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THE POULSEN SIMPLEX IS NOT A TENSOR PRODUCT

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ABSTRACT. It is shown that the Poulsen simplex is not a projective tensor product of non-trivial Choquet simplexes.

KEY WORDS AND PHRASES. Poulsen simplex, projective tensor product, projective limits.

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1. INTRODUCTION

The Poulson simplex P is an example of a metrisable Choquet simplex whose extreme points $\xi(P)$ are dense in P. Such a simplex was constructed by Poulsen in [11]. In [6] Lazar and Lindenstrauss showed how to represent metrisable Choquet simplexes S as projective limits of an affine projective system $\{\{\Delta_n : n\epsilon N\}, \{\pi_n : n\epsilon N\}\}$ where each Δ_n is an (n-1)-simplex with $\xi(\Delta_n)=\{e_1,\ldots,e_n\}\subset \xi(\Delta_{n+1})$ and with $\pi_n:\Delta_{n+1}\to\Delta_n$ described by the requirement that $\pi_n(e_j) = e_j$ if $1\leq j\leq n$ and $\pi_n(e_{n+1}) = a_{n1}e_1+\ldots+a_{nn}e_n$ where $a_{nj}\geq 0$ and $\Sigma a_{nj} = 1$. The triangular matrix $A=(a_{ij})$ is called a representing matrix for S. There are many representing matrices for S as there are many realizations of S as such a projective limit. It was established by Lindenstrauss, Olsen, and Sternfeld [7] that $S = \overline{\xi(S)}$ iff the sequence of rows of A form a dense subset of the positive face of the unit ball of ℓ^1 when each was regarded as a sequence. Lindenstrauss, Olsen and Sternfeld showed in [7] that, up to affine homeomorphism, P is the only metrisable Choquet simplex with $P=\overline{\xi(P)}$

It is well known that the Poulsen simplex P is prime in that A(K) is an anti-lattice (Asimow and Ellis, [2]). We show here that P is prime in the semigroup of convex metrisable compact sets with multiplication being projective tensor product. The proof involves a fairly straight forward application of the properties of projective limits and of representing matrices for metrisable simplexes.

2. MAIN RESULTS

We refer the reader to E.B. Davies and G.F. Vincent-Smith [3] for the details concerning projective tensor products both finite and infinite. For any family $\{S_i : i\varepsilon I\}$ of Choquet simplexes there is defined, up to affine homeomorphism, a Choquet simplex $\bigotimes_{i\varepsilon I} S_i$, the projective tensor product, which has the property that there is a continuous multi-affine embedding $\otimes$ of $\prod_{i\varepsilon I} S_i$ into $\bigotimes_{i\varepsilon I} S_i$ given by $(x_i : i\varepsilon I) \to \bigotimes_{i\varepsilon I} x_i$ so that if $m: \prod_{i\varepsilon I} S_i \to E$ is continuous multi-affine for some locally convex space E there exists a linear $n: \bigotimes_{i\varepsilon I} S_i \to E$ with $n \circ \otimes = m$. It is shown that $\xi(\bigotimes_{i\varepsilon I} S_i)$ is $\{\bigotimes_{i\varepsilon I} x_i : x_i \varepsilon \xi(S_i) : i\varepsilon I\}$ and that $\otimes : \prod_{i\varepsilon I} \xi(S_i)$ is a homeomorphism. It is easily checked that if each S_i is a projective limit of a sequence $\{S_{in} : n\varepsilon N\}$ of simplexes under projections $\{P_{in} : n\varepsilon N\}$ then $\bigotimes_{i\varepsilon I} S_i$ is a projective limit of $\{\bigotimes_{i\varepsilon I} S_{in} : n\varepsilon N\}$ under $\{P_n : n\varepsilon N\}$ where $P_n : \bigotimes_{i\varepsilon I} S_{in+1} \to \bigotimes_{i\varepsilon I} S_{in}$ is the map induced by the multi-affine transformation $(x_i : i\varepsilon I) \to \bigotimes_{i\varepsilon I} P_{in}(x_i)$ from $\prod_{i\varepsilon I} S_{in+1}$ to $\bigotimes_{i\varepsilon I} S_{in}$.

PROPOSITION. The Poulsen simplex is not a tensor product.

PROOF. Suppose that $P = X \otimes Y$ with X and Y each at least one dimensional metrisable Choquet simplexes. Let $A = (a_{ij})$ and $B = (b_{ij})$ be representing matrices for X and Y respectively. Let $p_n : \Delta_{n+1} \to \Delta_n$ and $q_n : \Delta_{n+1} \to \Delta_n$ for $n\varepsilon N$ be the sequences of projections associated with A and B respectively so that X is the projective limit of $\{\Delta_n : n\varepsilon N\}$ under $\{p_n : n\varepsilon N\}$ and Y is the projective limit of $\{\Delta_n : n\varepsilon N\}$ under $\{q_n : n\varepsilon N\}$. Then P is the projective limit of $\{\Delta_n \otimes \Delta_n : n\varepsilon N\}$ under the system $\{r_n : n\varepsilon N\}$ of projections where $r_n(e_i \otimes e_j) = e_i \otimes e_j$ if $1 \leq i,j \leq n$, $r_n(e_{n+1} \otimes e_j) = \sum_{i=1}^{n} a_{ni} e_i \quad e_j$, $r_n(e_i \quad e_{n+1}) = \sum_{j=1}^{n} b_{nj}\, e_i \quad e_j$ and $r_n(e_{n+1} \quad e_{n+1}) = \sum_{i,j=1}^{n} a_{ni} b_{nj} e_i \quad e_j$.

