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On the Constants for Some Sobolev Imbeddings*

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We consider the imbedding inequality $\| \; \|_{L^r(\mathbf{R}^d)} \leq S_{r,n,d} \| \; \|_{H^n(\mathbf{R}^d)}$; $H^n(\mathbf{R}^d)$ is the Sobolev space (or Bessel potential space) of L^2 type and (integer or fractional) order n. We write down upper bounds for the constants $S_{r,n,d}$, using an argument previously applied in the literature in particular cases. We prove that the upper bounds computed in this way are in fact the sharp constants if ($r=2$ or) $n > d/2$, $r=\infty$, and exhibit the maximising functions. Furthermore, using convenient trial functions, we derive lower bounds on $S_{r,n,d}$ for $n > d/2$, $2 < r < \infty$; in many cases these are close to the previous upper bounds, as illustrated by a number of examples, thus characterizing the sharp constants with little uncertainty.

Keywords: Sobolev spaces; Imbedding inequalities

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1. INTRODUCTION AND PRELIMINARIES

The imbedding inequality of $H^n(\mathbf{R}^d, \mathbf{C})$ into $L^r(\mathbf{R}^d, \mathbf{C})$ is a classical topic, and several approaches has been developed to derive upper bounds on the sharp imbedding constants $S_{r,n,d}$. A simple method, based on the Hausdorff-Young and Hölder inequalities, has been

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employed in the literature for special choices of r, n, d, as indicated in the references at the end of Section 2. Little seems to have been done to test reliability of the upper bounds derived in this way (*i.e.*, their precision in approximating the unknown sharp constants).

This paper is a contribution to the understanding of the Hausdorff-Young-Hölder (HYH) upper bounds, and aims to show their reliability for $n > d/2$. This case is interesting for a number of reasons, including application to PDE's; its main feature is that the H^n norm controls the L^r norms of *all* orders $r \geq 2$, up to $r = \infty$.

The paper is organized as follows. First of all, in Section 2 we write the general expression of the HYH upper bounds $S_{r,n,d} \leq S^+_{r,n,d}$ (containing all special cases of our knowledge in the literature). In Section 3 we show that the upper bounds $S^+_{r,n,d}$ are in fact the sharp constants if ($r = 2$, n arbitrary or) $n > d/2$, $r = \infty$, and exhibit the maximising functions; next, we assume $n > d/2$ and inserting a one parameter family of trial functions in the imbedding inequality, we derive lower bounds $S_{r,n,d} \geq S^-_{r,n,d}$ for arbitrary $r \in (2, \infty)$. In Section 4 we report numerical values of $S^{\pm}_{r,n,d}$ for representative choices of n, d and a wide range of r values; in all the examples the relative uncertainty on the sharp imbedding constants, *i.e.*, the ratio $(S^+_{r,n,d} - S^-_{r,n,d})/S^-_{r,n,d}$, is found to be $\ll 1$.

1.1. Notations for Fourier Transform and H^n Spaces

Throughout this paper, $d \in \mathbf{N} \setminus \{0\}$ is a fixed space dimension; the running variable in $\mathbf{R}^d$ is $x = (x_1, \ldots, x_d)$, and $k = (k_1, \ldots, k_d)$ when using the Fourier transform. We write $|\mathrm{x}|$ for the function $(x_1, \ldots, x_d) \mapsto \sqrt{x_1{}^2 + \cdots + x_d{}^2}$, and intend $|\mathrm{k}|$ similarly. We denote with $\mathcal{F}, \mathcal{F}^{-1}$: $\mathcal{S}'(\mathbf{R}^d, \mathbf{C}) \to \mathcal{S}'(\mathbf{R}^d, \mathbf{C})$ the Fourier transform of tempered distributions and its inverse, choosing normalizations so that (for f in L^1 ($\mathbf{R}^d$, $\mathbf{C}$)) it is $\mathcal{F}f(k) = (2\pi)^{-d/2} \times \int_{\mathbf{R}^d} dx e^{-ik \cdot x} f(x)$. The restriction of $\mathcal{F}$ to L^2 ($\mathbf{R}^d$, $\mathbf{C}$), with the standard inner product and the associated norm $\| \; \|_{L^2}$, is a Hilbertian isomorphism.

