PROPERTIES OF FIXED POINT SET OF A MULTIVALUED MAP

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Properties of the set of fixed points of some discontinuous multivalued maps in a strictly convex Banach space are studied; in particular, affirmative answers are provided to the questions related to set of fixed points and posed by Ko in 1972 and Xu and Beg in 1998. A result regarding the existence of best approximation is derived.

1. Introduction

The study of fixed points for multivalued contractions and nonexpansive maps using the Hausdorff metric was initiated by Markin [17]. Later, an interesting and rich fixed point theory for such maps has been developed. The theory of multivalued maps has applications in control theory, convex optimization, differential inclusions, and economics (see, e.g., [3, 8, 16, 22]).

The theory of multivalued nonexpansive mappings is harder than the corresponding theory of single-valued nonexpansive mappings. It is natural to expect that the theory of nonself-multivalued noncontinuous functions would be much more complicated.

The concept of a $*$-nonexpansive multivalued map has been introduced and studied by Husain and Latif [9] which is a generalization of the usual notion of nonexpansiveness for single-valued maps. In general, $*$-nonexpansive multivalued maps are neither nonexpansive nor continuous (see Example 3.7).

Xu [22] has established some fixed point theorems while Beg et al. [2] have recently studied the interplay between best approximation and fixed point results for $*$-nonexpansive maps defined on certain subsets of a Hilbert space and Banach space. For this class of functions, approximating sequences to a fixed point in Hilbert spaces are constructed by Hussain and Khan [10] and its applications to random fixed points and best approximations in Fréchet spaces are given by Khan and Hussain [12].

In this paper, using the best approximation operator, we (i) establish certain properties of the set of fixed points of a $*$-nonexpansive multivalued nonself-map in the setup of a strictly convex Banach space, (ii) prove fixed point results for $*$-nonexpansive random maps in a Banach space under several boundary conditions, and (iii) provide affirmative answers to the questions posed by Ko [14] and Xu and Beg [24] related to the set of fixed points.

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2. Notations and preliminaries

Let C be a subset of a normed space X. We denote by 2^X, $C(X)$, $K(X)$, $CC(X)$, $CK(X)$, and $CB(X)$ the families of all nonempty, nonempty closed, nonempty compact, nonempty closed convex, nonempty convex compact, and nonempty closed bounded subsets of X, respectively.

Define $d(x,C) = \inf_{y\in C} d(x,y)$. The Hausdorff metric on $CB(X)$ induced by the metric d on X is denoted by H and is defined as

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}. \tag{2.1}$$

A mapping $T : C \to CB(X)$ is a contraction if for any $x, y \in C$, $H(Tx,Ty) \le kd(x,y)$, where $0 \le k < 1$. If $k = 1$, then T is called a nonexpansive map. If $H(Tx,Ty) < d(x,y)$ whenever $x \ne y$ in C, then T is called a strictly nonexpansive mapping [14].

A multivalued map $T : C \to 2^X$ is said to be

(i) $*$-nonexpansive if for all $x, y \in C$ and $u_x \in Tx$ with $d(x,u_x) = d(x,Tx)$, there exists $u_y \in Ty$ with $d(y,u_y) = d(y,Ty)$ such that $d(u_x,u_y) \le d(x,y)$ (see [9, 10]),

(ii) strictly $*$-nonexpansive if for all $x \ne y$ in C and $u_x \in Tx$ with $d(x,u_x) = d(x,Tx)$, there exists $u_y \in Ty$ with $d(y,u_y) = d(y,Ty)$ such that $d(u_x,u_y) < d(x,y)$,

(iii) upper semicontinuous (usc) (lower semicontinuous (lsc)) if $T^{-1}(B) = \{x \in C : Tx \cap B \ne \phi\}$ is closed (open) for each closed (open) subset B of X. If T is both usc and lsc, then T is continuous,

(iv) asymptotically contractive [19] if there exist some $c \in (0,1)$ and $r > 0$ such that

$$\|y\| \le c\|x\|, \quad \forall y \in Tx,\ \forall x \in C \setminus rB_X, \tag{2.2}$$

where B_X is the closed unit ball of X.

