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NONLINEAR IMPULSIVE VOLTERRA INTEGRAL EQUATIONS IN BANACH SPACES AND APPLICATIONS[1]

DAJUN GUO[2]

Shangdong University, Jinan
Department of Mathematics
Shangdong 250100, PEOPLE'S REPUBLIC OF CHINA

ABSTRACT

In this paper, we first extend results on the existence of maximal solutions for nonlinear Volterra integral equations in Banach spaces to impulsive Volterra integral equations. Then, we give some applications to initial value problems for first order impulsive differential equations in Banach spaces. The results are demonstrated by means of an example of an infinite system for impulsive differential equations.

Key words: Impulsive Volterra integral equation, impulsive differential equation, cone and partial ordering, Darbo fixed point theorem.

AMS (MOS) subject classifications: 45N05, 47H07.

I. INTRODUCTION

Vaughn established an existence theorem of local maximal solutions for nonlinear Volterra integral equations in Banach spaces (see [3] or [2] Theorem 5.5.3). In this paper, we first extend Vaughn's result to nonlinear impulsive Volterra integral equations and obtain existence theorem of global maximal solutions and minimal solutions. And then, we offer some applications to initial value problems for first order impulsive differential equations in Banach spaces. Finally, we give an example of infinite system for impulsive differential equations.

Let the real Banach space E be partially ordered by a cone P of E, i.e. $x \leq y$ iff $y - x \in P$. Recall that cone P is said to be solid if its interior $int(P)$ is not empty. In this case, we write $x \ll y$ iff $y - x \in int(P)$ (see [1]). Let $J = [t_0, T]$, $t_0 < t_1 < \ldots < t_k < \ldots < t_m < T$ and $PC[J, E] = \{x$: x is a continuous map from J into E such that $x(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$ and its right limit at $t = t_k$ (denoted by $x(t_k^+)$) exists, $k = 1, 2, \ldots, m\}$. Evidently, $PC[J, E]$ is a Banach space with norm $\| x \|_p = \sup_{t \in J} \| x(t) \|$.

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Consider the nonlinear impulsive Volterra integral equation in E:

$$x(t) = x_0(t) + \int_{t_0}^{t} H(t,s,x(s))ds + \sum_{t_0 < t_k < t} a_k(t) I_k(x(t_k)), \tag{1}$$

where $x_0 \in PC[J,E]$, $H \in C[F \times E, E]$, $F = \{(t,s) \in J \times J: t \geq s\}$, $I_k \in C[E,E]$ and $a_k \in C[J_k^*, R^1]$, $J_k^* = [t_k, T]$ $(k = 1,2,\ldots,m)$. $x \in PC[J,E]$ is called a solution of Equation (1) if it satisfies (1) for all $t \in J$.

2. MAIN THEOREMS

In the following, let $a_k^* = \max_{t \in J_k^*} |a_k(t)|$ $(k = 1,2,\ldots,m)$, $T_r = \{x \in E: \|x\| \leq r\}$, $B_r = \{x \in PC[J,E]: \|x\|_p \leq r\}$ $(r > 0)$, $J_0 = [t_0, t_1]$, $J_1 = (t_1, t_2]$, $\ldots, J_{m-1} = (t_{m-1}, t_m]$, $J_m = (t_m, T]$ and α denotes the Kuratowski measure of noncompactness (see [2] Section 1.4). For $S \subset PC[J,E]$, we write $S(t) = \{x(t): x \in S\} \subset E$ $(t \in J)$ and $S(J) = \{x(t): x \in S, t \in J\}$ $\subset E$.

Consider the operator A defined by

$$Ax(t) = x_0(t) + \int_{t_0}^{t} H(t,s,x(s))ds + \sum_{t_0 < t_k < t} a_k(t) I_k(x(t_k)). \tag{2}$$

Lemma 1: *Suppose that, for any $r > 0$, H is uniformly continuous on $F \times T_r$, I_k is bounded on T_r, and there exists nonnegative constants L_r and $M_r^{(k)}$ with*

$$2(T - t_0)L_r + \sum_{k=1}^{m} a_k^* M_r^{(k)} < 1 \tag{3}$$

such that

$$\alpha(H(t,s,D)) \leq L_r \alpha(D),\ (t,s) \in F,\ D \subset T_r, \tag{4}$$

$$\alpha(I_k(D)) \leq M_r^{(k)} \alpha(D),\ D \subset T_r,\ k = 1,2,\ldots,m. \tag{5}$$

Then, for any $r > 0$, A is a strict set contraction from B_r into $PC[J,E]$, i.e. A is continuous and bounded and there exists a constant $0 \leq k_r < 1$ such that $\alpha(A(S)) \leq k_r \alpha(S)$ for any $S \subset B_r$.

Proof: It is easy to see that the uniform continuity of H on $F \times T_r$ implies the boundedness of H on $F \times T_r$, and so A is a bounded and continuous operator from B_r into $PC[J,E]$. By the uniform continuity of H and (4) and using Lemma 1.4.1 in [2], we have

$$\alpha(H(F \times D)) = \max_{(t,s) \in F} \alpha(H(t,s,D)) \leq L_r \alpha(D),\ D \subset T_r. \tag{6}$$

Now, let $S \subset B_r$ be arbitrarily given. By virtue of (2), it is easy to show that the elements of $A(S)$ are equicontinuous on each $J_k (k = 0, 1, \ldots, m)$, and so, by Lemma 1.4.1 in [2],

$$\alpha(A(S)) = \sup_{t \in J} \alpha(A(S(t))). \tag{7}$$

Using (6), (5) and the obvious formula

$$\int_{t_0}^{t} y(s)ds \in (t - t_0)\overline{co}\{y(s): t_0: t_0 \le s \le t\}, \quad y \in PC[J, E],\ t \in J,$$

we find

$$\begin{aligned} \alpha(A(S(t))) &\le (t - t_0)\alpha(\overline{co}\{H(t, s, x(s)): x \in S,\ t_0 \le s \le t\}) \\ &\quad + \sum_{t_0 < t_k < t} a_k^* \alpha(\{I_k(x(t_k)): x \in S\}) \\ &\le (T - t_0)\alpha(H(F \times S(J))) + \sum_{k=1}^{m} a_k^* \alpha(I_k(S(t_k))) \\ &\le (T - t_0)L_r \alpha(S(J)) + \sum_{k=1}^{m} a_k^* M_r^{(k)} \alpha(S(t_k)). \end{aligned} \tag{8}$$

For any given $\epsilon > 0$, there exists a partition $S = \bigcup_{j=1}^{p} S_j$ such that

$$diam(S_j) < \alpha(S) + \epsilon,\ (j = 1, 2, \ldots, p).$$

Since $S(t_k) = \bigcup_{j=1}^{p} S_j(t_k)$ and $diam(S_j(t_k)) \le diam(S_j)$, we have

$$\alpha(S(t_k)) \le \alpha(S) + \epsilon,\ (k = 1, 2, \ldots, m). \tag{9}$$

On the other hand, choosing $x_j \in S_j$ $(j = 1, 2, \ldots, p)$ and a partition $J_k = \bigcup_{i=1}^{n_k} J_k^{(i)}$ $(k = 0, 1, \ldots, m)$ such that

$$\| x_j(t) - x_j(t') \| < \epsilon,\ j = 1, 2, \ldots, p;\ t, t' \in J_k^{(i)}\ (k = 0, 1, \ldots, m;\ i = 1, 2, \ldots, n_k), \tag{10}$$

we have

$$S(J) = \bigcup \{S_j(J_k^{(i)}):\ i = 1, 2, \ldots, n_k;\ k = 0, 1, \ldots, m;\ j = 1, 2, \ldots, p\}.$$

For $x(t), \bar{x}(t') \in S_j(J_k^{(i)})$ (i.e. $x, \bar{x} \in S_j$, $t, t' \in J_k^{(i)}$), we find by (10)

$$\begin{aligned} \| x(t) - \bar{x}(t') \| &\le \| x(t) - x_j(t) \| + \| x_j(t) - x_j(t') \| + \| x_j(t') - \bar{x}(t') \| \\ &\le \| x - x_j \|_p + \epsilon + \| x_j - \bar{x} \|_p < 2diam(S_j) + \epsilon < 2\alpha(S) + 3\epsilon, \end{aligned}$$

which implies

$$a(S(J)) \leq 2a(S) + 3e. \tag{11}$$

Since e is arbitrary, it follows from (9) and (11) that

$$a(S(t_k)) \leq (S),\ (k = 1, 2, \ldots, m) \tag{12}$$

and

$$a(S(J)) \leq 2a(S). \tag{13}$$

Finally, (7), (8), (12) and (13) imply $a(A(S)) \leq k_r a(S)$, where

$$k_r = 2(T - t_0)L_r + \sum_{k=1}^{m} a_k^* M_r^{(k)}.$$

By (3), $k_r < 1$, and the lemma is proved.

Remark 1: The conditions of Lemma 1 are automatically satisfied if E is finite dimensional.

Lemma 2: *Let cone P be solid, $H(t,s,x)$ be nondecreasing in x (i.e. $x_1 \leq x_2$ implies $H(t,s,x_1) \leq H(t,s,x_2)$ for $(t,s) \in F$), I_k be strongly increasing (i.e. $x_1 \ll x_2$ implies $I_k(x_1) \ll I_k(x_2)$, $k = 1, 2, \ldots, m$), $a_k(t) \geq 0$ for $t \in J_k^*$ and $a_k(t_k) > 0$ $(k = 1, 2, \ldots, m)$. If $x_0, u, v \in PC[J, E]$ satisfy*

$$u(t) \leq x_0(t) + \int_{t_0}^{t} H(t,s,u(s))ds + \sum_{t_0 < t_k < t} a_k(t) I_k(u(t_k)),\ t \in J, \tag{14}$$

and

$$v(t) \gg x_0(t) + \int_{t_0}^{t} H(t,s,v(s))ds + \sum_{t_0 < t_k < t} a_k(t) I_k(v(t_k)),\ t \in J, \tag{15}$$

then $u(t_0) \ll v(t_0)$ implies $u(t) \ll v(t)$ for $t \in J$.

Proof: Suppose that the conclusion of the lemma is not true. Then the set $Z = \{t \in J\colon\ u(t) \ll v(t) \text{ does not hold}\}$ is not empty. Let $t^* = \inf Z$. From the continuity of u, v at t_0 and $u(t_0) \ll v(t_0)$ we know that $t_0 < t^* \leq T$. So $u(t) \ll v(t)$ for $t_0 \leq t < t^*$. It follows from the left continuous property of u, v at $t = t^*$ that $u(t^*) \leq v(t^*)$. Hence, by virtue of (14), (15) and the nondecreasing property of H and strongly increasing property of I_k,

$$u(t^*) \leq x_0(t^*) + \int_{t_0}^{t^*} H(t^*, s, u(s))ds + \sum_{t_0 < t_k < t^*} a_k^*(t^*) I_k(u(t_k))$$

$$\leq x_0(t^*) + \int_{t_0}^{t^*} H(t^*, s, v(s))ds + \sum_{t_0 \leq t_k < t^*} a_k(t^*) I_k(v(t_k)) \ll v(t^*). \tag{16}$$

There are two cases:

(a) $t^* \neq t_k (k = 1, 2, \ldots, m)$. In this case, u and v are continuous at $t = t^*$, and so (16) implies that there exists $\delta > 0$ such that $u(t) \ll v(t)$ for $t^* < t < t^* + \delta$, which contradicts the definition of t^*.

(b) $t^* = t_k$ for some k. In this case, we have by (14) and (15),

$$u(t_k^+) \leq x_0(t_k^+) + \int_{t_0}^{t_k} H(t_k, s, u(s))ds + \sum_{j=1}^{k} a_j(t_k) I_j(u(t_j)) \tag{17}$$

and

$$v(t_k^+) \geq x_0(t_k^+) + \int_{t_0}^{t_k} H(t_k, s, v(s))ds + \sum_{j=1}^{k} a_j(t_k) I_j(v(t_j)), \tag{18}$$

where $u(t_k^+)$, $v(t_k^+)$, $x_0(t_k^+)$ denote the right limits of u, v, x_0 at $t = t_k$, respectively. It follows from (16), (17), (18) and the strongly increasing property of I_k that $u(t_k^+) \ll v(t_k^+)$, and therefore, there exists $\delta^* > 0$ such that $u(t) \ll v(t)$ for $t^* = t_k < t < t_k + \delta^*$, which contradicts the definition of t^* too.

The proof is complete.

Remark 2: Lemma 2 is also true if (14) and (15) are replaced by

$$u(t) \ll x_0(t) + \int_{t_0}^{t} H(t, s, u(s))ds + \sum_{t_0 < t_k < t} a_k(t) I_k(u(t_k)), \ t \in J$$

and

$$v(t) \geq x_0(t) + \int_{t_0}^{t} H(t, s, v(s))ds + \sum_{t_0 < t_k < t} a_k(t) I_k(v(t_k)), \ t \in J.$$

Theorem 1: *Let the conditions of Lemma 1 be satisfied. Suppose that*

$$(T - t_0)c + \sum_{k=1}^{m} a_k^* c_k < 1, \tag{19}$$

where

$$c = \overline{\lim_{\| x \| \to \infty}} \left(\sup_{(t,s) \in F} \frac{\| H(t, s, x) \|}{\| x \|} \right) \tag{20}$$

and

$$c_k = \overline{\lim_{\| x \| \to \infty}} \frac{\| I_k(x) \|}{\| x \|} \ (k = 1, 2, \ldots, m). \tag{21}$$

Then, Equation (1) *has a solution in* $PC[J, E]$.

Proof: Choose $c' > c$ and $c'_k > c_k$ $(k = 1,2,\ldots,m)$ such that

$$b = (T - t_0)c' + \sum_{k=1}^{m} a_k^* c'_k < 1. \tag{22}$$

By virtue of (20) and (21), there exists $r > 0$ such that

$$\| H(t,s,x) \| < c' \| x \|, \ \| x \| \geq r, \ (t,s) \in F$$

and

$$\| I_k(x) \| < c'_k \| x \|, \ \| x \| \geq r \ (k = 1,2,\ldots,m),$$

so

$$\| H(t,s,x) \| \leq c' \| x \| + N, \ x \in E, \ (t,s) \in F \tag{23}$$

and

$$\| I_k(x) \| \leq c'_k \| x \| + N, \ x \in E, \tag{24}$$

where

$$N = max\{ \sup_{(t,s) \in F, x \in T_r} \| H(t,s,x) \|, \ \sup_{x \in T_r} \| I_k(x) \|, \ k = 1,2,\ldots,m\}.$$

Let

$$R = N'(1-b)^{-1}, \tag{25}$$

where b is defined by (22) and $N' = \| x_0 \|_p + (T - t_0 + \sum_{k=1}^{m} a_k^*)N$. For any $x \in B_R$ (i.e. $\| x \|_p \leq R$), we have by (2), (23), and (24),

$$\| Ax(t) \| \leq \| x_0(t) \| + \int_{t_0}^{t} (c' \| x(s) \| + N)ds + \sum_{k=}^{m} a_k^*(c'_k \| x(t_k) \| + N), \ t \in J$$

and so

$$\| Ax \|_p \leq \| x_0 \|_p + (T - t_0)(c' \| x \|_p + N) + \sum_{k=1}^{m} a_k^*(c'_k \| x \|_p + N)$$
$$= b \| x \|_p + N' \leq bR + N' = R.$$

Consequently, $A: B_R \rightarrow B_R$. On the other hand, by Lemma 1, A is a strict set contraction. Hence, the Darbo fixed point theorem (see [2] theorem 5.3.1) implies that A has a fixed point in B_R. The proof is complete.

Theorem 2: *Let the assumptions of Theorem 1 hold. Suppose that cone P is solid, $H(t,s,x)$ is nondecreasing in x and I_k are strongly increasing $(k = 1,2,\ldots,m)$. Then Equation (1) has maximal solution v and minimal solution u in $PC[J,E]$, i.e.*

$u(t) \leq x(t) \leq v(t)$ $(t \in J)$ for any solution x of Equation (1) *in* $PC[J,E]$.

Proof: Choose $y \in int(P)$, $\| y \| = 1$ and let

$$A_n x(t) = Ax(t) + \tfrac{1}{n}y$$
$$= x_0(t) + \tfrac{1}{n}y + \int_{t_0}^{t} H(t,s,x(s))ds + \sum_{t_0 < t_k < t} a_k(t) I_k(x(t_k)),\ (n = 1,2,3,\ldots). \tag{26}$$

By Theorem 1 and (25), A_n has a fixed point x_n in $PC[J,E]$ such that $\| x_n \|_p \leq R_n = N'_n(1-b)^{-1}$, where $N'_n = \| x_0 + n^{-1}y \|_p + (T - t_0 + \sum_{k=1}^{m} a_k^*)N$. So,

$$\| x_n \|_p \leq R^*,\ (n = 1,2,3,\ldots), \tag{27}$$

where

$$R^* = (1-b)^{-1}\{ \| x_0 \|_p + \| y \| + (T - t_0 + \sum_{k=1}^{m} a_k^*)N\} = const.$$

and

$$x_n(t) = Ax_n(t) + \tfrac{1}{n}y$$
$$= x_0(t) + \tfrac{1}{n}y + \int_{t_0}^{t} H(t,s,x_n(s))ds + \sum_{t_0 < t_k < t} a_k(t) I_k(x_n(t_k)),\ (n = 1,2,3,\ldots). \tag{28}$$

Let $S = \{x_n : n = 1,2,3,\ldots\}$. By virtue of (27) and Lemma 1, there exists $0 \leq k^* < 1$ such that $\alpha(A(S)) \leq k^* \alpha(S)$. On the other hand, (28) implies

$$\alpha(S) \leq \alpha(A(S)) + \alpha(\{\tfrac{1}{n}y : n = 1,2,3,\ldots\}) = \alpha(A(S)).$$

So, we have $\alpha(S) = 0$, and hence, there are $\{x_{n_i}\} \subset \{x_n\}$ and $v \in PC[J,E\}$ such that $\| x_{n_i} - v \|_p \to 0$ $(i \to \infty)$. Observing the uniform continuity of H on $F \times T_{R^*}$ and taking limit in (28) along n_i, we get

$$v(t) = x_0(t) + \int_{t_0}^{t} H(t,s,v(s))ds + \sum_{t_0 < t_k < t} a_k(t) I_k(v(t_k)),\ t \in J,$$

i.e. v is a solution of Equation (1).

Now, let x be any solution of Equation (1) in $PC[J,E]$, i.e.

$$x(t) = x_0(t) + \int_{t_0}^{t} H(t,s,x(s))ds + \sum_{t_0 < t_k < t} a_k(t) I_k(x(t_k)),\ t \in J. \tag{29}$$

By (28), we have

$$x_n(t) \gg x_0(t) + \int_{t_0}^{t} H(t,s,x_n(s))ds + \sum_{t_0 < t_k < t} a_k(t) I_k(x_n(t_k))\ t \in J. \tag{30}$$

In addition,

$$x_n(t_0) = x_0(t_0) + \tfrac{1}{n}y \gg x_0(t_0) = x(t_0). \tag{31}$$

So, (29), (30), (31) and Lemma 2 imply

$$x_n(t) \gg x(t),\ t \in J,\ n = 1,2,3,\ldots. \tag{32}$$

Taking the limit in (32) along n_i, we obtain $v(t) \geq x(t)$ for $t \in J$. Consequently, v is the maximal solution of Equation (1) in $PC[J,E]$.

Similarly, considering sequence of operators

$$A_n^* x(t) = Ax(t) - \tfrac{1}{n}y$$
$$= x_0(t) - \tfrac{1}{n}y + \int_{t_0}^{t} H(t,s,x(s))ds + \sum_{t_0 < t_k < t} a_k(t) I_k(x(t_k)) \quad (n = 1,2,3,\ldots)$$

instead of sequence (26), we can get the minimal solution u of Equation (1) in $PC[J,E]$. The proof is complete.

Theorem 3: *Let the assumptions of Theorem 2 hold. Let $m \in PC[J,E]$ and u and v be the minimal and maximal solutions of Equation* (1) *in $PC[J,E]$ respectively. The following conclusions hold:*

(*a*) *if*

$$m(t) \leq x_0(t) + \int_{t_0}^{t} H(t,s,m(s))ds + \sum_{t_0 < t_k < t} a_k(t) I_k(m(t_k)),\ t \in J, \tag{33}$$

then $m(t) \leq v(t)$ for $t \in J$.

(*b*) *if*

$$m(t) \geq x_0(t) + \int_{t_0}^{t} H(t,s,m(s))ds + \sum_{t_0 < t_k < t} a_k(t) I_k(m(t_k)),\ t \in J,$$

then $m(t) \geq u(t)$ for $u \in J$.

Proof: We need only to prove (*a*) since the proof of (*b*) is similar. As in the proof of Theorem 2, (30) holds, and, by (33),

$$x_n(t_0) = x_0(t_0) + \tfrac{1}{n}y \gg x_0(t_0) \geq m(t_0). \tag{34}$$

It follows from (30), (33), (34) and Lemma 2 that

$$x_n(t) \gg m(t),\ t \in J,\ n = 1,2,3,\ldots,$$

which implies by taking limit along n_i that $v(t) \geq m(t)$ for $t \in J$. The proof is complete.

3. APPLICATIONS

This section applies Theorem 2 to the IVP of the nonlinear impulsive differential equation in E:

$$\begin{cases} x' = f(t,x),\ t \neq t_k & (k = 1,2,\ldots,m), \\ \Delta x \mid_{t = t_k} = I_k(x(t_k)), & (k = 1,2,\ldots,m), \\ x(0) = x_0, & \end{cases} \tag{35}$$

where $f \in C[J \times E, E]$, $J = [0,T]$ $(T > 0)$, $x_0 \in E$, $I_k \in C[E,E]$, $0 < t_1 < \ldots < t_k < \ldots < t_m < T$. Let $J' = J \backslash \{t_1,\ldots,t_m\}$. $x \in PC[J,E] \cap C^1[J',E]$ is called a solution of IVP (35) if it satisfies (35).

Lemma 3: *Let $y, z \in PC[J,E]$ and M be a constant. Then, the IVP of linear impulsive differential equation*

$$\begin{cases} x' + Mx = y(t), t \neq t_k & (k = 1,2,\ldots,m), \\ \Delta x \mid_{t = t_k} = I_k(z(t_k)), & (k = 1,2,\ldots,m), \\ x(0) = x_0 & \end{cases} \tag{36}$$

has a unique solution in $PC[J,E] \cap C^1[J,E]$ which is given by

$$x(t) = x_0 e^{-Mt} + \int_0^t e^{-M(t-s)} y(s)ds + \sum_{0 < t_k < t} e^{-M(t-t_k)} I_k(z(t_k)). \tag{37}$$

Proof: Let $x(t)$ be defined by (37). Evidently, $x \in PC[J,E]$, $x(0) = x_0$ and $\Delta x \mid_{t = t_k} = I_k(z(t_k))$. By (37), we have

$$x(t)e^{Mt} = x_0 + \int_0^t e^{Ms} y(s)ds + \sum_{0 < t_k < t} e^{Mt_k} I_k(z(t_k)),$$

so, direct differentiation implies, for $t \neq t_k$,

$$(x'(t) + Mx(t))e^{Mt} = e^{Mt} y(t).$$

Hence, $x \in C^1[J',E]$ and it is a solution of (36).

It remains to show the uniqueness of solution. Let $x_1, x_2 \in PC[J,E] \cap C^1[J',E]$ be two solutions of (36) and let $\bar{x} = x_1 - x_2$. Then $\bar{x}' + M\bar{x} = \theta$ for $t \neq t_k$, where θ denotes the zero element of E, and so

$$(\bar{x}\, e^{Mt})' = (\bar{x}' + M\bar{x})e^{Mt} = \theta,\ t \neq t_k\ (k = 1, 2, \ldots, m). \tag{38}$$

Since $\bar{x}(0) = \theta$, it follows that

$$\bar{x}(t) = \bar{x}(0)e^{-Mt} = \theta \text{ for } 0 \leq t \leq t_1.$$

On the other hand,

$$\Delta x_1 \mid_{t = t_1} = I_1(z(t_1)) = \Delta x_2 \mid_{t = t_1},$$

hence

$$x_1(t_1^+) = x_1(t_1) + \Delta x_1 \mid_{t = t_1} = x_2(t_1) + \Delta x_2 \mid t_{t = t_1} = x_2(t_1^+). \tag{39}$$

It follows from (38) and (39) that

$$\bar{x}(t) = \bar{x}(t_1^+)e^{-M(t - t_1)} = \theta \text{ for } t_1 < t \leq t_2.$$

Similarly, we can show $\bar{x}(t) = \theta$ for $t_2 < t \leq T$, i.e. $x_1 = x_2$, and the lemma is proved.

Now, consider the impulsive Volterra integral equation

$$x(t) = x_0 e^{-Mt} + \int_0^t e^{-M(t-s)}[f(s, x(s)) + Mx(s)]ds + \sum_{0 < t_k < t} e^{-M(t - t_k)} I_k(x(t_k)). \tag{40}$$

Lemma 4: *$x \in PC[J, E] \cap C^1[J', E]$ is a solution of IVP (35) if and only if $x \in PC[J, E]$ is a solution of Equation (40).*

Proof: For $z \in PC[J, E]$, Lemma 3 implies that the linear problem

$$\begin{cases} x' = f(t, z) - M(x - z), t \neq t_k & (k = 1, 2, \ldots, m), \\ \Delta x \mid_{t = t_k} = I_k(z(t_k)), & (k = 1, 2, \ldots, m), \\ x(0) = x_0 & \end{cases} \tag{41}$$

has a unique solution of $PC[J, E] \cap C^1[J', E]$ which is given by

$$x(t) = x_0 e^{-Mt} + \int_0^t e^{-M(t-s)}[f(s, z(s)) + Mz(s)]ds + \sum_{0 \leq t_k < t} e^{-M(t - t_k)} I_k(z(t_k)). \tag{42}$$

Let $x = Bz$, i.e.

$$Bz(t) = x_0 e^{-Mt} + \int_0^t e^{-M(t-s)}[f(s,z(s)) + Mz(s)]ds$$

$$+ \sum_{0 < t_k < t} e^{-M(t-t_k)} I_k(z(t_k)).$$

It is easy to see from (41) that x is a solution of IVP (35) if and only if $x = Bz = z$, that is, x is a solution of Equation (40). The lemma is proved.

Theorem 4: *Let cone P be solid and I_k be strongly increasing $(k = 1,2,\ldots,)$. Assume that, for any $r > 0$, f is uniformly continuous on $J \times T_r$ and there exist nonnegative constants L_r and $M_r^{(k)}$ with*

$$2T(L_r + M) + \sum_{k=1}^{m} M_r^{(k)} < 1 \tag{43}$$

such that

$$\alpha(f(t,D)) \leq L_r \alpha(D),\ t \in J,\ D \subset T_r \tag{44}$$

and

$$\alpha(I_k(D)) \leq M_r^{(k)} \alpha(D),\ D \subset T_r \quad (k = 1,2,\ldots,m), \tag{45}$$

where M is a nonnegative constant independent of r. Assume further that,

$$T(c + M) + \sum_{k=1}^{m} c_k < 1, \tag{46}$$

where

$$c = \overline{\lim_{\|x\| \to \infty}} \Big(\sup_{t \in J} \frac{\| f(t,x) \|}{\| x \|} \Big),\ c_k = \overline{\lim_{\|x\| \to \infty}} \frac{\| I_k(x) \|}{\| x \|} \ (k = 1,2,\ldots,m),$$

and

$$f(t,x) - f(t,y) \geq -M(x-y) \text{ for } x \geq y \quad (x,y \in E). \tag{47}$$

Then IVP (35) has maximal solution v and minimal solution u in $PC[J,E] \cap C^1[J',E]$, i.e. $u(t) \leq x(t) \leq v(t)$ $(t \in J)$ for any solution x of IVP (35) in $PC[J,E] \cap C^1[J',E]$.

Proof: By Lemma 4, $x \in PC[J,E] \cap C^1[J',E]$ is a solution of IVP (35) iff $x \in PC[J,E]$ is a solution of Equation (40). Evidently, (40) is an integral equation of type (1), where $t_0 = 0$,

$$x_0(t) = x_0 e^{-Mt},\ H(t,s,x) = e^{-M(t-s)}[f(s,x) + Mx],$$

$$a_k(t) = e^{-M(t-t_k)} \quad (k = 1,2,\ldots,m).$$

Since

$$\alpha(H(t,s,D)) \leq \alpha(f(s,D)) + M\alpha(D) \leq (L_r + M)\alpha(D),\ (t,s) \in F,\ D \subset T_r,$$

$$a_k(t) > 0 \text{ for } t \in I_k, a_k(t_k) = a_k^* = 1,$$

$$\overline{\lim_{\|x\| \to \infty}} \left(\sup_{(t,s) \in F} \frac{\| H(t,s,x) \|}{\| x \|} \right) \leq \overline{\lim_{\|x\| \to \infty}} \left(\sup_{s \in J} \frac{\| f(s,x) \| + M \| x \|}{\| x \|} \right) = c + M,$$

and, by virtue of (47), $H(t,s,x)$ is nondecreasing in x, all conditions of Theorem 2 are satisfied, and so, Theorem 2 implies that Equation (40) has maximal solution v and minimal solution u in $PC[J,E]$. The proof is complete.

Example: Consider the IVP in terms of the infinite system of impulsive differential equations:

$$x_n' = \frac{1}{n+3}(2t - x_n) = \sqrt[3]{\frac{t^2 - x_{n+1}}{n}} + \frac{1}{8n^2} ln(1 + e^t x_{2n}), \quad 0 \leq t \leq 1, t \neq \frac{1}{2}$$

$$\Delta x_n |_{t=\frac{1}{2}} = \frac{1}{4} x_n(\tfrac{1}{2}), \tag{48}$$

$$x_n(0) = cos\ n, (n = 1,2,3,\ldots).$$

A solution $x = (x_1, \ldots, x_n, \ldots)$ of IVP (48) is called bounded if $\sup_n | x_n(t) | < \infty$ for $0 \leq t \leq 1$.

Conclusion: IVP (48) has maximal bounded solution $v = (v_1, \ldots, v_n, \ldots)$ and minimal bounded solution $u = (u_1, \ldots, u_n, \ldots)$, i.e. for any bounded solution $x = (x_1, \ldots, x_n, \ldots)$ of (48), $u(t) \leq x(t) \leq v(t)$ $(t \in [0,1])$ holds.

Proof: Let $E = \{x = (x_1, \ldots, x_n, \ldots): \sup_n | x_n | < \infty\}$ with norm $\| x \| = \sup_n | x_n |$, $P = \{x = (x_1, \ldots, x_n, \ldots\} \in E;\ x_n \geq 0,\ n = 1,2,3,\ldots\}$. It is well known that E is a Banach space and P is a solid cone in E with $int(P) = \{x = (x_1, \ldots, x_n, \ldots) \in E; \inf_n x_n > 0\}$. Evidently, the bounded solution $x = (x_1, \ldots, x_n, \ldots)$ of IVP (48) is equivalent to the solution $x \in PC[J,E] \cap C^1[J', E]$ of the following IVP in E:

$$\begin{cases} x' = f(t,x), & 0 \leq t \leq 1, t \neq \frac{1}{2}; \\ \Delta x |_{t=\frac{1}{2}} = \frac{1}{4} x(\frac{1}{2}), & \\ x(0) = x_0, & \end{cases} \tag{49}$$

where $x_0 = (cos\ 1, \ldots, cos\ n, \ldots) \in E$, $f = (f_1, \ldots, f_n, \ldots)$, in which

$$f_n(t,x) = \frac{1}{n+3}(2t - x_n) - \sqrt[3]{\frac{t^2 - x_{n+1}}{n}} + \frac{1}{8n^2} ln(1 + e^t x_{2n}), \ (n = 1,2,3,\ldots), \tag{50}$$

$T = 1$, $J = [0,1]$, $m = 1$, $t_1 = \frac{1}{2}$ and

$$I_1(x) = \frac{1}{4} x. \tag{51}$$

Obviously, (49) is of the form (35). It is clear that $f \in C[J \times E, E]$ and, for any $r > 0$, f is uniformly continuous on $J \times T_r$, $I_1 \in C[E, E]$, I_1 is strongly increasing and

$$\alpha(I_1(D)) = \frac{1}{4}\alpha(D) \text{ for bounded } D \subset E. \tag{52}$$

We now show

$$\alpha(f(t, D)) = 0 \text{ for } t \in J \text{ and bounded } D \subset E. \tag{53}$$