For real $n \geq 0$, let us introduce the operators

$$\mathcal{S}'(\mathbf{R}^d, \mathbf{C}) \to \mathcal{S}'(\mathbf{R}^d, \mathbf{C}), \quad g \mapsto \sqrt{1 - \Delta}^{\pm n} g := \mathcal{F}^{-1}\left(\sqrt{1 + |\mathrm{k}|^2}^{\pm n} \mathcal{F}g\right) \tag{1.1}$$

(in case of integer, even exponent n, we have a power of 1 minus the distributional Laplacian Δ, in the elementary sense). The n-th order Sobolev (or Bessel potential [1]) space of L^2 type and its norm are

$$H^n(\mathbf{R}^d, \mathbf{C}) := \{f \in \mathcal{S}'(\mathbf{R}^d, \mathbf{C}) \,|\, \sqrt{1-\Delta}^n f \in L^2(\mathbf{R}^d, \mathbf{C})\} = \\ = \{\sqrt{1-\Delta}^{-n} u \,|\, u \in L^2(\mathbf{R}^d, \mathbf{C})\} = \\ = \left\{ f \in \mathcal{S}'(\mathbf{R}^d, \mathbf{C}) \,\Big|\, \sqrt{1+|\mathrm{k}|^2}^{\,n} \mathcal{F}f \in L^2(\mathbf{R}^d, \mathbf{C}) \right\}, \qquad (1.2)$$

$$\|f\|_{H^n} := \|\sqrt{1-\Delta}^n f\|_{L^2} = \left\| \sqrt{1+|\mathrm{k}|^2}^{\,n} \mathcal{F}f \right\|_{L^2}. \qquad (1.3)$$

Of course, if $n \le n'$, it is $H^{n'}(\mathbf{R}^d, \mathbf{C}) \subset H^n(\mathbf{R}^d, \mathbf{C})$ and $\| \;\|_{H^n} \le \| \;\|_{H^{n'}}$; also, $H^0 = L^2$.

1.2. Connection with Bessel Functions

For $\nu > 0$, and in the limit case zero, let us put, respectively,

$$G_{\nu,d} := \mathcal{F}^{-1}\left(\frac{1}{\sqrt{1+|\mathrm{k}|^2}^{\,\nu}} \right) = \frac{|\mathrm{x}|^{\nu/2-d/2}}{2^{\nu/2-1}\Gamma(\nu/2)} K_{\nu/2-d/2}(|\mathrm{x}|); \qquad (1.4)$$

$$G_{0,d} := \mathcal{F}^{-1}(1) = (2\pi)^{d/2}\delta.$$

Here, Γ is the factorial function; $K_{(\;)}$ are the modified Bessel functions of the third kind, or Macdonald functions, see *e.g.* [2]; δ is the Dirac distribution. The expression of $G_{\nu,d}$ *via* a Macdonald function [1] comes from the known computational rule for the Fourier transforms of radially symmetric functions [3]. With the above ingredients, we obtain another representation of H^n spaces [1]; in fact, explicitating $\sqrt{1-\Delta}^{-n} u$ in Eq. (1.2) and recalling that $\mathcal{F}^{-1}$ sends pointwise product into $(2\pi)^{-d/2}$ times the convolution product $*$, we see that

$$H^n(\mathbf{R}^d, \mathbf{C}) = \left\{ \frac{1}{(2\pi)^{d/2}} G_{n,d} * u \,\Big|\, u \in L^2(\mathbf{R}^d, \mathbf{C}) \right\} \qquad (1.5)$$

for each $n \ge 0$. All this is standard; in this paper we will show that, for $n > d/2$, the function $G_{2n,d}$ also plays a relevant role for $H^n(\mathbf{R}^d, \mathbf{C})$, being an element of this space and appearing to be a maximiser for the inequality $\| \ \|_{L^\infty} \le$ const $\| \ \|_{H^n}$. Incidentally we note that (for all $n \ge 0$) the relation $(1+|\mathrm{k}|^2)^{-n} = \sqrt{1+|\mathrm{k}|^2}^{-n} \sqrt{1+|\mathrm{k}|^2}^{-n}$ gives, after application of $\mathcal{F}^{-1}$, $G_{2n,d} = (2\pi)^{-d/2} G_{n,d} * G_{n,d}$.

For future conveniency, let us recall a case in which the expression of $G_{\nu,d}$ simply involves an exponential $\times$ a polynomial in $|\mathrm{x}|$. This occurs if $\nu/2 - d/2 = m + 1/2$, with m a nonnegative integer: in fact, it is well known [2] that

$$\rho^{m+1/2} K_{m+1/2}(\rho) = \sqrt{\frac{\pi}{2}} e^{-\rho} \sum_{i=0}^{m} \frac{(2m-i)!}{i!(m-i)!} \frac{\rho^i}{2^{m-i}} \quad (m \in \mathbf{N}, \rho \in \mathbf{R}). \tag{1.6}$$

2. HYH UPPER BOUNDS FOR THE IMBEDDING CONSTANTS

It is known [1,4] that $H^n(\mathbf{R}^d, \mathbf{C})$ is continuously imbedded into $L^r(\mathbf{R}^d, \mathbf{C})$ if $0 \le n < d/2$, $2 \le r \le d/(d/2 - n)$ or $n = d/2$, $2 \le r < \infty$ or $n > d/2$, $2 \le r \le \infty$. We are interested in the sharp imbedding constants

$$S_{r,n,d} := \mathrm{Inf}\{S \ge 0 | \ \|f\|_{L^r} \le S \|f\|_{H^n} \text{ for all } f \in H^n(\mathbf{R}^d, \mathbf{C})\}. \tag{2.1}$$

Let us derive general upper bounds on the above constants, with the HYH method mentioned in the Introduction; this result will be expressed in terms of the functions Γ and E, the latter being defined by

$$E(s) := s^s \quad \text{for } s \in (0, +\infty), \ E(0) := \lim_{s \to 0^+} E(s) = 1. \tag{2.2}$$

PROPOSITION 2.1 *Let $n = 0$, $r = 2$ or $0 < n < d/2$, $2 \le r < d/(d/2 - n)$ or $n = d/2$, $2 \le r < \infty$ or $n > d/2$, $2 \le r \le \infty$. Then $S_{r,n,d} \le S^+_{r,n,d}$, where*

$$S^+_{r,n,d} := \frac{1}{(4\pi)^{d/4 - d/(2r)}} \left(\frac{\Gamma((n/(1-2/r)) - (d/2))}{\Gamma((n/(1-2/r)))} \right)^{1/2 - 1/r} \times$$
$$\times \left(\frac{E(1/r)}{E(1-1/r)} \right)^{d/2} \quad \text{if } r \ne 2, \infty, \tag{2.3}$$

$$S^+_{2,n,d} := 1, \quad S^+_{\infty,n,d} := \frac{1}{(4\pi)^{d/4}} \left(\frac{\Gamma(n - d/2)}{\Gamma(n)} \right)^{1/2}. \tag{2.4}$$

Proof Of course, it amounts to showing that $\|f\|_{L^r} \leq S^+_{r,n,d} \|f\|_{H^n}$ for all $f \in H^n(\mathbf{R}^d, \mathbf{C})$. For $r = 2$ and any n this follows trivially, because $\|f\|_{L^2} = \|f\|_{H^0} \leq 1 \times \|f\|_{H^n}$.

From now on we assume $r \neq 2$ (intending $1/r := 0$ if $r = \infty$); p, s are such that

$$\frac{1}{r} + \frac{1}{p} = 1; \quad \frac{1}{s} + \frac{1}{2} = \frac{1}{p}, \quad i.e., \quad s = \frac{2}{1 - 2/r}. \tag{2.5}$$

Let $f \in H^n$ $(\mathbf{R}^d, \mathbf{C})$. Then, the Hausdorff-Young inequality for $\mathcal{F}$ and the (generalized) Hölder's inequality for $\mathcal{F}f = \sqrt{1 + |\mathrm{k}|^2}^{\,-n} \left(\sqrt{1 + |\mathrm{k}|^2}^{\,n} \mathcal{F}f \right)$ give

$$\|f\|_{L^r} \leq C_{r,d} \|\mathcal{F}f\|_{L^p}, \quad C_{r,d} := \frac{1}{(2\pi)^{d/2 - d/r}} \left(\frac{E(1/r)}{E(1 - 1/r)} \right)^{d/2}, \tag{2.6}$$

$$\|\mathcal{F}f\|_{L^p} \leq \left\| \frac{1}{\sqrt{1 + |\mathrm{k}|^2}^{\,n}} \right\|_{L^s} \left\| \sqrt{1 + |\mathrm{k}|^2}^{\,n} \mathcal{F}f \right\|_{L^2} =$$

$$= \left(\int_{\mathbf{R}^d} \frac{dk}{\sqrt{1 + |k|^2}^{\,ns}} \right)^{1/s} \|f\|_{H^n} \tag{2.7}$$

($C_{r,d}$ is the sharp Hausdorff-Young constant: see [5, 6] Chapter 5 and references therein. Our expression for $C_{r,d}$ differs by a factor from the one in [6] due to another normalization for the Fourier transform).

Of course, statements (2.6), (2.7) are meaningful if the integral in Eq. (2.7) converges; in fact this is the case, because the definition of s and the assumptions on r, n, d imply $ns > d$. Summing up, we have

$$\|f\|_{L^r} \leq \frac{1}{(2\pi)^{d/2 - d/r}} \left(\frac{E(1/r)}{E(1 - 1/r)} \right)^{d/2} \times$$

$$\times \left(\int_{\mathbf{R}^d} \frac{dk}{\sqrt{1 + |k|^2}^{\,ns}} \right)^{1/s} \|f\|_{H^n}, \tag{2.8}$$

with s as in (2.5). Now, the thesis is proved if we show that

$$\text{constant in Eq.(2.8)} = S^+_{r,n,d}; \tag{2.9}$$

to check this, it suffices to write

$$\int_{\mathbf{R}^d} \frac{dk}{\sqrt{1+|k|^2}^{ns}} = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^{+\infty} d\xi \frac{\xi^{d-1}}{\sqrt{1+\xi^2}^{ns}} = = \pi^{d/2} \frac{\Gamma(ns/2 - d/2)}{\Gamma(ns/2)}. \tag{2.10}$$

and to explicitate s. ■

Remarks

(i) Let us indicate the special cases of our knowledge, in which some HYH upper bounds $S^+_{r,n,d}$ have been previously given in the literature. Reference [6] derives these bounds for $d=1$, $n=1/2$, $d=2$, $n=1$ and $2 \le r < \infty$ (with a misprint). The inequality in [7], page 55 is strictly related to the case $n=2$, $d \ge 4$, $2 \le r < d/(d/2-2)$. The upper bound $S^+_{\infty,n,d}$ is given for arbitrary $n > d/2$ by many authors, see *e.g.* [8, 9].

To our knowledge, little was done to discuss reliability of the HYH upper bounds; the next two sections will be devoted to this topic, in the $n > d/2$ case. First of all, we will emphasize that $S^+_{\infty,n,d}$ is in fact the *sharp* imbedding constant for any $n > d/2$ (this is shown in [6] for $d=1$, $n=1$ only, with an *hoc* technique). $S^+_{2,n,d}$ is also the sharp constant (for any n), by an obvious argument; our analysis will show that, for $n > d/2$ and intermediate values $2 < r < \infty$, $S^+_{r,n,d}$ gives a generally good approximation of the sharp constant.

(ii) Discussing reliability of the bounds $S^+_{r,n,d}$ for $n \le d/2$ would require a separate analysis, which is outside the purposes of this paper; let us only present a few comments.

The upper bound $S^+_{r,n,d}$ is certainly far from the sharp constant for $0 < n < d/2$ and r close to $d/(d/2-n)$: note that $S^+_{r,n,d}$ diverges for r approaching this limit, in spite of the validity of the imbedding inequality even at the limit value. As a matter of fact other approaches, not using the HYH scheme, are more suitable to estimate the imbedding constants if $0 < n < d/2$, $r \simeq d/(d/2-n)$. We refer, in particular, to methods based on the Hardy-Littlewood-Sobolev inequality [8]: the sharp constants for that inequality were found variationally in [10]. Let us also mention the papers [11], prior to [5], and [12]; the inequalities considered therein, for which the sharp constants were determined, are strictly related to the limit case $r = d/(d/2-n)$ with $n = 1$ and 2, respectively.

The HYH upper bounds $S^+_{r,n,d}$ might be close to the sharp imbedding constants $S_{r,n,d}$ in the critical case $n = d/2$, but this topic will not be discussed in the sequel.

3. CASES WHERE $S^+_{r,n,d}$ IS THE SHARP CONSTANT. LOWER BOUNDS ON THE SHARP CONSTANTS FOR $n > d/2$ AND ARBITRARY r

Let us begin with the aforementioned statement that

PROPOSITION 3.1 *$S^+_{r,n,d}$ is the sharp imbedding constant if $n \geq 0$, $r = 2$ or $n > d/2$, $r = \infty$. In fact:*

(i) *for any $n \geq 0$ and nonzero $f \in H^n$ $(\mathbf{R}^d, \mathbf{C})$, it is*

$$\lim_{\lambda \mapsto 0^+} \frac{\|f^{(\lambda)}\|_{L^2}}{\|f^{(\lambda)}\|_{H^n}} = 1 = S^+_{2,n,d}, \quad \textit{where}$$
$$f^{(\lambda)}(x) := f(\lambda x) \textit{ for } x \textit{ in } \mathbf{R}^d, \ \lambda \in (0, +\infty). \tag{3.1}$$

(ii)

$$\|f\|_{L^\infty} = S^+_{\infty,n,d}\|f\|_{H^n} \quad \textit{for } n > d/2 \textit{ and}$$
$$f := \mathcal{F}^{-1}\left(\frac{1}{(1+|k|^2)^n}\right) = G_{2n,d}. \tag{3.2}$$

Proof

(i) Given any $f \in H^n$ $(\mathbf{R}^d, \mathbf{C})$, define $f^{(\lambda)}$ as above; by elementary rescaling of the integration variables, we find

$$(\mathcal{F}f^{(\lambda)})(k) = \frac{1}{\lambda^d}(\mathcal{F}f)\left(\frac{k}{\lambda}\right) \quad \text{for } k \in \mathbf{R}^d; \tag{3.3}$$

$$\|f^{(\lambda)}\|_{H^n} = \frac{1}{\lambda^d}\sqrt{\int_{\mathbf{R}^d} dk(1+|k|^2)^n\left|\mathcal{F}f\left(\frac{k}{\lambda}\right)\right|^2} =$$

$$= \frac{1}{\lambda^{d/2}}\sqrt{\int_{\mathbf{R}^d} dh(1+\lambda^2|h|^2)^n|\mathcal{F}f(h)|^2}; \tag{3.4}$$

$$\|f^{(\lambda)}\|_{H^n} \underset{\lambda\to 0^+}{\sim} \frac{1}{\lambda^{d/2}}\sqrt{\int_{\mathbf{R}^d} dh|\mathcal{F}f(h)|^2} = \frac{1}{\lambda^{d/2}}\|f\|_{L^2} = \|f^{(\lambda)}\|_{L^2}. \tag{3.5}$$

(ii) Let $n > d/2$; then $1/(1+|k|^2)^n \in L^1$ $(\mathbf{R}^d, \mathbf{C})$, so f in Eq. (3.2) is continuous and bounded. For all $x \in \mathbf{R}^d$ (and for the everywhere continuous representative of f) it is

$$f(x) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbf{R}^d} dk \frac{e^{ik\cdot x}}{(1+|k|^2)^n},$$
$$|f(x)| \le \int_{\mathbf{R}^d} dk \frac{1}{(1+|k|^2)^n} = f(0), \tag{3.6}$$

so that

$$\|f\|_{L^\infty} = f(0) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbf{R}^d} dk \frac{1}{(1+|k|^2)^n}. \tag{3.7}$$

Also, it is $f \in H^n$ $(\mathbf{R}^d, \mathbf{C})$ and

$$\|f\|_{H^n} = \sqrt{\int_{\mathbf{R}^d} dk(1+|k|^2)^n|\mathcal{F}f(k)|^2} = \sqrt{\int_{\mathbf{R}^d} dk \frac{1}{(1+|k|^2)^n}}. \tag{3.8}$$

The last two equations give

$$\frac{\|f\|_{L^\infty}}{\|f\|_{H^n}} = \frac{1}{(2\pi)^{d/2}}\sqrt{\int_{\mathbf{R}^d} dk \frac{1}{(1+|k|^2)^n}}, \tag{3.9}$$

and by comparison with Eqs. (2.8), (2.9) we see that the above ratio is just $S^+_{\infty,n,d}$. ■

As an example, let us write down the maximising function $f = G_{2n,d}$ of item (ii) when $n = d/2 + 1/2$ or $n = d/2 + 1$. According to Eqs. (1.4), (1.6), we have

$$G_{2(d/2+1/2),d} = \frac{|x|^{1/2} K_{1/2}(|x|)}{2^{d/2-1/2}\Gamma(d/2+1/2)} = \frac{\sqrt{\pi} e^{-|x|}}{2^{d/2}\Gamma(d/2+1/2)};$$
$$G_{2(d/2+1),d} = \frac{|x| K_1(|x|)}{2^{d/2}\Gamma(d/2+1)}. \tag{3.10}$$

From now on $n > d/2$; we attack the problem of finding lower bounds on $S_{r,n,d}$ for $2 < r < \infty$. To obtain them, one can insert into the imbedding inequality (2.1) a trial function; the previous considerations suggest to employ the one parameter family of rescaled functions

$$G^{(\lambda)}_{2n,d}(x) := G_{2n,d}(\lambda x) \qquad (\lambda \in (0, \infty)). \tag{3.11}$$

Of course, the sharp constant satisfies

$$S_{r,n,d} \geq \mathrm{Sup}_{\lambda > 0} \frac{\| G^{(\lambda)}_{2n,d} \|_{L^r}}{\| G^{(\lambda)}_{2n,d} \|_{H^n}}; \tag{3.12}$$

one should expect the above supremum to be attained for $\lambda \simeq 0$ if $r \simeq 2$, and for $\lambda \simeq 1$ if r is large. Evaluation of the above ratio of norms leads to the following.

Proposition 3.2 *For $n > d/2$, $2 < r < \infty$ it is $S_{r,n,d} \geq S^-_{r,n,d}$, where*

$$S^-_{r,n,d} := \left(\frac{\Gamma(d/2)}{2\pi^{d/2}} \right)^{1/2-1/r} \frac{I^{1/r}_{r,n,d}}{2^{n-1}\Gamma(n)\sqrt{\Phi_{r,n,d}}}, \tag{3.13}$$

$$I_{r,n,d} := \int_0^{+\infty} dt\ t^{d-1}\ (t^{n-d/2} K_{n-d/2}(t))^r, \tag{3.14}$$

$$\Phi_{r,n,d} := \operatorname*{Inf}_{\lambda > 0} \varphi_{r,n,d}(\lambda),$$
$$\varphi_{r,n,d}(\lambda) := \frac{1}{\lambda^{d-2d/r}} \int_0^{+\infty} ds\ s^{d-1} \frac{(1+\lambda^2 s^2)^n}{(1+s^2)^{2n}}. \tag{3.15}$$

Proof From the explicit expression (1.4), it follows (using the variable $t = \lambda|x|$)

$$\|G^{(\lambda)}_{2n,d}\|^r_{L^r} = \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{1}{2^{r(n-1)}\Gamma(n)^r \lambda^d} \times \\ \times \int_0^{+\infty} dt\ t^{d-1} \Big(t^{n-d/2} K_{n-d/2}(t) \Big)^r . \tag{3.16}$$