The map $T : C \to CB(X)$ is called (i) H-continuous (continuous with respect to Hausdorff metric H) if and only if for any sequence $\{x_n\}$ in C with $x_n \to x$, we have $H(Tx_n,Tx) \to 0$ (the two concepts of set-valued continuity are equivalent when T is compact-valued (cf. [8, Theorem 20.3, page 94])); (ii) demiclosed at 0 if the conditions $x_n \in C$, x_n converges weakly to x, $y_n \in Tx_n$ and $y_n \to 0$ imply that $0 \in Tx$. An element x in C is called a fixed point of a multivalued map T if and only if $x \in Tx$. The set of all fixed points of T will be denoted by $F(T)$.

For each $x \in X$, let $P_C(x) = \{z \in C : d(x,z) = d(x,C)\}$. Any $z \in P_C(x)$ is called a point of best approximation to x from C. If $P_C(x) \ne \phi$ (singleton) for each $x \in X$, then C is called a proximinal (Chebyshev) set, respectively. If C is proximinal, then the mapping $P_C : X \to 2^C$ is well defined and is called the metric projection.

The space X is said to have the Oshman property (see [18]) if it is reflexive and the metric projection on every closed convex subset is usc.

For the multivalued map T and each $x \in C$, we follow Xu [22] to define the best approximation operator, $P_T(x) = \{u_x \in Tx : d(x,u_x) = d(x,Tx)\}$, (possibly empty set).

A single-valued (multivalued) map $f : C \to X$ ($F : C \to 2^X$) is said to be a selector of T if $f(x) \in Tx$ ($Fx \subseteq Tx$), respectively, for each $x \in C$.

The space X is said to have the Opial condition if for every sequence $\{x_n\}$ in X weakly convergent to $x \in X$, the inequality

$$\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y|| \tag{2.3}$$

holds for all $y \neq x$.

Every Hilbert space and the spaces ℓ_p $(1 \le p < \infty)$ satisfy the Opial condition.

The inward set $I_C(x)$ of C at $x \in X$ is defined by $I_C(x) = \{x + \gamma(y - x) : y \in C$ and $\gamma > 0\}$. We will denote the closure of C by $\mathrm{cl}(C)$.

Let $(\Omega, \mathcal{A})$ denote a measurable space with $\mathcal{A}$ sigma algebra of subsets of Ω. A mapping $T : \Omega \to 2^X$ is called measurable if for any open subset B of X, $T^{-1}(B) = \{\omega \in \Omega : T(\omega) \cap B \neq \phi\} \in \mathcal{A}$. A mapping $\zeta : \Omega \to X$ is said to be a measurable selector of a measurable mapping $T : \Omega \to 2^X$ if ζ is measurable and for any $\omega \in \Omega$, $\zeta(\omega) \in T(\omega)$. A mapping $T : \Omega \times C \to 2^X$ is a random operator if for any $x \in C$, $T(\cdot, x)$ is measurable. A mapping $\zeta : \Omega \to C$ is said to be a random fixed point of T if ζ is a measurable map such that for every $\omega \in \Omega$, $\zeta(\omega) \in T(\omega, \zeta(\omega))$.

A random operator $T : \Omega \times C \to 2^X$ is said to be continuous (nonexpansive, $*$-nonexpansive, convex, etc.) if for each $\omega \in \Omega$, $T(\omega, \cdot)$ is continuous (nonexpansive, $*$-nonexpansive, convex, etc.).

The following results are needed.

