Journal of Applied Mathematics and Stochastic Analysis 6, Number 2, Summer 1993, 153-160

A NOTE ON CONTROLLABILITY OF NEUTRAL VOLTERRA INTEGRODIFFERENTIAL SYSTEMS[1]

K. BALACHANDRAN and P. BALASUBRAMANIAM

Bharathiar University
Department of Mathematics
Coimbatore 641 046
Tamil Nadu, INDIA

ABSTRACT

Sufficient conditions for the controllability of nonlinear neutral Volterra integrodifferential systems are established. The results are obtained using the Schauder fixed point theorem.

Key words: Controllability, neutral Volterra systems, fixed point method.

AMS (MOS) subject classifications: 93B05.

1. INTRODUCTION

The theory of functional differential equations has been studied by several authors [6, 9, 11, 15]. The problem of controllability of linear neutral systems has been investigated by Banks and Kent [3], and Jacobs and Langenhop [12]. Motivation for physical control systems and its importance in other fields can be found in [11, 13]. Angell [1] and Chukwu [4] discussed the functional controllability of nonlinear neutral systems, and Underwood and Chukwu [16] studied the null controllability for such systems. Further, Chukwu [5] considered the Euclidian controllability of a neutral system with nonlinear base. Onwuatu [14] discussed the problem for nonlinear systems of neutral functional differential equations with limited controls. Gahl [10] derived a set of sufficient conditions for the controllability of nonlinear neutral systems through the fixed point method developed by Dauer [7]. Recently, Balachandran [2] established sufficient conditions for the controllability of nonlinear neutral Volterra integrodifferential systems and infinite delay neutral Volterra systems. In this paper we shall derive a new set of sufficient conditions for the controllability of nonlinear neutral Volterra integrodifferential systems by suitably adopting the technique of Do [8].

[1]Received: November, 1991. Revised: March, 1993.

2. PRELIMINARIES

Let $J = [0, t_1], t_1 > 0$ and let Q be the Banach space of all continuous functions

$$(x, u): J \times J \rightarrow R^n \times R^m$$

with the norm defined by

$$\| (x, u) \| = \| x \| + \| u \|$$

where $\| x \| = \sup_{t \in J} | x(t) |$. Define the norm of a continuous $n \times m$ matrix valued function $D: J \rightarrow R^n \times R^m$ by

$$\| D(t) \| = \max_i \sum_{j=1}^{m} \max_{t \in J} | d_{ij}(t) |$$

where d_{ij} are the elements of D.

Consider the linear neutral Volterra integrodifferential system of the form

$$\frac{d}{dt}[x(t) - \int_0^t C(t-s)x(s)ds - g(t)] = Ax(t) + \int_0^t G(t-s)x(s)ds + B(t)u(t) \qquad (1)$$

and the nonlinear system

$$\frac{d}{dt}[x(t) - \int_0^t C(t-s)x(s)ds - g(t)]$$

$$= Ax(t) + \int_0^t G(t-s)x(s)ds + B(t)u(t) + f(t, x(t), u(t)), \qquad (2)$$

where $x(t) \in R^n, u(t) \in R^m, C(t)$ and $G(t)$ are $n \times n$ continuous matrix valued functions and $B(t)$ is a continuous $n \times m$ matrix valued functions, A a constant $n \times n$ matrix and f and g are, respectively, continuous and absolutely continuous vector functions. We consider the controllability on a bounded interval J of system (2).

Definition 1 [18]: A function $x: J \rightarrow R^n$ is said to be a solution of the initial value problem (1) or (2) through $(0, x(0))$ on J, if

(*i*) x is continuous on J,

(*ii*) $[x(t) - \int_0^t C(t-s)x(s)ds - g(t)]$ is absolutely continuous on J,

(*iii*) (1) or (2) holds almost everywhere on J.

Definition 2: The system (2) is said to be controllable on J if for every $x(0)$, $x_1 \in R^n$ there exists a control function $u(t)$ defined on J such that the solution of (2) satisfies

$x(t_1) = x_1$.

The solution of (1) can be written, as in [17], in the form

$$x(t) = Z(t)[x(0) - g(0)] + g(t) + \int_0^t \dot{Z}(t-s)g(s)ds + \int_0^t Z(t-s)B(s)u(s)ds,$$

where $\dot{Z}(t-s) = \frac{\partial Z}{\partial t}(t-s)$ and $Z(t)$ is an $n \times n$ continuously differentiable matrix satisfying the equation

$$\frac{d}{dt}[Z(t) - \int_0^t C(t-s)Z(s)ds] = AZ(t) - \int_0^t G(t-s)Z(s)ds$$

with $Z(0) = I$ and the solution of nonlinear system (2) is given by

$$x(t) = Z(t)[x(0) - g(0)\} + g(t) + \int_0^t \dot{Z}(t-s)g(s)ds$$
$$+ \int_0^t Z(t,s)[B(s)u(s) + f(s, x(s), u(s))]ds.$$

Define the matrix W by

$$W(t) = \int_0^t Z(t-s)B(s)B^*(s)Z^*(t-s)ds,$$

where the $*$ denotes the transpose matrix. We know that system (1) is controllable on J if and only if W is nonsingular [2].

It is clear that x_1 can be obtained if there exist continuous functions $x(\cdot)$ and $u(\cdot)$ such that

$$u(t) = B^*(t)Z^*(t_1 - t)W^{-1}(t_1)[x_1 - Z(t_1)(x(0) - g(0)) - g(t_1)$$
$$- \int_0^{t_1} \dot{Z}(t_1 - s)g(s)ds - \int_0^{t_1} Z(t_1 - s)f(s, x(s), u(s))ds] \tag{3}$$

and

$$x(t) = Z(t)[x(0) - g(0)] + g(t) + \int_0^t \dot{Z}(t-s)g(s)ds$$
$$+ \int_0^t Z(t-s)[B(s)u(s) + f(s, x(s), u(s))]ds. \tag{4}$$

Now, we will find conditions for the existence of such $x(\ )$ and $u(\ )$. If $\alpha_i \in L^1(J)$ $(i = 1, 2, \dots, q)$ then $\| \alpha_i \|$ $(i = 1, 2, \dots, q)$ is the L^1 norm of $\alpha_i(s)$ $(i = 1, 2, \dots, q)$. That is,

$$\| \alpha_i \| = \int_0^{t_1} | \alpha_i(s) | \, ds \quad (i = 1,2,\ldots,q).$$

Next, for our convenience, let us introduce the following notations:

$$K = max\{ \| Z(t-s) \| : 0 \le s \le t \le t_1\},$$

$$k = max\{ \| Z(t-s)B(s) \| \, t_1, 1\},$$

$$a_1 = 3k\{ \| B^*(s)Z^*(t_1 - s) \| \; \| W^{-1}(t_1) \| \; \| Z(t_1 - s) \| \; \| \alpha_i \| \}, \quad (i = 1,2,\ldots,q)$$

$$b_i = 3K \| \alpha_i \|, \qquad (i = 1,2,\ldots,q)$$

$$c_i = max\{a_i, b_i\}, \qquad (i = 1,2,\ldots,q)$$

$$d_1 = 3k \| B^*(s)Z^*(t_1 - s) \| \; \| W^{-1}(t_1) \| [\, | x_1 | + \| Z(t_1) \| \; | x(0) - g(0) | + | g(t_1) | + | \int_0^{t_1} \dot{Z}(t_1 - s)g(s)ds | \,],$$

$$d_2 = 3 \| Z(t_1) \| \; | x(0) - g(0) | + | g(t_1) | + | \int_0^{t_1} \dot{Z}(t_1 - s)g(s)ds |,$$

$$d = max\{d_1, d_2\}.$$

3. MAIN RESULTS

Now, we will prove the following main theorem, which is a generalization of Theorem 2 of Balachandran [2].

Theorem: *Let measurable functions* $\phi_i: R^n \times R^m \rightarrow R^+$ $(i = 1,2,\ldots,q)$ *and* L^1 *functions* $\alpha_i: J \rightarrow R^+$ $(i = 1,2,\ldots,q)$ *be such that*

$$| f(t,x,u) | \le \sum_{i=1}^{q} \alpha_i(t)\phi_i(x,u) \quad \textit{for every } (t,x,u) \in J \times R^n \times R^m.$$

Then the controllability of (1) *implies the controllability of* (2) *if*

$$\lim_{r \to \infty} sup\left(r - \sum_{i=1}^{q} c_i sup\{\phi_i(x,u): \| (x,u) \| \le r\}\right) = \infty. \tag{5}$$

Proof: Define $T: Q \rightarrow Q$ by

$$T(x,u) = (y,v),$$

where

$$v(t) = B^*(t)Z^*(t_1 - t)W^{-1}(t_1)[x_1 - Z(t_1)(x(0) - g(0)) - g(t_1)$$

$$- \int_0^{t_1} \dot{Z}(t_1 - s)g(s)ds - \int_0^{t_1} Z(t_1 - s)f(s, x(s), u(s))ds] \tag{6}$$

and

$$y(t) = Z(t)[x(0) - g(0)] + g(t) + \int_0^t \dot{Z}(t-s)g(s)ds$$
$$+ \int_0^t Z(t-s)[B(s)v(s) + f(s, x(s), u(s))]ds. \tag{7}$$

Based on our assumptions, T is continuous. Clearly the solutions $u(\cdot)$ and $x(\cdot)$ to (3) and (4) are fixed points of T. We will prove the existence of such fixed points by using the Schauder fixed point theorem.

Let $\psi_i(r) = sup\{\phi_i(x,u) \colon \| (x,u) \| \le r\}$. Since (5) holds, there exists $r_0 > 0$ such that

$$\sum_{i=1}^{q} c_i \psi_i(r_0) + d \le r_0.$$

Now, let

$$Q_{r_0} = \{(x,u) \in Q \colon \| (x,u) \| \le r_0\}.$$

If $(x,u) \in Q_{r_0}$, from (6) and (7), we have

$$\| v \| \le \| B^*(t)Z^*(t_1 - t) \| \, \| W^{-1}(t_1) \| [\, | x_1 | + \| Z(t_1) \| \, | x(0) - g(0) |$$
$$+ | g(t_1) | + | \int_0^{t_1} \dot{Z}(t_1 - s)g(s)ds | + \int_0^{t_1} \| Z(t_1 - s) \| \sum_{i=1}^{q} \alpha_i(s)\phi_i(x(s), u(s))ds]$$
$$\le (d_1/3k) + (1/3k) \sum_{i=1}^{q} a_i \psi_i(r_0)$$

$$\le (1/3k)(d + \sum_{i=1}^{q} c_i \psi_i(r_0))$$

$$\le (r_0/3k) \le (r_0/3)$$

and

$$\| y \| \le \| Z(t) \| \, | x(0) - g(0) | + | g(t) | + | \int_0^t \dot{Z}(t-s)g(s)ds |$$

$$+ \int_0^t \| Z(t-s)B(s) \| \, \| v \| ds + \int_0^t \| Z(t,s) \| \sum_{i=1}^{q} \alpha_i(s)\phi_i(x(s), u(s))ds$$
$$\le (d/3) + k \| v \| + K \sum_{i=1}^{q} \| \alpha_i \| \psi_i(r_0)$$

$$\leq (d/3) + k \| v \| + (1/3)\sum_{i=1}^{q} c_i\psi_i(r_0)$$

$$\leq (1/3)(d + \sum_{i=1}^{q} c_i\psi_i(r_0)) + k \| v \|$$

$$\leq (r_0/3) + (r_0/3) = 2(r_0/3).$$

Hence T maps Q_{r_0} into itself. Further, it is easy to see that $T(Q_r)$ is equicontinuous for all $r > 0$ [8]. By the Ascoli-Arzela theorem, $\overline{T(Q_{r_0})}$ is compact in Q. Since Q_{r_0} is nonempty, closed, bounded and convex, by the Schauder fixed point theorem, solutions of (3) and (4) exist. Hence the proof is complete.

To apply the above theorem we have to construct α_i's and ϕ_i's such that (5) is satisfied. These constructions are different for different situations. However, an obvious construction of α_i's and ϕ_i's is easily achieved by taking $q = 1$, $\alpha_1 = \alpha = 1$ and

$$\phi_1(x,u) = \phi(x,u) = sup\{ \mid f(t,x,u) \mid : t \in J\}.$$

In this case (5) holds if

$$\lim_{r \to \infty} inf(1/r)sup\{\phi(x,u) \colon \| (x,u) \| \leq r\} < 1/c_1.$$

Now, we will state a corollary which is a particular case of the above theorem and it was proved by Balachandran [2].

Corollary: *If the continuous function f satisfies the condition*

$$\lim_{\| (x,u) \| \to \infty} \frac{\mid f(t,x,u) \mid}{\| (x,u) \|} = 0$$

uniformly in $t \in J$ *and if system* (1) *is controllable on* J*, then system* (2) *is controllable on* J.

REFERENCES

[1] Angell, T.S., On controllability for nonlinear hereditary systems: A fixed point approach, *Nonlinear Analysis: Theory, Methods and Applications* **4** (1980), 529-548.

[2] Balachandran, K., Controllability of neutral Volterra integrodifferential systems, *J. of Australian Math. Soc.*, Ser. **B**, **34** (1992), 18-25.

[3] Banks, H.T. and G.A. Kent, Control of functional differential equations of retarded and neutral type to target sets in function space. *SIAM J. on Control* **10**, (1972), 562-593.

[4] Chukwu, E.N., Functional inclusion and controllability of nonlinear neutral functional differential systems, *J. of Optim. Theory and Appl.* **29** (1979), 291-300.

[5] Chukwu, E.N., On the Euclidian controllability of a neutral system with nonlinear base, *Nonlinear Analysis: Theory, Methods and Applications* **11** (1987), 115-123.

[6] Chukwu, E.N. and H.C. Simpson, Perturbations on nonlinear systems of neutral type, *J. of Diff. Equations* **82** (1989), 28-59.

[7] Dauer, J.P., Nonlinear perturbations of quasilinear control systems, *J. of Math. Anal. and Appl.* **54** (1976), 717-725.

[8] Do, V.N., Controllability of semilinear systems, *J. of Optim. Theory and Appl.* **65** (1990), 41-52.

[9] Fuchs, E., The degeneracy property in linear autonomous functional differential equations of neutral type, *J. of Math. Anal. and Appl.* **90** (1983), 527-549.

[10] Gahl, R.D., Controllability of nonlinear systems of neutral type, *J. of Math. Anal. and Appl.* **63** (1978), 33-42.

[11] Hale, J.K., *Theory of Functional Differential Equations*, Springer Verlag, New York 1977.

[12] Jacobs, M.Q. and C.E. Langenhop, Criteria for function space controllability of linear neutral systems, *SIAM J. on Control and Optim.* **14** (1976), 1009-1048.

[13] O'Conner, D.A., *State Controllability and Observability for Linear Neutral Systems*, Washington University, D.Sc. Thesis 1978.

[14] Onwuatu, J.U., On the null controllability in function space of nonlinear systems of neutral functional differential equations with limited controls, *J. of Optim. Theory and Appl.* **42** (1984), 397-420.

[15] Tadmore, G., Functional differential equations of retarded and neutral type: analytic solutions and piecewise continuous controls, *J. of Diff. Equations* **51** (1984), 151-181.

[16] Underwood, R.G. and E.N. Chukwu, Null controllability of nonlinear neutral differential equations, *J. of Math. Anal. and Appl.* **129** (1988), 326-345.

[17] Wu, J., Globally stable periodic solutions of linear neutral Volterra integrodifferential equations, *J. of Math. Anal. and Appl.* **130** (1988), 474-483.

[18] Wu, J., Unified treatment of local theory of NFDES with infinite delay, *Tamkang J. of Math.* **22** (1991), 51-72.