In fact, let $x^{(p)} = (x_1^{(p)}, \ldots, x_n^{(p)}, \ldots) \in D$ and $y^{(p)} = f(t, x^{(p)}), y^{(p)} = (y_1^{(p)}, \ldots, y_n^{(p)}, \ldots)$ $(p = 1, 2, 3, \ldots)$. Then there exists positive constant d such that

$$|x_n^{(p)}| \leq d\ n, (p = 1, 2, 3, \ldots).$$

Since, by (50),

$$y_n^{(p)} = f_n(t, x^{(p)}) = \frac{1}{n+3}(2t - x_n^{(p)}) - \sqrt[3]{\frac{t^2 - x_{n+1}^{(p)}}{n}} + \frac{1}{8n^2} ln(1 + e^t x_{2n}^{(p)}),$$

we have

$$|y_n^{(p)}| \leq \frac{2+d}{n+3} + \sqrt[3]{\frac{1+d}{n}} + \frac{1}{8n^2} ln(1 + e^d),\ (n, p = 1, 2, 3, \ldots). \tag{54}$$

Hence $\{y_n^{(p)}\}$ is bounded, so, by the diagonal method, we can select a subsequence $\{p_i\}$ of $\{p\}$ such that

$$\lim_{i \to \infty} y_n^{(p_i)} = y_n \quad (n = 1, 2, 3, \ldots). \tag{55}$$

It follows from (54) that

$$|y_n| \leq \frac{2+d}{n+3} + \sqrt[3]{\frac{1+d}{n}} + \frac{1}{8n^2} ln(1 + e^d), \quad (n = 1, 2, 3, \ldots). \tag{56}$$

Consequently, $y = (y_1, \ldots, y_n, \ldots) \in E$. For any given $\epsilon > 0$, (54) and (56) imply that there exists a positive integer N such that

$$|y_n^{(p_i)}| < \frac{\epsilon}{3},\ |y_n| < \frac{\epsilon}{3} \text{ for } n > N,\ i = 1, 2, 3, \ldots. \tag{57}$$

On the other hand, (55) implies that there exists i_0 such that

$$|y_n^{(p_i)} - y_n| < \frac{\epsilon}{3} \text{ for } i > i_0,\ n = 1, 2, \ldots, N. \tag{58}$$

It follows from (57) and (58) that

$$\| y^{(p_i)} - y \| = \sup_n |y_n^{(p_i)} - y_n| < \epsilon \text{ for } i > i_0,$$

i.e. $\| y^{(p_i)} - y \| \to 0$. Hence, $f(t, D)$ is relatively compact and (53) holds. For $x = (x_1, \ldots, x_n, \ldots) \in E$, $y = (y_1, \ldots, y_n, \ldots) \in E$, $x \geq y$ (i.e. $x_n \geq y_n, n = 1, 2, 3, \ldots$) and $t \in J$, we have by (50),

$$f_n(t,x) - f_n(t,y) = -\frac{1}{n+3}(x_n - y_n) + \frac{1}{3\sqrt{n}}\left(\sqrt[3]{x_{n+1} - t^2} - \sqrt[3]{y_{n+1} - t^2}\right)$$

$$+ \frac{1}{8n^2}(ln(1 + e^t x_{2n}) - ln(1 + e^t y_{2n}))$$

$$\geq -\frac{1}{4}(x_n - y_n),\ n = 1, 2, 3, \ldots,$$

so, (47) is satisfied for $M = \frac{1}{4}$ and (43) is also satisfied since, by (52) and (53), $L_r = 0$, $M_r^{(1)} = \frac{1}{4}$ and $2T(L_r + M) + M_r^{(1)} = \frac{1}{2} + \frac{1}{4} < 1$.

Now, we check (46). By virtue of (50), we have

$$\| f(t,x) \| \leq \frac{1}{4}(2t + \| x \|) + \sqrt[3]{t^2 + \| x \|} + \frac{1}{8}ln(1 + e^{t \| x \|}),$$

and so

$$c = \overline{\lim_{\| x \| \to \infty}} \left(\sup_{t \in J} \frac{\| f(t,x) \|}{\| x \|}\right)$$

$$\leq \lim_{\| x \| \to \infty} \| x \|^{-1}\left(\frac{1}{4}(2 + \| x \|) + \sqrt[3]{1 + \| x \|} + \frac{1}{8}ln(1 + e^{\| x \|})\right) = \frac{3}{8}.$$

On the other hand, (51) implies

$$c_1 = \overline{\lim_{\| x \| \to \infty}} \frac{\| I_1(x) \|}{\| x \|} = \frac{1}{4}.$$

Hence $T(c + M) + c_1 \leq \frac{7}{8} < 1$, i.e. (46) holds. Finally, our conclusion follows from Theorem 4.

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