Proposition 2.1 (see [3, Proposition 2.2]). *Let E be a metric space. If $T : \Omega \to C(E)$ is a multivalued mapping, then the following conditions are equivalent:*

(i) *T is measurable;*
(ii) *$\omega \to d(x, T(\omega))$ is a measurable function of ω for each $x \in E$;*
(iii) *there exists a sequence $\{f_n(\omega)\}$ of measurable selectors of T such that $\mathrm{cl}\{f_n(\omega)\} = T(\omega)$ for all ω in Ω.*

Theorem 2.2 (see [24, Theorem 3.1]). *Let C be a nonempty separable weakly compact convex subset of a Banach space X. Suppose that the map $T : \Omega \times C \to K(C)$ is a nonexpansive random mapping. If for each $\omega \in \Omega$, $I - T(\omega, \cdot)$ is demiclosed at 0, then the fixed point set function F of T given by $F(\omega) = \{x \in C : x \in T(\omega, x)\}$ is measurable (and hence T has a random fixed point).*

3. $*$-nonexpansive maps

The properties of the set of fixed points of single-valued and multivalued maps have been considered by a number of authors (see, e.g., Agarwal and O'Regan [1], Browder [4], Bruck [5], Espínola et al. [6], Ko [14], Schöneberg [20], and Xu and Beg [24]). For a wide class of unbounded closed convex sets C in a Banach space, there exist nonexpansive maps $T : C \to K(C)$ which fail to have a fixed point (see [13]).

We obtain some properties of the set of fixed points of a $*$-nonexpansive map on a Banach space with values which are not necessarily subsets of the domain.

Markin [17], Xu [22], and Jachymski [11] have utilized "selections;" we employ "nonexpansive selector," P_T, of a $*$-nonexpansive map T to study the structure of the set of fixed points of T. Consequently, we obtain generalized and improved versions of many results in the current literature.

In Theorem 8.2, Browder [4] has established the following result.

Theorem 3.1. *Let C be a nonempty closed, convex, subset of a strictly convex Banach space X and let $T : C \to C$ be a nonexpansive map. Then the set $F(T)$ of fixed points of T is closed and convex.*

Ko [14] pointed out that Theorem 3.1 need not hold for multivalued nonexpansive mappings as follows.

Example 3.2 (see [14, Example 3]). Consider $C = [0,1] \times [0,1]$ with the usual norm. Define $T : C \to CK(C)$ by

$$T(x,y) = \text{ the triangle with vertices } (0,0),\ (x,0),\ \text{and } (0,y). \tag{3.1}$$

Note that T is nonexpansive and the norm in $\mathbb{R}^2$ is strictly convex. But the set $F(T) = \{(x,y) : (x,y) \in C \text{ and } xy = 0\}$ is not convex.

The following generalization of Theorem 3.1 for $*$-nonexpansive continuous mappings is obtained in [15].

Theorem 3.3. *Let X be a strictly convex Banach space and C a nonempty weakly compact convex subset of X. Let $T : C \to CC(C)$ be a $*$-nonexpansive map such that $F(T)$ is nonempty. Then the set $F(T)$ is convex and is closed if T is continuous.*

We present a new proof, through the best approximation operator, of Theorem 3.3 without assuming any type of continuity of the map T and obtain the following structure theorem.

Theorem 3.4. *Let X be a strictly convex Banach space and C a nonempty weakly compact convex subset of X. Let $T : C \to CC(C)$ be a $*$-nonexpansive map such that $F(T)$ is nonempty. Then the set $F(T)$ is closed and convex.*

Proof. For each $x \in C$, its image Tx is weakly compact and convex and thus each Tx is Chebyshev. Hence, each u_x in $P_T(x)$ is unique. Thus by the definition of $*$-nonexpansiveness of T, there is $u_y = P_T(y) \in Ty$ for all y in C such that

$$||P_T(x) - P_T(y)|| = ||u_x - u_y|| \leq \|x - y\|. \tag{3.2}$$

Hence, $P_T : C \to C$ is a nonexpansive selector of T (see also [22]).By the definition of P_T, we have for each $y \in C$,

$$d(y, P_T(y)) = d(y, u_y) = d(y, Ty). \tag{3.3}$$

Equation (3.3) now implies that $F(T) = F(P_T)$. Thus $F(P_T)$ and hence $F(T)$ is closed and convex by Theorem 3.1. □

The following example illustrates our results.

Example 3.5. Let $T : [0,1] \to 2^{[0,1]}$ be a multivalued map defined by

$$Tx = \begin{cases} \left[0, \dfrac{1}{2}\right], & x \neq \dfrac{1}{2}, \\ [0,1], & x = \dfrac{1}{2}. \end{cases} \tag{3.4}$$

Then

$$P_T(x) = \begin{cases} x, & x \in \left[0, \dfrac{1}{2}\right], \\ \dfrac{1}{2}, & x \in \left(\dfrac{1}{2}, 1\right]. \end{cases} \tag{3.5}$$

This implies that T is a $*$-nonexpansive map. Further, T is usc but not lsc (see [8, Remark 15.2, page 71]) and hence T is not continuous according to both definitions as T is compact-valued. Note that $F(T) = [0, 1/2]$ is closed and convex.

If T is a single-valued strictly nonexpansive map, then $F(T)$ is a singleton. In general, this is not true for a multivalued nonexpansive map [17]. The set $F(T)$ is said to be singleton in a generalized sense if there exists $x \in F(T)$ such that $F(T) \subseteq Tx$. Ko has given an example of a strictly nonexpansive mapping $T : C \to CC(C)$, in a strictly convex Banach space, for which the set $F(T)$ is not singleton in a generalized sense (cf. [17, Example 4]). Ko raised the following question: is $F(T)$ singleton in a generalized sense if T is nonexpansive, I is the identity operator, and $I - T$ is convex?

The following proposition provides an affirmative answer to this question for strictly $*$-nonexpansive multivalued mappings.

Proposition 3.6. *Let C be a nonempty closed convex subset of a reflexive strictly convex Banach space X and let $T : C \to CC(C)$ be a strictly $*$-nonexpansive map such that $F(T)$ is nonempty. Then the set $F(T)$ is singleton in a generalized sense.*

Proof. Any closed convex subset of a reflexive strictly convex Banach space is Chebyshev, so each Tx is Chebyshev. Thus as in the proof of Theorem 3.4, $P_T : C \to C$ is a strictly nonexpansive selector of T satisfying (3.3). Hence, $F(T) = F(P_T)$ is singleton in a generalized sense as required. □

The following example supports the above proposition.

Example 3.7. Let $T : [0,1] \to 2^{[0,1]}$ be a multivalued map defined by

$$Tx = \begin{cases} \left\{\dfrac{1}{2}\right\}, & x \in \left[0, \dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, 1\right], \\ \left[\dfrac{1}{4}, \dfrac{3}{4}\right], & x = \dfrac{1}{2}. \end{cases} \tag{3.6}$$

Then $P_T(x) = \{1/2\}$ for every $x \in [0,1]$. This implies that T is a strictly $*$-nonexpansive map:

$$\begin{aligned} H\left(T\left(\frac{1}{3}\right), T\left(\frac{1}{2}\right)\right) &= H\left(\left\{\frac{1}{2}\right\}, \left[\frac{1}{4}, \frac{3}{4}\right]\right) \\ &= \max\left\{\sup_{a\in\{1/2\}} d\left(a, \left[\frac{1}{4}, \frac{3}{4}\right]\right), \sup_{b\in[1/4,3/4]} d\left(b, \frac{1}{2}\right)\right\} \\ &= \max\left\{0, \frac{1}{4}\right\} = \frac{1}{4} > \frac{1}{6} = \left|\frac{1}{3} - \frac{1}{2}\right|. \end{aligned} \tag{3.7}$$

This implies that T is not nonexpansive. Obviously, T is compact-valued. Next we show that T is not lsc.

