) = x_0,
\]

is oscillatory if \( B(np, (n+1)p-l)/B(np, np-l) \) is not eventually positive.

**Proof.** Let \( x(t) \) be a solution of (2.15). The continuity of \( x(t) \) at \( t = (n+1)p - l \) in (2.6) gives

\[
x((n+1)p-l) = B(np, (n+1)p-l) x_n.
\]

Again using (2.6) with \( t = np - l \) we have that

\[
x(np-l) = B(np, np-l) x_n.
\]

From (2.16) and (2.17) we obtain that

\[
x((n+1)p-l) = \frac{B(np, (n+1)p-l)}{B(np, np-l)} x(np-l).
\]

It is easy to see that the sequence \( \{x(np-l)\} \) oscillates if \( B(np, (n+1)p-l)/B(np, np-l) \) is not eventually positive. Therefore \( x(t) \) oscillates if \( \{x(np-l)\} \) oscillates. This completes the proof.

**Corollary 2.4.** If \( b(t) \leq 0 \), the sign of every solution of (2.15) is identical with the sign of its initial value.

The proof of Corollary 2.4 is obvious from (2.3).
**Remark 2.5.** Corollary 2.4 can be recounted that if \( b(t) \leq 0 \), then all solutions of (2.15) are nonoscillatory. The following example is to illustrate Theorem 2.3.

**Example 2.6.** Consider the equation
\[
\dot{x}(t) + (2 - t)x\left(2\left[\frac{t+1}{2}\right]\right) = 0, \quad t > 0
\] (2.19)
with \( x(0) = x_0 \). Theorem 2.1 asserts that (2.19) subject to \( x(0) = x_0 \) has a unique solution on \([0, \infty)\). The solution of (2.19) is given by
\[
x(t) = \left[1 - 2(t - n) + \frac{t^2 - n^2}{2}\right]\left(\prod_{j=1}^{n} \frac{4j - 5}{7 - 4j}\right)x_0 \quad \text{for } t \in [2n - 1, 2n + 1].
\] (2.20)
From Theorem 2.3, all solutions of (2.19) are oscillatory if
\[
\frac{B(2n, 2(n+1) - 1)}{B(2n, 2n - 1)} = -1 + \frac{6}{7 - 4n}
\] (2.21)
is not eventually positive.

3. The case (1.1). To simplify the notation, define
\[
B(a, b) = 1 - \int_{a}^{b} b(s) \exp\left(\int_{a}^{s} a(u) \, du\right) \, ds,
\]
\[
F(a, b) = \int_{a}^{b} f(s) \exp\left(\int_{a}^{s} a(u) \, du\right) \, ds,
\]
\[
\int_{-p}^{0} a(s) \, ds = 0, \quad x(np) = x_n,
\]
\[
I_n = [np - l, (n + 1)p - l] \quad \text{for } n = 1, 2, \ldots.
\]
We state some theorems for (1.1). The proofs of Theorems 3.1, 3.2, and 3.3 can be obtained by the techniques used in the proofs of Theorems 2.1, 2.2, and 2.3 of Section 2.

**Theorem 3.1.** Let \( a(t), b(t), \) and \( f(t) \) be locally integrable on \([0, \infty)\). Then (1.1) and (1.2) has a unique solution on \([0, \infty)\) given by
\[
x(t) = B(g(t), t) \exp\left(-\int_{g(t)}^{t} a(s) \, ds\right)
\times \left(\prod_{j=1}^{(1/p)} \exp\left(-\int_{(j-1/p)}^{jp} a(s) \, ds\right) \frac{B((j-1)p, jp - l)}{B(jp, jp - l)}\right)
\times \left[x_0 - \sum_{j=1}^{(1/p)} \left(\prod_{i=1}^{j} \exp\left(-\int_{(i-1/p)}^{ip} a(s) \, ds\right) \frac{B^{-1}(i-1)p, ip - l)}{B^{-1}(ip, ip - l)}\right) \frac{F(ip, (j-1)p)}{B(jp, jp - l)}\right]
\times \exp\left(-\int_{g(t)}^{t} a(s) \, ds\right) F(g(t), t),
\] (3.2)
where \( B(a, b) \) and \( F(a, b) \) are defined in (3.1).
In addition, if \( a(t), b(t), \) and \( f(t) \) are integrable on \((-\infty, 0]\), this solution can be continued backwards on \((-\infty, 0]\) and is given by

\[
x(t) = B(g(t), t) \exp \left( -\int_{g(t)}^{t} a(s) \, ds \right) \\
\times \left( \prod_{j=1}^{-\frac{g(t)}{p}} \exp \left( -\int_{-(j-1)p}^{-(j-1)p} a(s) \, ds \right) \frac{B((-j-1)p, -jp - l)}{B(-jp, -jp - l)} \right) \\
\times \left[ x_0 - \sum_{j=1}^{-\frac{g(t)}{p}} \left( \prod_{i=1}^{j} \exp \left( -\int_{-(i-1)p}^{-(i-1)p} a(s) \, ds \right) \frac{B^{-1}((-i-1)p, -ip - l)}{B^{-1}(-ip, -ip - l)} \right) F(-jp, -(j-1)p) \right] \\
+ \exp \left( -\int_{g(t)}^{t} a(s) \, ds \right) F(g(t), t).
\]

(3.3)

**Theorem 3.2.** Let \( a(t) \) and \( b(t) \) be locally integrable on \([0, \infty)\). Assume that

\[
|a(t)| < A, \quad |b(t)| < B_1 (A, B_1 > 0) \quad \text{for} \quad t \in [0, \infty) \quad \text{and}
\]

\[
\left| \frac{B((n-1)p, np - l)}{B(np, np - l)} \right| < \alpha < 1 \quad \text{for} \quad n \in \{1, 2, \ldots\}. \quad (3.4)
\]

(a) If \( f(t) \equiv 0 \) then the trivial solution of (1.1) is globally asymptotically stable.

(b) If \( \lim_{t \to \infty} f(t) = 0 \) then every solution of (1.1) tends to zero as \( t \to \infty \).

**Theorem 3.3.** Let \( a(t) \) and \( b(t) \) be locally integrable on \([0, \infty)\). Every solution of the equation

\[
\dot{x}(t) + a(t)x(t) + b(t)x(g(t)) = 0, \quad x(0) = x_0
\]

(3.5)
is oscillatory if \( B(np, (n+1)p - l)/B(np, np - l) \) is not eventually positive.

In summary, equations with piecewise constant argument are interesting in their own right, and have some curious and unpredictable properties. The systems of nonautonomous differential equations of alternately retarded and advanced type can be studied in similar ways.

**References**


QIONG MENG: DEPARTMENT OF MATHEMATICS, SHANXI UNIVERSITY, TAIYUAN, SHANXI 030006, CHINA
E-mail address: qiongmeng@sohu.com

JURANG YAN: DEPARTMENT OF MATHEMATICS, SHANXI UNIVERSITY, TAIYUAN, SHANXI 030006, CHINA
E-mail address: jryan@sxu.edu.cn
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Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

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<thead>
<tr>
<th>Manuscript Due</th>
<th>February 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>May 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>August 1, 2009</td>
</tr>
</tbody>
</table>

Guest Editors

José Roberto Castilho Piqueira, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

Elbert E. Neher Macau, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil; elbert@lac.inpe.br

Celso Grebogi, Department of Physics, King’s College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk