J. of Inequal. & Appl., 1999, Vol. 4, pp. 215–240
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A Weighted Isoperimetric Inequality and Applications to Symmetrization

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(Received 20 October 1998; Revised 15 January 1999)

We prove an inequality of the form $\int_{\partial\Omega} a(|x|)\mathcal{H}_{n-1}(\mathrm{d}x) \geq \int_{\partial B} a(|x|)\mathcal{H}_{n-1}(\mathrm{d}x)$, where Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary, B is a ball centered in the origin having the same measure as Ω. From this we derive inequalities comparing a weighted Sobolev norm of a given function with the norm of its symmetric decreasing rearrangement. Furthermore, we use the inequality to obtain comparison results for elliptic boundary value problems.

Keywords: Weighted isoperimetric inequality; Weighted Sobolev norm; Symmetric decreasing rearrangement; Comparison theorem

AMS Subject Classification: 26D10, 35B05, 35J25

1 INTRODUCTION

Consider a boundary integral of the type

$$p_a(\Omega) := \int_{\partial\Omega} a(x)\,\mathcal{H}_{n-1}(\mathrm{d}x), \tag{1}$$

where a is a given nonnegative function on $\mathbf{R}^n$ and Ω is a smooth open set. It can be seen as a weighted perimeter of Ω. The classical isoperimetric

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theorem in Euclidean space says that, if $a \equiv 1$, then

$$p_a(\Omega^\sharp) \leq p_a(\Omega), \tag{2}$$

where $\Omega^\sharp$ is the ball centered at the origin having the same Lebesgue measure of Ω (see [27]). By employing the so-called *method of level sets* one can infer a lot of further functional inequalities from the isoperimetric theorem, thus comparing underlying problems with simpler – one-dimensional – ones. The literature for this theme is large. As an orientation we refer to the monographies [5,15,23] and to the articles [1,12,26].

Recently Rakotoson and Simon [24,25] studied the problem of minimizing $p_a(\Omega)$ over the class of open sets with given, *fixed* measure.

We are interested in the question, for which general type of weights a (2) might hold. In Section 2 we prove inequality (2) for *radial* weights $a = a(|x|)$ satisfying some further conditions. In Section 3 using the method of level sets, we show integral inequalities comparing some *weighted* Sobolev norm of a function with a corresponding norm of its symmetric decreasing rearrangement. In Section 4 an extension of one of these inequalities to BV-spaces leads to a general version of our weighted isoperimetric inequality for Caccioppoli sets. We also include a discussion of the equality case in the inequality. We mention that weighted norm inequalities which are similar to ours, are known for the so-called starlike rearrangements (see [6,7,16,18,19]) and for the Steiner symmetrization (see [9]). As an application of the weighted isoperimetric inequalitiy (2), in Section 5 we derive a comparison result for elliptic PDE. To be more specific, let us consider the problem

$$\begin{cases} Lu = -(a_{ij}u_{x_j})_{x_i} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where

(i) Ω is an open bounded subset of $\mathbf{R}^n$,
(ii) a_{ij} are real valued measurable functions on Ω which satisfy

$$a_{ij}(x)\xi_i\xi_j \geq \nu(|x|)|\xi|^2 \quad \forall \xi \in \mathbf{R}^n, \text{ for a.e. } x \in \Omega,$$

with $\nu(|x|) \geq 0$ on Ω,

(iii) f and ν^{-1} in suitable Lebesgue spaces which guarantee the existence of a weak solution.

Assuming that the weighted isoperimetric inequality (2) holds with $a=\sqrt{\nu(|x|)}$, we prove that $u^\sharp \le v$, where v is the solution of a problem whose data are radially symmetric. Here $u^\sharp$ denotes the Schwarz symmetrization of u (see Section 3 for definition). Results in this order of ideas are contained, for example, in [17,28] when the operator L is uniformly elliptic and in [2]. Such result allows us to estimate any Orlicz norm of u by simply evaluating the norm of v.

2 THE SMOOTH CASE

For any measurable set E with finite Lebesgue measure let $E^\sharp$ denote the ball B_R with center at the origin and $m(E)=m(B_R)$. Here and in what follows $m(E)$ denotes the Lebesgue measure of E.

Throughout the paper we will assume that $a:[0,+\infty[\to[0,+\infty[$ satisfies

$$a(t),\quad (t\ge 0),\quad \text{is nondecreasing and} \tag{4}$$

$$(a(z^{1/n})-a(0))z^{1-(1/n)}\ (z\ge 0),\quad \text{is convex.} \tag{5}$$

Frequently we will write

$$a_1(t):=a(t)-a(0),\quad (t\ge 0).$$

Remark 2.1 Note that (5) is satisfied, for instance, in the cases

$$a(t)=t^p,\ (t\ge 0),\quad \text{for } p\ge 1,$$

or, more generally, if $a(t)$ $(t\ge 0)$, is nondecreasing and convex.

For $n\ge 2$ we shall use n-dimensional polar coordinates $(r,\theta_1,\dots,\theta_{n-1})$, to represent any point $x=(x_1,\dots,x_n)\in\mathbf{R}^n$ (compare [16]):

$$\begin{aligned} |x| &= r,\\ x_1 &= r\cos\theta_1,\\ x_k &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{k-1}\cos\theta_k \quad \text{for } k=2,\dots,n-1,\\ x_n &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-1}, \end{aligned} \tag{6}$$

where $r \geq 0$, $0 \leq \theta_k \leq \pi$ for $k = 1, \ldots, n-2$, and $-\pi \leq \theta_{n-1} \leq \pi$. Let θ denote the vector of the angular coordinates $(\theta_1, \ldots, \theta_{n-1})$ and T the $(n-1)$-dimensional hypercube $[0, \pi]^{n-2} \times [-\pi, \pi]$.

There are functions $h, h_m \in C(T)$ satisfying

$$h(\theta) > 0, \quad h_m(\theta) > 0 \qquad \text{a.e. in } T\ (m = 1, \ldots, n-1),$$

such that, if Σ is any smooth $(n-1)$-dimensional hypersurface with representation

$$\Sigma: \quad \{(r, \theta) \colon r = \rho(\theta),\ \theta \in \bar{T}_0\},$$

where T_0 is an open subset of T with Lipschitz boundary and $\rho \in C^1(\bar{T}_0)$, then

$$\int_\Sigma a(|x|)\ \mathcal{H}_{n-1}(\mathrm{d}x) = \int_{T_0} a(\rho) \left\{ 1 + \rho^{-2} \sum_{m=1}^{n-1} \left(\frac{\partial \rho}{\partial \theta_m} \right)^2 h_m \right\}^{1/2} \rho^{n-1} h \, \mathrm{d}\theta. \tag{7}$$

Note that

$$\mathcal{H}_{n-1}(B_1) = n\omega_n = \int_T h(\theta)\, \mathrm{d}\theta, \tag{8}$$

where $\omega_n = \pi^{n/2}[\Gamma(n/2 + 1)]^{-1}$ is the measure of the n-dimensional unit ball.

THEOREM 2.1 *Let Ω be a bounded open set with Lipschitz boundary. Then*

$$\int_{\partial\Omega} a(|x|)\ \mathcal{H}_{n-1}(\mathrm{d}x) \geq \int_{\partial\Omega^\sharp} a(|x|)\ \mathcal{H}_{n-1}(\mathrm{d}x)$$
$$= n\,\omega_n^{1/n} a\big((\omega_n^{-1} m(\Omega))^{1/n}\big) \big(m(\Omega)\big)^{1-1/n}. \tag{9}$$

Proof To show inequality in (9), we divide the proof into three steps.

Step 1 Let $n \geq 2$ and suppose that

$$\begin{aligned} &\partial\Omega \quad \text{is piecewise affin and} \\ &\{(r, \theta) \colon\ r > 0\} \cap \partial\Omega \quad \text{is a discrete set for every } \theta \in T. \end{aligned} \tag{10}$$

Let us observe that, to show (9), it is sufficient to prove the following inequality:

$$I \geq I^{\sharp}, \tag{11}$$

where

$$I := \int_{\partial\Omega} a_1(|x|)\, \mathcal{H}_{n-1}\,(\mathrm{d}x),$$

$$I^{\sharp} := \int_{\partial\Omega^{\sharp}} a_1(|x|)\, \mathcal{H}_{n-1}\,(\mathrm{d}x).$$

Indeed, (11) and the isoperimetric inequality (Appendix 2) yield

$$\begin{aligned}\int_{\partial\Omega} a(|x|)\, \mathcal{H}_{n-1}(\mathrm{d}x) &= I + a(0) \int_{\partial\Omega} \mathcal{H}_{n-1}\,(\mathrm{d}x)\\ &\geq I^{\sharp} + a(0) \int_{\partial\Omega^{\sharp}} \mathcal{H}_{n-1}\,(\mathrm{d}x)\\ &= \int_{\partial\Omega^{\sharp}} a(|x|)\, \mathcal{H}_{n-1}\,(\mathrm{d}x).\end{aligned}$$

In view of the assumption (10), we have the following representations:

$$\begin{aligned}\partial\Omega &= \{(r,\theta)\colon\ r = r_{ij}(\theta),\ \theta \in \bar{T}_i,\ j = 1,\dots,2k_i,\ i = 1,\dots,l\},\\ \bar{\Omega} &= \{(r,\theta)\colon\ r_{i,2\kappa-1}(\theta) \leq r \leq r_{i,2\kappa}(\theta),\ \theta \in \bar{T}_i,\\ &\qquad \kappa = 1,\dots,k_i,\ i = 1,\dots,l\},\end{aligned} \tag{12}$$

where the sets T_i $(i = 1,\dots,l)$, are open, pairwise disjoint subsets of T with Lipschitz boundary,

$$\begin{aligned}& r_{ij} \in C^1(\bar{T}_i), \quad (j = 1,\dots,2k_i),\\ & r_{i,1}(\theta) < \cdots < r_{i,2k_i}(\theta), \quad \text{for } \theta \in T_i,\\ & r_{i,1}\begin{cases} = 0 & \text{if } 0 \in \Omega\\ > 0 & \text{if } 0 \notin \Omega, \end{cases} \quad (i = 1,\dots,l).\end{aligned} \tag{13}$$

Using (7) and (12), we compute

$$I = \sum_{i=1}^{l} \sum_{j=1}^{2k_i} \int_{T_i} a_1(r_{ij}) \left\{ 1 + (r_{ij})^{-2} \sum_{m=1}^{n-1} \left(\frac{\partial r_{ij}}{\partial \theta_m} \right)^2 h_m \right\}^{1/2} (r_{ij})^{n-1} h \, \mathrm{d}\theta. \quad (14)$$

By setting

$$z_{ij} := (r_{ij})^n, \quad (j = 1, \ldots, 2k_i, \; i = 1, \ldots, l), \quad (15)$$

we obtain from (13) and (14)

$$I \geq \sum_{i=1}^{l} \sum_{j=1}^{2k_i} \int_{T_i} a_1((z_{ij})^{(1/n)})(z_{ij})^{1-(1/n)} h \, \mathrm{d}\theta$$

$$\geq \sum_{i=1}^{l} \int_{T_i} a_1((z_{i,2k_i})^{(1/n)})(z_{i,2k_i})^{1-(1/n)} h \, \mathrm{d}\theta =: I_1. \quad (16)$$

Let $\Omega^\sharp = B_R$, $(R > 0)$. By using (18), (15) and (12), we see that

$$m(B_R) = \omega_n R^n = (1/n) \sum_{i=1}^{l} \sum_{j=1}^{2k_i} \int_{T_i} z_{ij}(-1)^j h \, \mathrm{d}\theta,$$

and hence, by (13),

$$R = \left((n\omega_n)^{-1} \sum_{i=1}^{l} \sum_{j=1}^{2k_i} \int_{T_i} z_{ij}(-1)^j h \, \mathrm{d}\theta \right)^{1/n}$$

$$\leq \left((n\omega_n)^{-1} \sum_{i=1}^{l} \int_{T_i} z_{i,2k_i} h \, \mathrm{d}\theta \right)^{1/n} =: R_1. \quad (17)$$

Furthermore, we have by (4) and (17)

$$I^\sharp = n\omega_n a_1(R) R^{n-1} \leq n\omega_n a_1(R_1) R_1^{n-1}. \quad (18)$$

Now, in view of the assumption (5), we may apply Jensen's inequality (see Appendix 1) to obtain from (16) and (17)

$$
\begin{aligned}
I_1 &\geq n\omega_n a_1\left(\left[(n\omega_n)^{-1}\sum_{i=1}^{l}\int_{T_i} z_{i,2k_i} h\,\mathrm{d}\theta\right]^{1/n}\right)\\
&\quad\times\left((n\omega_n)^{-1}\sum_{i=1}^{l}\int_{T_i} z_{i,2k_i} h\,\mathrm{d}\theta\right)^{1-(1/n)}\\
&= n\omega_n a_1(R_1)R_1^{n-1}.
\end{aligned}
$$

Together with (16) and (18), this proves (11) in the case under consideration.

Step 2 Let $n = 1$ and suppose that

$$\Omega = \bigcup_{\kappa=1}^{k}(x_{2\kappa-1}, x_{2\kappa}) \quad \text{where } x_1 < \cdots < x_{2k}. \tag{19}$$

As in the previous case, we prove that $I \geq I^{\sharp}$. Then we compute

$$I = \sum_{i=1}^{2k} a_1(|x_i|) \tag{20}$$

and

$$I^{\sharp} = 2a_1\left(\frac{1}{2}\sum_{i=1}^{2k} x_i(-1)^i\right). \tag{21}$$

By (19) we have that

$$x_{2k} - x_1 \geq \sum_{i=2}^{2k-1} x_i(-1)^{i-1} \geq 0.$$

In view of (20), (21) and (5) this means that

$$
\begin{aligned}
I &\geq a_1(|x_{2k}|) + a_1(|x_1|)\\
&\geq 2a_1(\tfrac{1}{2}(|x_{2k}| + |x_1|)) \geq I^{\sharp}.
\end{aligned}
$$

Step 3 Let $\partial\Omega$ be Lipschitz. We can find a sequence of sets $\{\Omega_k\}$ satisfying (19) if $n=1$, respectively (10) if $n \geq 2$, and such that

$$\lim_{k\to\infty} m((\Omega_k \backslash \Omega) \cup (\Omega \backslash \Omega_k)) = 0,$$

$$\lim_{k\to\infty} \mathcal{H}_{n-1}(\Omega_k) = \mathcal{H}_{n-1}(\Omega).$$

By previous steps, the inequality (9) holds for Ω_k. Since $a(|x|)$ is continuous, this means that

$$\begin{aligned}\int_\Omega a(|x|)\mathcal{H}_{n-1}(\mathrm{d}x) &= \lim_{k\to\infty}\int_{\Omega_k} a(|x|)\ \mathcal{H}_{n-1}(\mathrm{d}x)\\ &\geq \lim_{k\to\infty}\int_{(\Omega_k)^\sharp} a(|x|)\ \mathcal{H}_{n-1}(\mathrm{d}x) = \int_{\Omega^\sharp} a(|x|)\ \mathcal{H}_{n-1}(\mathrm{d}x).\end{aligned}$$

Remark 2.2 The proof of Theorem 2.1 much simplifies if Ω is starlike with respect to the origin. We leave it to the reader to confirm that the assumption (4) is superfluous in this case.

3 WEIGHTED SOBOLEV INEQUALITIES

We recall some definitions and basic properties (see [15,26]).

Let $u: \mathbf{R}^n \to \mathbf{R}$ be a measurable function which decays at infinity, i.e. $m\{x: |u(x)| > t\}$ is finite for every positive t. The map

$$\mu_u(t) = m\{x: |u(x)| > t\}, \quad (t \geq 0),$$

is called the *distribution function* of u; it is a decreasing and right-continuous in $[0, +\infty)$.

The function u^* defined by

$$u^*(s) = \inf\{t \geq 0: \ \mu_u(t) \leq s\}, \quad (s \geq 0),$$

is called the *decreasing rearrangement* of u; it is a decreasing and right-continuous function on $[0, +\infty)$. Furthermore it satisfies the following properties:

$$\begin{aligned} \mu_u(u^*(s)) \le s \quad & \forall s \ge 0, \\ \mu_u(u^*(s)-) \ge s \quad & \forall s \in [0, m(\operatorname{supp} u)], \\ b - a = m\{x \in \mathbf{R}^n\colon\ & u^*(a) \ge |u(x)| > u^*(b)\} \\ \text{if } 0 \le a < b \le & m(\operatorname{supp} u); \end{aligned} \tag{22}$$

in other words, u^* is an inverse function of μ_u. The function $u^\sharp$, defined by

$$u^\sharp(x) = u^*(\omega_n |x|^n), \quad (x \in \mathbf{R}^n),$$

is called the *Schwarz symmetrization* of u. It is nonnegative, radial and radially decreasing; moreover u and $u^\sharp$ are *equidistributed*, i.e.

$$m\{x\colon\ |u(x)| > t\} = m\{x\colon\ u^\sharp(x) > t\} \quad \forall t > 0. \tag{23}$$

The mapping $u \longmapsto u^\sharp$ is a *contraction in* $L^p(\mathbf{R}^n)$ for $1 \le p < +\infty$ (compare [15]), i.e.

$$\text{if } u, v \in L^p(\mathbf{R}^n), \text{ then } \|u^\sharp - v^\sharp\|_{L^p(\mathbf{R}^n)} \le \|u - v\|_{L^p(\mathbf{R}^n)}. \tag{24}$$

Now we prove the following theorem:

THEOREM 3.1 *Let* $G: [0, +\infty[\to [0, +\infty[$ *be nondecreasing and convex with* $G(0) = 0$ *and let* $u: \mathbf{R}^n \to \mathbf{R}_0^+$ *be Lipschitz continuous and decays at infinity, i.e.* $m\{x\colon |u(x)| > t\} < \infty$ *for every* $t > 0$. *Then*

$$\int_{\mathbf{R}^n} G(a(|x|)|\nabla u(x)|)\, \mathrm{d}x \ge \int_{\mathbf{R}^n} G(a(|x|)|\nabla u^\sharp(x)|)\, \mathrm{d}x, \tag{25}$$

provided the left integral in (25) *converges.*

Proof The proof is divided in three steps.

Step 1 We claim that for every $s \in (0, m(\operatorname{supp} u))$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s} \int_{\{x\colon |u(x)| > u^*(s)\}} G(a(|x|)|\nabla u(x)|)\, \mathrm{d}x \\ &\quad \ge G\left(\frac{\mathrm{d}}{\mathrm{d}s} \int_{\{x\colon |u(x)| > u^*(s)\}} a(|x|)|\nabla u(x)|\, \mathrm{d}x \right), \end{aligned} \tag{26}$$

where $\operatorname{supp} u$ denotes the support of the function u. Let $0 \le s < s+h \le m(\operatorname{supp} u)$. Then Jensen's inequality (Appendix 1)) gives

$$\frac{1}{h}\int_{\{x:\ u^*(s+h)\ge|u(x)|>u^*(s)\}} G(a(|x|)|\nabla u(x)|)\,\mathrm{d}x$$

$$\ge G\left(\frac{1}{h}\int_{\{x:\ u^*(s+h)\ge|u(x)|>u^*(s)\}} a(|x|)|\nabla u(x)|\,\mathrm{d}x\right).$$

Sending $h \to 0$, and by taking into account (22), we obtain (26).

Step 2 We claim that for every $s \in (0, m(\operatorname{supp} u))$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_{\{x:\ |u(x)|>u^*(s)\}} a(|x|)|\nabla u(x)|\,\mathrm{d}x \ge -n\omega_n^{1/n}s^{1-1/n}a(\omega_n^{-1/n}s^{1/n})\,\frac{\mathrm{d}u^*}{\mathrm{d}s}. \tag{27}$$

Let $0 \le s < s+h \le m(\operatorname{supp} u)$. Then we have

$$\frac{1}{h}\int_{\{x:\ u^*(s)\ge|u(x)|>u^*(s+h)\}} a(|x|)|\nabla u(x)|\,\mathrm{d}x$$

$$= \frac{1}{h}\int_{u^*(s+h)}^{u^*(s)} \mathrm{d}t \int_{\{x:\ |u(x)|=t\}} a(|x|)\ \mathcal{H}_{n-1}(\mathrm{d}x)$$

(by the coarea formula (Appendix 3))

$$\ge \frac{1}{h}\int_{u^*(s+h)}^{u^*(s)} n\omega_n^{1/n}\mu_u(t)^{1-1/n}a(\omega_n^{-1/n}\mu_u(t)^{1/n})\,\mathrm{d}t \quad \text{(by Theorem 2.1)}$$

$$\ge \frac{1}{h}(u^*(s) - u^*(s+h))n\omega_n^{1/n} \inf_{t\in[u^*(s+h),u^*(s)]} \mu_u(t)^{1-1/n}a(\omega_n^{-1/n}\mu_u(t)^{1/n}).$$

Passing to the limit $h \to 0$, this yields (27).

Step 3 We have that

$$\int_{\mathbf{R}^n} G(a(|x|)|\nabla u(x)|)\,\mathrm{d}x$$

$$= \int_0^{+\infty} \mathrm{d}s \frac{\mathrm{d}}{\mathrm{d}s}\left\{\int_{\{x:\ |u(x)|>u^*(s)\}} G(a(|x|)|\nabla u(x)|)\,\mathrm{d}x\right\}$$

(by the coarea formula)

$$\geq \int_0^{+\infty} \mathrm{d}s\ G\left(\frac{\mathrm{d}}{\mathrm{d}s}\int_{\{x:\ |u(x)|>u^*(s)\}} a(|x|)|\nabla u(x)|\,\mathrm{d}x\right) \quad \text{(by (26))}$$

$$\geq \int_0^{+\infty} \mathrm{d}s\ G\left(-n\omega_n^{1/n}s^{1-(1/n)}a(\omega_n^{-1/n}s^{1/n})\frac{\mathrm{d}u^*}{\mathrm{d}s}\right). \quad \text{(by (27))}$$

But since u^* is radially decreasing, this last expression is equal to

$$\int_{\mathbf{R}^n} G(a(|x|)|\nabla u^\sharp(x)|)\,\mathrm{d}x.$$

By specializing $G(t) = t^p$ in Theorem 3.1, we get the following

COROLLARY 3.1 *Let* $u \in W^{1,p}(\mathbf{R}^n)$ *for some* $p \in [1, +\infty)$. *Then*

$$\int_{\mathbf{R}^n} a^p(|x|)|\nabla u(x)|^p\,\mathrm{d}x \geq \int_{\mathbf{R}^n} a^p(|x|)|\nabla u^\sharp(x)|^p\,\mathrm{d}x, \tag{28}$$

provided the left integral in (28) *converges.*

Proof If u is Lipschitz continuous and decays at infinity, then (28) follows from Theorem 3.1.

In the general case we choose a sequence $\{u_k\} \subset C_0^\infty(\mathbf{R}^n)$ such that

$$u_k \longrightarrow u \quad \text{in } W^{1,p}(\mathbf{R}^n).$$

By (24) we have that

$$(u_k)^\sharp \longrightarrow u^\sharp \quad \text{in } L^p(\mathbf{R}^n), \tag{29}$$

Since $\|\nabla(u_k)^\sharp\|_{L^p(\mathbf{R}^n)} \le \|\nabla(u_k)^\sharp\|_{L^p(\mathbf{R}^n)}$, the functions $(u_k)^\sharp$ are equibounded in $W^{1,p}(\mathbf{R}^n)$. Together with (29) this implies that for a subsequence $\{(u_{k'})^\sharp\}$,

$$(u_{k'})^\sharp \rightharpoonup u^\sharp \quad \text{weakly in } W^{1,p}(\mathbf{R}^n).$$

In view of the weak lower semi-continuity of the integral in (28) we obtain

$$\begin{aligned}\int_{\mathbf{R}^n} a^p(|x|)|\nabla u^\sharp(x)|^p\,\mathrm{d}x &\le \liminf_{k'\to\infty}\int_{\mathbf{R}^n} a^p(|x|)|\nabla(u_{k'})^\sharp(x)|^p\,\mathrm{d}x\\ &\le \lim_{k\to\infty}\int_{\mathbf{R}^n} a^p(|x|)|\nabla u_k(x)|^p\,\mathrm{d}x\\ &= \int_{\mathbf{R}^n} a^p(|x|)|\nabla u(x)|^p\,\mathrm{d}x.\end{aligned}$$

Remark 3.1 We did not use assumption (4) in the proof of Theorem 3.1. In view of Remark 2.2, the results of this section remain true, if a satisfies (5) but not (4), and if the level sets of u are starlike with respect to the origin, i.e.

$$v_e(t) := u(te), \ (t \ge 0), \quad \text{is nonincreasing for every } e \in \mathbf{R}^n. \tag{30}$$

4 THE GENERAL CASE

Our aim is to generalize Theorem 2.1 to Caccioppoli sets. The theory of these sets is imbedded in the framework of spaces $BV(\Omega)$, where Ω is an open set of $\mathbf{R}^n$. Recall that any measurable set $E \subset \Omega$ satisfying $\|D\chi_E\|_{BV(\Omega)} < +\infty$, is called a *Caccioppoli set*, and the quantity

$$p(E) := \|D\chi_E\|_{BV(\mathbf{R}^n)}$$

is called the *perimeter* of E (in the sense of De Giorgi). As an extension of this definition, for any function $u \in BV(\mathbf{R}^n)$ we set

$$f_a(u) := \sup\left\{\int_{\mathbf{R}^n} u(x)\mathrm{div}(a(|x|)\varphi(x))\,\mathrm{d}x,\right.\\ \left.\varphi \in C_0^\infty(\mathbf{R}^n,\mathbf{R}^n),\ |\varphi| \le 1\right\},$$

and for any Caccioppoli set E we call the quantity

$$p_a(E) := f_a(\chi_E)$$

the *weighted perimeter of E* (with weight a) (see also [3,24,25]). Note that f_a is a nonnegative, convex and weakly lower semi-continuous functional on $BV(\mathbf{R}^n)$, and, since

$$f_a(u) \leq \sup\{a(|x|):\ x \in \operatorname{supp} u\} \|Du\|_{BV(\mathbf{R}^n)} \quad \forall u \in BV(\mathbf{R}^n),$$

$f_a(u)$ is at least finite if $\operatorname{supp} u$ is bounded. Furthermore,

$$\text{if } u \in W^{1,1}(\mathbf{R}^n) \text{ and } f_a(u) < +\infty, \text{ then}$$
$$f_a(u) = \int_{\mathbf{R}^n} a(|x|)|\nabla u(x)|\,\mathrm{d}x. \tag{31}$$

LEMMA 4.1 *If E is a bounded open set with Lipschitz boundary, then*

$$p_a(E) = \int_{\partial E} a(|x|)\ \mathcal{H}_{n-1}\,(\mathrm{d}x). \tag{32}$$

Proof It is well-known that $p(E \cap U) = \mathcal{H}_{n-1}(\partial(E \cap U))$ for every open set U (see [14]). Since $a(|x|)$ is continuous, this yields (32).

LEMMA 4.2 *Let* $\{u_k\} \subset W^{1,1}(\mathbf{R}^n)$, $u \in BV(\mathbf{R}^n)$,

$$u_k \longrightarrow u \quad \textit{in } L^1(\mathbf{R}^n)$$

and

$$\lim_{k\to\infty} \|\nabla u_k\|_{L^1(\mathbf{R}^n)} = \|Du\|_{BV(\mathbf{R}^n)}. \tag{33}$$

Then

$$\lim_{k\to\infty} \int_{\mathbf{R}^n} a(|x|)|\nabla u_k(x)|\,\mathrm{d}x = f_a(u). \tag{34}$$

Proof (33) implies

$$\lim_{k\to\infty} \|\nabla u_k\|_{L^1(U)} = \|Du\|_{BV(U)} \quad \text{for every open set } U,$$

(compare [14]). Since $a(|x|)$ is continuous, this yields (34).

THEOREM 4.1 *Let* $u \in BV(\mathbf{R}^n)$ *and* $f_a(u^\sharp) < +\infty$. *Then*

$$f_a(u) \geq f_a(u^\sharp). \tag{35}$$

Proof We choose a sequence $\{u_k\} \subset W^{1,1}(\mathbf{R}^n)$, such that (33) is satisfied. From (24) we see that

$$(u_k)^\sharp \longrightarrow u^\sharp \quad \text{in } L^1(\mathbf{R}^n). \tag{36}$$

Since

$$\|\nabla (u_k)^\sharp\|_{L^1(\mathbf{R}^n)} \leq \|\nabla u_k\|_{L^1(\mathbf{R}^n)}, \quad (k = 1, 2, \ldots),$$

the functions $(u_k)^\sharp$ are equibounded in $W^{1,1}(\mathbf{R}^n)$. It follows that for a subsequence $\{(u_{k'})^\sharp\}$,

$$(u_{k'})^\sharp \rightharpoonup u^\sharp \quad \text{weakly in } BV(\mathbf{R}^n).$$

Since the functional f_a is weakly lower semi-continuous, this implies

$$f_a(u^\sharp) \leq \liminf_{k'\to\infty} f_a((u_{k'})^\sharp). \tag{37}$$

But by (31) and Corollary 3.1 we have that

$$f_a((u_k)^\sharp) = \|a|\nabla (u_k)^\sharp|\|_{L^1(\mathbf{R}^n)} \leq \|a|\nabla u_k|\|_{L^1(\mathbf{R}^n)} = f_a(u_k).$$

Together with (36) and (37) this concludes the proof of the Theorem.

By choosing $u = \chi_E$ in (35), we obtain a generalized form of Theorem 2.1.

THEOREM 4.2 (*Weighted isoperimetric inequality*) *Let E be a Caccioppoli set in* $\mathbf{R}^n$. *Then*

$$\begin{aligned} p_a(E) &\geq p_a(E^\sharp) \\ &= n\omega_n^{1/n} a((\omega_n^{-1} m(E))^{1/n})(m(E))^{1-(1/n)}. \end{aligned} \tag{38}$$

Next we analyze the case of equality in (38). We need two auxiliary lemmata.

LEMMA 4.3 *Let A, B be Caccioppoli sets with $p_a(A) < \infty$ and $p_a(B) < \infty$. Then*

$$p_a(A \cap B) + p_a(A \cup B) \leq p_a(A) + p_a(B). \tag{39}$$

Proof If A and B are bounded, open sets with Lipschitz boundary, then (39) follows by Lemma 4.1.

In the general case we find sequences $\{A_k\}$ and $\{B_k\}$ of bounded, open sets with Lipschitz boundary, and such that

$$\lim_{k\to\infty} m((A_k \backslash A) \cup (A \backslash A_k)) = 0,$$

$$\lim_{k\to\infty} m((B_k \backslash B) \cup (B \backslash B_k)) = 0,$$

$$\lim_{k\to\infty} \mathcal{H}_{n-1}(\partial A_k) = p(A)$$

and

$$\lim_{k\to\infty} \mathcal{H}_{n-1}(\partial B_k) = p(B),$$

(compare [14]). Since $a(|x|)$ is continuous, this yields

$$\lim_{k\to\infty} p_a(A_k) = \lim_{k\to\infty} \int_{\partial A_k} a(|x|)\ \mathcal{H}_{n-1}(\mathrm{d}x) = p_a(A)$$

and

$$\lim_{k\to\infty} p_a(B_k) = \lim_{k\to\infty} \int_{\partial B_k} a(|x|)\ \mathcal{H}_{n-1}(\mathrm{d}x) = p_a(B).$$

By the weak lower semi-continuity of p_a we infer that

$$\begin{aligned} p_a(A) + p_a(B) &= \lim_{k\to\infty} (p_a(A_k) + p_a(B_k)) \\ &\geq \liminf_{k\to\infty}\ p_a(A_k \cap B_k) + \liminf_{k\to\infty}\ p_a(A_k \cup B_k) \\ &\geq p_a(A \cap B) + p_a(A \cup B). \end{aligned}$$

LEMMA 4.4 *Let $g:[0,+\infty[\ \rightarrow [0,+\infty[$ be a convex function. Then*

$$g(\alpha - s) + g(\beta + s) \geq g(\alpha) + g(\beta) \quad \text{for } 0 \leq s \leq \alpha \leq \beta. \tag{40}$$

Proof First suppose that g is differentiable. We set

$$\varphi(t) := g(\alpha - t) + g(\beta + t) - g(\alpha) - g(\beta), \quad (0 \leq t \leq \alpha).$$

Then $\phi(0) = 0$ and, by convexity,

$$\varphi'(t) = -g'(\alpha - t) + g'(\beta + t) \geq 0 \quad \text{for } 0 \leq t \leq \alpha.$$

This yields (40).

In the general case we can argue by approximation.

THEOREM 4.3 *Let $a(t) > 0$ for $t > 0$ and, for some Caccioppoli set E,*

$$p_a(E) = p_a(E^\sharp). \tag{41}$$

Then E is equivalent to a ball. Furthermore, if either

$$(i) \quad n = 1 \text{ and } a(t) \text{ is strictly convex, or} \tag{42}$$

$$(ii) \quad n \geq 2 \text{ and } a(t) \text{ is strictly increasing } (t > 0), \tag{43}$$

then E is equivalent to $E^\sharp$.

Proof The proof is divided into five steps.

Step 1 Suppose that for some $\delta > 0$,

$$\begin{aligned} &m(E \cap B_{2\delta}) = m(B_{2\delta}), \text{ or} \\ &m(E \cap B_{2\delta}) = 0. \end{aligned} \tag{44}$$

By setting

$$\tilde{a}(t) := \begin{cases} 0 & \text{if } 0 \leq t \leq \delta \\ a(t) - a(\delta) & \text{if } \delta < t, \end{cases}$$

we obtain by (41) and (44),

$$a(\delta)p(E) + p_{\tilde{a}}(E) = p_a(E) = p_a(E^\sharp) = a(\delta)p(E^\sharp) + p_{\tilde{a}}(E^\sharp). \tag{45}$$

Furthermore, since $\tilde{a}$ satisfies (4) and (5), we have that

$$p_{\tilde{a}}(E) \leq p_{\tilde{a}}(E^{\sharp}).$$

This implies, together with (45) and the isoperimetric inequality (Appendix 2), that

$$p(E) = p(E^{\sharp}).$$

By once more applying the isoperimetric theorem, we infer that E must be equivalent to a ball.

Step 2 Next suppose that $a(0) > 0$. We have that

$$a(0)p(E) + p_{a_1}(E) = p_a(E) = p_a(E^{\sharp}) = a(0)p(E^{\sharp}) + p_{a_1}(E^{\sharp}),$$

and since a_1 satisfies (5), we may argue as in step 1 to infer that E is equivalent to a ball.

Step 3 Now suppose that $a(0) = 0$, and that (44) is not satisfied. Then

$$0 < m(E \cap B_\delta) < m(B_\delta) \quad \forall \delta > 0.$$

We choose $\varepsilon > 0$ such that $E \cup B_\varepsilon$ is not equivalent to a ball. The function

$$g(z) := a(z^{1/n})z^{1/n-1}, \quad (z > 0),$$

is convex by (5). In view of Lemma 4.4 this yields

$$g(m(E)) + g(m(B_\varepsilon)) \leq g(m(E \cap B_\varepsilon)) + g(m(E \cup B_\varepsilon)).$$

On the other hand, we have that

$$\begin{aligned}
&n\omega_n^{1/n}(g(m(E \cap B_\varepsilon)) + g(m(E \cup B_\varepsilon)))\\
&= p_a((E \cap B_\varepsilon)^{\sharp}) + p_a((E \cup B_\varepsilon)^{\sharp})\\
&\leq p_a(E \cap B_\varepsilon) + p_a(E \cup B_\varepsilon) \quad \text{(by Theorem 4.2)}\\
&\leq p_a(E) + p_a(B_\varepsilon) \quad \text{(by Lemma 4.3)}\\
&= p_a(E^{\sharp}) + p_a(B_\varepsilon) \quad \text{(by (41))}\\
&= n\omega_n^{1/n}(g(m(E)) + g(m(B_\varepsilon))).
\end{aligned}$$

Hence we must have

$$p_a((E\cap B_\varepsilon)^\sharp)+p_a((E\cup B_\varepsilon)^\sharp)=p_a(E\cap B_\varepsilon)+p_a(E\cup B_\varepsilon),$$

which means that

$$p_a((E\cup B_\varepsilon)^\sharp)=p_a(E\cup B_\varepsilon),$$

by Theorem 4.2. In view of step 1 we infer that $E\cup \mathrm{B}_\varepsilon$ is equivalent to a ball, a contradiction.

Thus we have proved that E is equivalent to a ball.

Step 4 Now suppose (42).

Since the sets E and $E^\sharp$ are equivalent to intervals $(-R+s,+R+s)$ and $(-R,+R)$, respectively ($R>0$, $s\in\mathbf{R}$), we compute

$$a(|-R+s|)+a(|R+s|)=p_a(E)=p_a(E^\sharp)=2a(R).$$

On the other hand, if $|s|>0$, (42) yields

$$a(|-R+s|)+a(|R+s|)=a(R+|s|)+a(-R+|s|)>2a(R).$$

Hence $s=0$, i.e. E is equivalent to $E^\sharp$.

Step 5 Finally assume (43).

The sets E and $E^\sharp$ are equivalent to balls $B_R(x_0)$ and B_R, respectively ($R>0, x_0\in\mathbf{R}^n$). We fix a coordinate system $x=(x_1,x')$ ($x'\in\mathbf{R}^{n-1}$), such that $x_0=(s,0,\dots,0)$, $s=|x_0|$. Then we compute

$$\begin{aligned}&\int_{\partial B_R(x_0)} a(|x|)\mathcal{H}_{n-1}(\mathrm{d}x)\\ &=\int_{|x'|<R}\left[a\left(\left\{|x'|^2+\left(s-\sqrt{R^2-|x'|^2}\right)^2\right\}^{1/2}\right)\right.\\ &\qquad\left.+a\left(\left\{|x'|^2+\left(s+\sqrt{R^2-|x'|^2}\right)^2\right\}^{1/2}\right)\right]\\ &\qquad\cdot\{1+|x'|^2(R^2-|x'|^2)^{-1}\}^{1/2}\,\mathrm{d}x'. \end{aligned}\tag{46}$$

Assume that $s=|x_0|>R$. Then the term $[\dots]$ in (46) increases strictly as s increases. In view of Theorem 2.1 this means that $p_a(B_R)>p_a(B_R)$, a contradiction. Hence we must have $|x_0|\le R$, that is $B_R(x_0)$ is starlike with respect to the origin. Following step 1 of the proof of Theorem 2.1

we compute

$$I = \int_{\partial B_R(x_0)} a_1(|x|)\mathcal{H}^{n-1}(\mathrm{d}x) \tag{47}$$

$$\rightleftharpoons \int_T a_1(\tau)\left\{1 + \tau^{-2}\sum_{m=1}^{n-1}\left(\frac{\partial r}{\partial \theta_m}\right)^2 h_m\right\}^{1/2} r^{n-1} h\,\mathrm{d}\theta, \tag{48}$$

where $\tau = \tau(\theta)$, $(\theta \in T)$, is a representation for $\partial B_R(x_0)$, and

$$I^\sharp := \int_{\partial B_R} a_1(|x|)\mathcal{H}_{n-1}(\mathrm{d}x) = n\omega_n a_1(R)R^{n-1}. \tag{49}$$

Note that (43) means that $a_1(t) > 0$ for $1 > 0$.

Since

$$n\omega_n R^n = \int_I \tau^n h\ \mathrm{d}\theta,$$

we obtain, using (47), (49), (5) and Jensen's inequality,

$$\begin{aligned} I &\geq \int_T a_2(\tau)\tau^{n-1}h\ \mathrm{d}\theta \\ &\geq n\omega_n a_1(R)R^{n-1} = I^\sharp, \end{aligned}$$

where the first in equality is strict when $|x_0| \neq 0$. This again means that E is equivalent to $E^\sharp$.

The theorem is proved.

5 COMPARISON RESULTS FOR PDE

Let us consider the following Dirichlet problem:

$$\begin{cases} Lu = -(a_{ij}u_{x_j})_{x_i} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{50}$$

where

(i) Ω is an open bounded subset of $\mathbf{R}^n$,

(ii) a_{ij} are real valued measurable functions on Ω which satisfy

$$a_{ij}(x)\xi_i\xi_j \geq \nu(|x|)|\xi|^2 \quad \forall \xi \in \mathbf{R}^n,\ \text{for a.e. } x \in \Omega,$$

where ν is a nonnegative measurable function on Ω such that $\nu \in L^1(\Omega)$, $\nu^{-1} \in L^t(\Omega)$ for some $t > 1$ if $n \geq 2$ and $\nu^{-1} \in L^1(\Omega)$ if $n = 1$,

(iii) $f \in L^q(\Omega)$, with q such that $1/q = 1/2 - 1/(2t) + 1/n$ if $n \geq 2$ and $f \in L^1(\Omega)$ if $n = 1$.

A solution of the problem (50) is a function $u \in W_0^{1,2}(\nu, \Omega)^\dagger$ which verifies the following condition:

$$\int_\Omega a_{ij} u_{x_j} \varphi_{x_i} \, dx = \int_\Omega f \varphi \, dx \quad \forall \varphi \in C_0^\infty(\Omega). \tag{51}$$

The assumptions (i)–(iii) guarantee the existence of such a solution (see [21,29]).

From Theorem 4.2 we derive the following comparison result:

THEOREM 5.1 *Let u be the solution of* (50). *Furthermore let $w \in W_0^{1,2}(\nu, \Omega^\#)$ be the solution of the following problem:*

$$\begin{cases} -(\nu(|x|) w_{x_i})_{x_i} = f^\# & \text{in } \Omega^\#, \\ w = 0 & \text{on } \partial\Omega^\#, \end{cases} \tag{52}$$

If $\sqrt{\nu(t)}$, $(t \geq 0)$, verifies the assumptions (4) *and* (5) *then we have*:

$$u^\#(x) \leq w(x) \quad \text{for a.e. } x \in \Omega^\#. \tag{53}$$

Furthermore, for every $q \in]0, 2]$, it results:

$$\int_\Omega \nu(|x|)^{q/2} |\nabla u|^q \, dx \leq \int_{\Omega^\#} \nu(|x|)^{q/2} |\nabla w|^q \, dx. \tag{54}$$

Proof Let $t \in [0, \operatorname{ess\,sup} |u|[$ and $h > 0$. We choose as test function in (51)

$$\varphi_h = \begin{cases} \operatorname{sign} u & \text{if } |u| > t + h, \\ \dfrac{u - t \operatorname{sign} u}{h} & \text{if } t < |u| \leq t + h, \\ 0 & \text{otherwise.} \end{cases}$$

Then we get

$$\frac{1}{h} \int_{t<|u|\leq t+h} a_{ij} u_{x_i} u_{x_j} \, dx = \int_{|u|>t+h} f \operatorname{sign} u \, dx + \frac{1}{h} \int_{t<|u|\leq t+h} f(u - t \operatorname{sign} u) \, dx.$$

† We denote by $W_0^{1,p}(\nu, \Omega)$, $1 \leq p < \infty$ the weighted Sobolev space, that is the closure of $C_0^\infty(\Omega)$ under the norm $(\int_\Omega \nu(x) |\nabla u(x)|^p \, dx)^{1/p}$.

Using (ii), Hardy's inequality and letting h go to zero, we have (see also [2]):

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{|u|>t}\nu(|x|)|\nabla u|^2\,\mathrm{d}x \leq -\frac{\mathrm{d}}{\mathrm{d}t}\int_{|u|>t}a_{ij}(x)u_{x_i}u_{x_j}\,\mathrm{d}x$$

$$= \int_{|u|>t}f(x)\,\mathrm{d}x \leq \int_0^{\mu_u(t)}f^*(\sigma)\,\mathrm{d}\sigma. \tag{55}$$

Moreover, by the Cauchy–Schwarz inequality, we have

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{|u|>t}\sqrt{\nu(|x|)}|\nabla u|\,\mathrm{d}x$$

$$\leq \left(-\frac{\mathrm{d}}{\mathrm{d}t}\int_{|u|>t}\nu(|x|)|\nabla u|^2\,\mathrm{d}x\right)^{1/2}(-\mu_u'(t))^{1/2}. \tag{56}$$

On the other hand, from coarea formula (see Appendix 3) we obtain

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{|u|>t}\sqrt{\nu(|x|)}|\nabla u|\,\mathrm{d}x = \int_{|u|=t}\sqrt{\nu(|x|)}\,\mathcal{H}_{n-1}\,(\mathrm{d}x). \tag{57}$$

Now Theorem 4.2 gives

$$\int_{u^\#=t}\sqrt{\nu(|x|)}\,\mathcal{H}_{n-1}\,(\mathrm{d}x) \leq \int_{|u|=t}\sqrt{\nu(|x|)}\,\mathcal{H}_{n-1}\,(\mathrm{d}x),$$

that is,

$$\sqrt{\nu\left(\frac{\mu_u(t)^{1/n}}{\omega_n^{1/n}}\right)}\,n\omega_n^{1/n}\mu_u(t)^{1-1/n} \leq \int_{|u|=t}\sqrt{\nu(|x|)}\,\mathcal{H}_{n-1}\,(\mathrm{d}x). \tag{58}$$

On combining (55)–(58), we obtain

$$-\frac{1}{\mu_u'(t)}\nu\left(\frac{\mu_u(t)^{1/n}}{\omega_n^{1/n}}\right)n^2\omega_n^{2/n}\mu_u(t)^{2-2/n} \leq \int_0^{\mu_u(t)}f^*(\sigma)\,\mathrm{d}\sigma. \tag{59}$$

Let us consider problem (52). Obviously, since $w(x) = w^{\#}(x)$, the arguments leading to (59) proceed in the same way except that equalities now replace inequalities in the details. Thus, in place of (59), we obtain the equality

$$-\frac{1}{\mu_w'(t)}\nu\left(\frac{\mu_w(t)^{1/n}}{\omega_n^{1/n}}\right)n^2\omega_n^{2/n}\mu_w(t)^{2-2/n} = \int_0^{\mu_w(t)} f^*(\sigma)\,\mathrm{d}\sigma, \qquad (60)$$

where μ_w is the distribution function of w. Setting

$$F(\lambda) = \frac{\int_0^\lambda f^*(\sigma)\,\mathrm{d}\sigma}{\nu(\lambda^{1/n}/\omega_n^{1/n})n^2\omega_n^2\lambda^{2-2/n}}, \quad \lambda \in]0, |\Omega|],$$

(59) and (60) give:

$$\mu_u'(t)F(\mu_u(t)) \leq \mu_w'(t)F(\mu_w(t)). \qquad (61)$$

Let $\tilde{F}$ be a primitive of F. Then, integrating (61) between 0 and t, we get

$$\tilde{F}(\mu_u(t)) \leq \tilde{F}(\mu_w(t)).$$

If $f \not\equiv 0$, then $\mathrm{d}\tilde{F}/\mathrm{d}\lambda = F(\lambda) > 0$ for all $\lambda > 0$. Hence $\tilde{F}$ is strictly increasing and:

$$\mu_u(t) \leq \mu_w(t).$$

This yields (53). Furthermore, we have by Hölder's inequality:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{|u|>t}\nu(|x|)^{q/2}|\nabla u|^q\,\mathrm{d}x$$
$$\leq \left(-\frac{\mathrm{d}}{\mathrm{d}t}\int_{|u|>t}\nu(|x|)|\nabla u|^2\,\mathrm{d}x\right)^{q/2}(-\mu_u'(t))^{1-q/2}.$$

Using (55), we derive from this:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{|u|>t}\nu(|x|)^{q/2}|\nabla u|^q\,\mathrm{d}x \leq \left(\int_0^{\mu_u(t)} f^*(s)\,\mathrm{d}s\right)^{q/2}(-\mu_u'(t))^{1-q/2}.$$

Integrating this between 0 and $+\infty$ yields:

$$\int_\Omega \nu(|x|)^{q/2}|\nabla u|^q \,\mathrm{d}x \leq \int_0^{+\infty}\left(\frac{1}{-\mu'(t)}\int_0^{\mu_u(t)} f^*(s)\,\mathrm{d}s\right)^{9/2}(-\mathrm{d}\mu(t))$$

from which we obtain, by (59):

$$\int_\Omega \nu(|x|)^{q/2}|\nabla u|^q \,\mathrm{d}x$$
$$\leq \frac{1}{n^q\omega_n^{q/n}}\int_0^{+\infty}\left(\frac{1}{s^{1-1/n}\sqrt{\gamma\left(s^{1/n}/\omega_n^{1/n}\right)}}\left(\int_0^s f^*(r)\,\mathrm{d}r\right)\right)^q \mathrm{d}s,$$

and (54) follows.

Remark 5.1 Alvino and Trombetti [2] obtained another comparison result for the solution of problem (50). They proved the inequality

$$u^\# \leq v, \tag{62}$$

where v is the solution of the Dirichlet problem

$$\begin{cases} -(\tilde{\nu}(|x|)v_{x_i})_{x_i} = f^\# & \text{in } \Omega^\#,\\ v = 0 & \text{on } \partial\Omega^\#, \end{cases} \tag{63}$$

and $\tilde{\nu}(|x|)$ is a function defined on $[0, |\Omega|]$, such that

$$\int_0^{\mu_u(t)} \frac{1}{\tilde{\nu}}(s)\,\mathrm{d}s = \int_{|u|>t}\frac{1}{\nu}(x)\,\mathrm{d}x \quad \text{for a.e. } t \in [0, \text{ess sup}\,|u|[.$$

According to Lemma 2.1 in [2], the function $1/\tilde{\nu}$ is a weak limit of a sequence of functions having the same rearrangement as $1/\nu$. Let us observe that, since $\tilde{\nu}$ depends on u, in (62) $u^\#$ is compared with the solution of a problem which depends on u. In Theorem 5.1 the problem (51) does not depend on u but further assumptions on ν are requested.

Remark 5.2 Theorem 5.1 can be extended to nonlinear elliptic problems of the type:

$$\begin{cases} -\mathrm{div}(A(x,u,\nabla u)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{64}$$

where

(i) $A : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ is a Caratheodory function such that

$$A(x,s,\xi)\xi \geq \nu(|x|)|\xi|^p$$

where $\nu \in L^s(\Omega)$, $s \geq 1$ and $1/\nu \in L^t(\Omega)$, $t > 1$, $1 < p < (n(t-1))/(t-n)$.

(ii) $f \in L^q(\Omega)$, with q such that $1/q = ((p-1)/p)(1-1/t) + (1/n)$.

Let us denote by $u \in W_0^{1,p}(\nu,\Omega)$ a solution of (64) and by $z \in W_0^{1,p}(\nu,\Omega^{\#})$ the solution of the following problem:

$$\begin{cases} -\mathrm{div}(\nu(|x|)|\nabla z|^{p-2}\nabla z) = f^{\#} & \text{in } \Omega^{\#}, \\ z = 0 & \text{on } \partial\Omega^{\#}. \end{cases}$$

If $(\nu(t))^{1/p}$ $(t \geq 0)$, verifies the assumptions (4) and (5), we have

$$u^{\#}(x) \leq z(x) \quad \text{for a.e. } x \in \Omega^{\#}.$$

Arguing as in Theorem 3.1 in [8], we can prove that problem (52) is the unique problem such that equality holds in (53). More precisely we have

THEOREM 5.2 *Let u and w the solutions of* (50) *and* (52) *respectively. If* $u^{\#} = w$ *a.e. in* Ω, *then* $\Omega = \Omega^{\#} + x_0$, $f = f^{\#}(\cdot + x_0)$ *and* $a_{ij}(x + x_0)x_j = \nu(|x|)x_i$ *for some* $x_0 \in \mathbf{R}^n$.

APPENDIX

We recall some well known theorems.

(1) *Jensen's inequality* (see e.g. [20])
Let $E \subset \mathbf{R}^n$ be measurable with finite measure, let f, h be integrable on E, $h \geq 0$, and let $G : \mathbf{R} \to [0, +\infty[$ be convex. Then

$$\frac{\int_E G(f(x))h(x)\,dx}{\int_E h(x)\,dx} \geq G\left(\frac{\int_E f(x)h(x)\,dx}{\int_E h(x)\,dx}\right). \tag{65}$$

(2) *Isoperimetric theorem in* $\mathbf{R}^n$ (see e.g. [27])
If $E \subset \mathbf{R}^n$ is measurable with finite measure, then

$$\mathcal{H}_{n-1}(\partial E) \geq n\omega_n^{1/n}(m(E))^{1-(1/n)}. \tag{66}$$

Furthermore, if (66) is valid with equality sign, then E is equivalent to a ball.

(3) *Coarea formula* (see e.g. [13])
If u is Lipschitz continuous and f is integrable, then

$$\int_{\mathbf{R}^n} f(x)|\nabla u(x)|\,\mathrm{d}x = \int_0^\infty \mathrm{d}t \int_{\{x\in\mathbf{R}^n:\ |u(x)|=t\}} f(x)\ \mathcal{H}_{n-1}(\mathrm{d}x). \tag{67}$$

Acknowledgment

The second author wants to thank the University of Naples for a visiting appointment.

References

[1] A. Alvino, P.-L. Lions and G. Trombetti: On optimization problems with prescribed rearrangements. *Nonlin. Analysis T.M.A.* **13** (1989), 185–220.
[2] A. Alvino and G. Trombetti: Sulle migliori costanti di maggiorazione per una classe di equazioni ellittiche degeneri. *Ricerche Mat.* **27** (1978), 413–428.
[3] A. Amar and G. Bellettini: A notion of total variation depending on a metric with discontinuous cofficient. *Ann. Inst. Henri Poincar'e* **11** (1994), 91–133.
[4] A. Baernstein II: A unified approach to symmetrization. in: *Partial Differential Equations of Elliptic Type*, Eds. A. Alvino *et al.*, Symposia matematica **35**, Cambridge Univ. Press 1995, 47–91.
[5] C. Bandle: *Isoperimetric Inequalities and Applications*. Pitman, London 1980.
[6] C. Bandle and M. Marcus: Radial averaging transformations with various metrics. *Pac. J. Math.* **46** (1973), 337–348.
[7] C. Bandle and M. Marcus: Radial averaging transformations and generalized capacities. *Math. Z.* **145** (1975), 11–17.
[8] M.F. Betta and A. Mercaldo: Uniqueness results for optimization problems with prescribed rearrangement. *Potential Anal.* **5** (1996), 183–205.
[9] F. Brock: Weighted Dirichlet-type inequalities for Steiner symmetrization. *Calc. Var.* **8** (1999), 15–25.
[10] J.E. Brothers and W.P. Ziemer: Minimal rearrangements of Sobolev function. *J. reine angew. Math.* **384** (1988), 153–179.
[11] A. Cianchi, D.E. Edmunds and P. Gurka: On weighted Poincaré inequalities. *Math. Nachr.* **180** (1996), 15–41.
[12] V.N. Dubinin: Symmetrization in geometric theory of functions of complex variables. (in Russian), *Uspehi Mat. Nauk* **49** (1994), 3–76; translation in: *Russian Math. Surveys* **49**: 1 (1994), 1–79.

[13] H. Federer: *Geometric Measure Theory*. Grundlehren der Mathematischen Wissenschaften, Vol. 153, Springer-Verlag, New York, 1969.
[14] E. Giusti: *Minimal Surfaces and Functions of Bounded Variation*. Birkhäuser-Verlag, Boston, 1984.
[15] B. Kawohl: *Rearrangements and Convexity of Level Sets in PDE*. Lecture Notes in Math., Vol. 1150, Springer-Verlag, Berlin, 1985.
[16] B. Kawohl: On starshaped rearrangement and applications. *Trans. Amer. Math. Soc.*, **296**(1) (1986), 377–386.
[17] P.L. Lions: Quelques remarques sur la symètrisation de Schwarz. in: *Nonlinear Partial Differential Equations and their Application*, Collège de France, Seminar n. 1, Pitman, London, 1980, 308–319.
[18] M. Marcus: Transformations of domains in the plane and applications in the theory of functions. *Pac. J. Math.* **14** (1964), 613–626.
[19] M. Marcus: Radial averaging of domains, estimates for Dirichlet integrals and applications. *J. Anal. Math.* **27** (1974), 47–93.
[20] D.S. Mitrinovic, J.E. Pecaric and A.M. Fink: *Classical and New Inequalities in Analysis*. Kluwer Academic Publishers, 1993.
[21] M.K.W. Murthy and G. Stampacchia: Boundary value problems for some degenerate elliptic operators. *Ann. Mat. Pura Appl.* **80** (1968), 1–122.
[22] A. Kufner and B. Opic: *Hardy-type Inequalities*. Pitman Research Notes in Math. **219**, Longman Scientific & Technical, Harlow, 1990.
[23] G. Polya and G. Szegö: Isoperimetric inequalities in mathematical physics. *Ann. Math. Stud.* **27** (1952), Princeton Univ. Press.
[24] J.M. Rakotoson and B. Simon: Relative rearrangement on a measure space. Application to the regularity of weighted monotone rearrangement, Part 1. *Appl. Math. Lett.* **6** (1993), 75–78.
[25] J.M. Rakotoson and B. Simon: Relative rearrangement on a measure space. Application to the regularity of weighted monotone rearrangement, Part 2. *Appl. Math. Lett.* **6** (1993), 79–82.
[26] G. Talenti: Inequalities in rearrangement invariant function spaces. in: *Nonlinear Analysis, Function Spaces and Applications*, Vol. 5, Proceedings of the Spring School held in Prague, May 1994, Eds. M. Krbec *et al.*, Prometheus Publishers, 1995.
[27] G. Talenti: *The Standard Isoperimetric Theorem*. Handbook of Convex Geometry, Volume A, (P.M. Gruber and J.M. Wills, Eds.), North-Holland, Amsterdam, 1993, pp. 73–124.
[28] G. Talenti: Elliptic equations and rearrangements. *Ann. Scuola Norm. Sup. Pisa* **3** (1976), 697–718.
[29] N. Trudinger: Linear elliptic operators with measurable coefficients. *Ann. Scuola Norm. Sup. Pisa* **27** (1973), 265–308.