Let $V_{1/4}$ be any small open neighbourhood of 1/4. Then the set

$$T^{-1}(V_{1/4}) = \{x \in [0,1] : Tx \cap V_{1/4} \neq \phi\} = \left\{\frac{1}{2}\right\} \tag{3.8}$$

is not open. Thus T is not continuous in the sense of both definitions.

Note that $F(T) = \{1/2\}$ is singleton in a generalized sense.

The conclusion of Proposition 3.6 does not hold for $*$-nonexpansive maps as follows.

Example 3.8. Let $C = [0,\infty)$ and $T : C \to CK(C)$ be defined by

$$Tx = [x, 2x] \quad \text{for } x \in C. \tag{3.9}$$

Then $P_T(x) = \{x\}$ for every $x \in C$. This clearly implies that T is $*$-nonexpansive but not nonexpansive (cf. [22]). Note that $F(T) = C$ and there does not exist any x in $F(T)$ such that $F(T) \subseteq Tx$. Thus $F(T)$ is not singleton in a generalized sense.

The above example also indicates that the fixed point set of a $*$-nonexpansive map need not be bounded in general. However, if T is asymptotically contractive, then we have the following affirmative result.

Theorem 3.9. *Let X be a uniformly convex Banach space and C a nonempty closed convex subset of X. Let $T : C \to CC(C)$ be a $*$-nonexpansive map which is asymptotically contractive on C. Then $F(T)$ is nonempty closed, convex, and bounded.*

Proof. The map T has a nonexpansive selector f which is also asymptotically contractive by the asymptotic contractivity of T. Further, $F(T) = F(f)$ is nonempty closed bounded and convex (see [19, Corollary 3 and Remark (a)]). □

We are now ready to derive a version of the Ky-Fan best approximation theorem [7] (compare the result with [10, Theorem 3.1] and [21, Theorem 4.3]).

Theorem 3.10. *Let C be a nonempty closed convex subset of a strictly convex Banach space X with the Oshman property. If $T : C \to CC(X)$ is an H-continuous (or a $*$-nonexpansive) map and $T(C)$ is relatively compact, then there exists $y \in C$ such that*

$$d(y, Ty) = \|y - fy\| = d(fy, \text{cl}(I_C(y))), \quad \text{for some continuous selector } f \text{ of } T. \tag{3.10}$$

Proof. The Hausdorff continuity of T implies that $f = P_T : C \to X$ is a continuous selector of T. Since $T(C)$ is relatively compact and $f(C) \subseteq T(C)$, therefore $f(C)$ is relatively compact. By [18, Theorem 3(0)] and (3.3), we obtain

$$d(y,Ty) = d(y,fy) = d\big(fy,\mathrm{cl}\,(I_C(y))\big), \quad \text{for some } y \in C. \tag{3.11}$$

The proof for $*$-nonexpansive map is similar. □

As an application of Theorems 3.4 and 3.10, we obtain the following extension of [9, Theorem 3.2], [22, Corollary 1], and Theorem 3.3.

Corollary 3.11. *Let C be a nonempty closed convex subset of a strictly convex Banach space X with the Oshman property. If $T : C \to CC(C)$ is a $*$-nonexpansive map and $T(C)$ is relatively compact, then $F(T)$ is nonempty closed and convex.*

Remark 3.12. The map T in Example 3.2 is neither $*$-nonexpansive nor has a nonexpansive selection f with $F(f) = F(T)$; for if T is so, then using the same argument as in the proof of Theorem 3.4, T should have a nonexpansive selector $P_T : C \to C$ such that $F(T) = F(P_T)$, which should be convex by Theorem 3.1; a contradiction.

Xu [23] obtained the randomization of a remarkable fixed point theorem for multivalued nonexpansive maps due to Lim [16]. Further, Xu and Beg stated that it is unknown whether the fixed point set function F in this case is measurable (see [24, page 69]). We prove that the fixed point set function F is measurable if the underlying map is $*$-nonexpansive.