For any $n\varepsilon N$, let $D_{n^2} = \Delta_n \otimes \Delta_n$. For $1 \leq k \leq n$ define D_{n^2+k} to be $\mathrm{conv}(D_{n^2+k-1}, e_k \otimes e_{n+1}) \subset \Delta_{n+1} \otimes \Delta_{n+1}$. Define D_{n^2+n+k} to be $\mathrm{conv}(D_{D^2+n+k-1}, e_{n+1} \otimes e_k)$. Define, for $1 \leq k \leq n$, the affine surjection $R_{n^2+k-1} : D_{n^2+k-1} \to D_{n^2+k-1}$ to be the identity on D_{n^2+k-1} and to be $\sum_{j=1}^{n} b_{nj}\, e_k \otimes e_j$ on $e_k \otimes e_{n+1}$. Similarly define, for $1 \leq k \leq n$, the affine surjection $R_{n^2+n+k-1} : D_{n^2+n+k} \to D_{n^2+n+k-1}$ by setting $R_{n^2+n+k-1}$

equal to the identity on $D_{n^2+n+k-1}$ and by setting $R_{n^2+n+k-1}(e_{n+1} \otimes e_k)$ equal to $\sum_{i=1}^{n} a_{ni} (e_i \otimes e_k)$. Finally, set R_{n^2+2n} equal to the affine surjection from $D_{(n+1)^2}$ to D_{n^2+2n} which is equal to the identity on D_{n^2+2n} and has $R_{n^2+2n}(e_{n+1} \otimes e_{n+1}) = \sum_{i,j=1}^{n} a_{ni} b_{nj}\, e_i \otimes e_j$. We then have P equal to the projective limit of $\{D_m : m \in N\}$ under $\{R_m : m \in N\}$.

The projections $\{R_m : m \in N\}$ have a representing matrix $C = (c_{ij})$ which is triangular and has its rows and columns most conveniently indexed by ordered pairs (i,j). In this set up the entries in row $(n+1,k)$ for $k=1,\dots,n$ are a_{ni} in column (i,k) and 0 otherwise. The entry in column (k,j) for row $(k,n+1)$ is b_{nj} and 0 otherwise. The entry in row $(n+1,n+1)$ in column (i,j) for $1 \le i,\ j \le n$ is $a_{ni} b_{nj}$ with 0 entries elsewhere. Except for the rows (n,n) each row lies in the subspace of $\ell^1(N \times N)$ $\{x : x_{(2,1)} = 0\}$ or the subspace $\{x : x_{(1,2)} = 0\}$ Since the union of these two subspaces is closed and nowhere dense in the politive face of the unit ball of $\ell^1(N \times N)$ the rows indexed by $\{(n,n) : n \in N\}$ must be dense in the positive face of the unit ball of $\ell^1(N \times N)$ in order that all of the rows be dense (if we are to have the representing matrix of a Poulsen simplex.)

Let $M_1^+(n,n)$ denote all nxn matrices (c_{ij}) with $c_{ij} \le 0$ and $\sum_{i,j=1}^{n} c_{ij} = 1$. $M_1^+(n,n)$ is naturally embedded in $\ell^1(N \times N)$ by setting $c_{ij}=0$ if $i>n$ or $j>n$. In order that the (n,n) rows of the representing matrix of $\{R_m : m \in N\}$ be dense it is necessary and sufficient that for all n every $(c_{ij}) \in M_1^+(n,n)$ be approximable by matrices of the form (d_{ij}) where $d_{ij} = a_{mi} b_{mi}$ $1 \le i,j \le n$. This in turn implies that each $(c_{ij}) \in M^+(n,n)$ be approximable by matrices $(f_{ij}) \in M_1^+(n,n)$ where $f_{ij} = a_i b_j$ with a a_i the i-th row sum and b_j the j-th column sum. The set of such matrices (f_{ij}) is (2n-2)- dimensional whereas $M_1^+(n,n)$ is (n^2-1)-dimensional. Since $n^2-1 > 2n-2$ if $n>1$ this is impossible. This establishes that $X \otimes Y = P$ is impossible.

COROLLARY. P is not $\bigotimes_{n=1}^{\infty} X_n$ for any sequence of non-zero dimensional simplexes $\{X_n : n \in N\}$.

PROOF. $\bigotimes_{n=1}^{\infty} X_n = X_1 \otimes [\bigotimes_{n=2}^{\infty} X_n]$.

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