Theorem 3.13. *Let C be a nonempty separable closed bounded convex subset of a Banach space X and $T : \Omega \times C \to K(C)$ a $*$-nonexpansive random operator. Then the fixed point set function F of T given by $F(\omega) = \{x \in C : x \in T(\omega,x)\}$ is measurable (and hence T has a random fixed point) provided one of the following conditions holds:*

(i) *$T(\omega,\cdot)$ is convex for each $\omega \in \Omega$ and X is a uniformly convex space,*

(ii) *C is weakly compact and $I - T$ is demiclosed at 0,*

(iii) *C is weakly compact and X satisfies the Opial condition.*

Proof. (i) As before, for each $\omega \in \Omega$, $P_T(\omega,\cdot) : \Omega \times C \to C$ is a nonexpansive selector of $T(\omega,\cdot)$ and for each $y \in C$, $\omega \in \Omega$,

$$d\big(y,P_T(\omega,y)\big) = d\big(y,u_y\big) = d\big(y,T(\omega,y)\big). \tag{3.12}$$

By Proposition 2.1, $T(\cdot,x)$ is measurable if and only if for each x in C, the function $d(x,T(\cdot,x))$ is measurable. Thus by (3.12), for each x in C, $d(x,P_T(\cdot,x))$ is measurable and hence again by Proposition 2.1, $P_T(\cdot,x)$ is measurable (see also [12, Proposition 3.6]). Thus $P_T : \Omega \times C \to C$ is a nonexpansive random operator.

We observe that if X is a uniformly convex space, then $I - P_T(\omega,\cdot)$ is demiclosed. Also (3.12) implies that fixed point set function G of P_T given by $G(\omega) = \{x \in C : x = P_T(\omega,x)\}$ is equal to $F(\omega) = \{x \in C : x \in T(\omega,x)\}$ for each $\omega \in \Omega$. Consequently, G, and hence F, is measurable by Theorem 2.2.

(ii) Note that $P_T(\omega,\cdot):C\to K(C)$ is a nonexpansive selector of $T(\omega,\cdot)$. Also for each $y\in C$, $\omega\in\Omega$,

$$d(y,P_T(\omega,y))\le d(y,u_y)=d(y,T(\omega,y))\le d(y,P_T(\omega,y)). \tag{3.13}$$

The measurability of P_T follows from the arguments adopted in part (i) using (3.13) instead of (3.12). The demiclosedness of $I-T(\omega,\cdot)$ at 0 implies that $I-P_T(\omega,\cdot)$ is also demiclosed at 0 as follows.

Suppose that $x_n\to x_0$ weakly and $y_n\in I-P_T(\omega,x_n)$ with $y_n\to 0$ strongly. Note that $y_n\in I-P_T(\omega,x_n)\subseteq I-T(\omega,x_n)$ and $I-T(\omega,\cdot)$ is demiclosed at 0 so $0\in I-T(\omega,x_0)$ for each $\omega\in\Omega$. This implies that $x_0\in T(\omega,x_0)$ and hence $0=d(x_0,T(\omega,x_0))=d(x_0,P_T(\omega,x_0))$ for each $\omega\in\Omega$. Thus $x_0\in P_T(\omega,x_0)$ implies that $I-P_T(\omega,\cdot)$ is demiclosed at 0 for each $\omega\in\Omega$. Thus G, and hence, F is measurable by Theorem 2.2.

(iii) It is well known that if C is a weakly compact subset of a Banach space X satisfying the Opial condition and $f:C\to K(C)$ is nonexpansive, then $I-f$ is demiclosed on C. Hence, $I-P_T(\omega,\cdot)$ is demiclosed for each $\omega\in\Omega$ and the conclusion now follows from part (ii). □

Remark 3.14. It is not common at all that a nonexpansive multivalued mapping admits a single-valued nonexpansive selection (cf. Example 3.2 and Remark 3.12). However, in the general setup of metric linear spaces, $*$-nonexpansive maps have nonexpansive selector satisfying a very useful relation (3.3).

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