By (3.3) with $f = G_{2n,d}$, it is $(\mathcal{F}G^{(\lambda)}_{2n,d})(k) = \lambda^{-d}(1 + |k|^2/\lambda^2)^{-n}$, whence (using the variable $s = |k|/\lambda$)

$$\|G^{(\lambda)}_{2n,d}\|^2_{H^n} = \frac{1}{\lambda^{2d}} \int_{\mathbf{R}^d} dk \frac{(1 + |k|^2)^n}{(1 + |k|^2/\lambda^2)^{2n}} = \\ = \frac{2\pi^{d/2}}{\Gamma(d/2)\lambda^d} \int_0^{+\infty} ds\ s^{d-1} \frac{(1 + \lambda^2 s^2)^n}{(1 + s^2)^{2n}} . \tag{3.17}$$

Eqs. (3.16), (3.17) imply

$$\frac{\|G^{(\lambda)}_{2n,d}\|_{L^r}}{\|G^{(\lambda)}_{2n,d}\|_{H^n}} = \left(\frac{\Gamma(d/2)}{2\pi^{d/2}} \right)^{1/2-1/r} \frac{I^{1/r}_{r,n,d}}{2^{n-1}\Gamma(n)\sqrt{\varphi_{r,n,d}(\lambda)}}, \tag{3.18}$$

and (3.12) yields the thesis. ■

Remarks

(i) For n integer, the integral in the definition (3.15) of $\varphi_{r,n,d}$ is readily computed expanding $(1 + \lambda^2\ s^2)^n$ with the binomial formula, and integrating term by term. The integral of each term is expressible *via* the Beta function $B(z, w) = \Gamma(z)\ \Gamma(w)/\Gamma(z + w)$, the final result being

$$\varphi_{r,n,d}(\lambda) = \frac{1}{2\lambda^{d-2d/r}} \sum_{\ell=0}^{n} \binom{n}{\ell} B\Big(\ell + \frac{d}{2}, 2n - \frac{d}{2} - \ell\Big) \lambda^{2\ell} \quad (n \in \mathbf{N}). \tag{3.19}$$

For arbitrary, possibly noninteger n, the integral in (3.15) can be expressed in terms of the Gauss hypergeometric function $F = {}_2F_1$, and the conclusion is

$$\varphi_{r,n,d}(\lambda) = \frac{1}{2\lambda^{d-2d/r}} \left(B\left(2n - \frac{d}{2}, \frac{d}{2}\right) F\left(\frac{d}{2}, -n, 1 + \frac{d}{2} - 2n; \lambda^2\right) + \right.$$
$$+ \lambda^{4n-d} B\left(n - \frac{d}{2}, \frac{d}{2} - 2n\right) \times$$
$$\left. \times F\left(2n, n - \frac{d}{2}, 1 - \frac{d}{2} + 2n; \lambda^2\right)\right) \tag{3.20}$$

(in the singular cases $2n - d/2 - 1 \in \mathbf{N}$, the first hypergeometric in (3.20) must be appropriately intended, as a limit from nonsingular values).

(ii) Concerning $I_{r,n,d}$, there is one case in which the integral (3.14) is elementary, namely $n = d/2 + 1/2$. In fact, this case involves the function $t^{1/2} K_{1/2}(t) = \sqrt{\pi/2}\, \mathrm{e}^{-t}$, so that

$$I_{r,d/2+1/2,d} = \left(\frac{\pi}{2}\right)^{r/2} \int_0^{+\infty} dt\, t^{d-1} \mathrm{e}^{-rt} = \left(\frac{\pi}{2}\right)^{r/2} \frac{\Gamma(d)}{r^d}. \tag{3.21}$$

More generally, if $n = d/2 + m + 1/2$, $m \in \mathbf{N}$, the integral defining $I_{r,n,d}$ involves the function $t^{m+1/2}\, K_{m+1/2}(t)$, which has the elementary expression (1.6); for n as above and r integer, expanding the power $(t^{m+1/2}\, K_{m+1/2}(t))^r$ we can reduce $I_{r,n,d}$ to a linear combination of integrals of the type $\int_0^{+\infty} t^\alpha \mathrm{e}^{-rt} = \Gamma(\alpha + 1)/r^{\alpha+1}$. In other cases, $I_{r,n,d}$ can be evaluated numerically.

4. EXAMPLES

We present four examples (A), (B), (C), (D), each one corresponding to fixed values of (n, d) with $n > d/2$, and r ranging freely. Of course, in all these cases the analytical expression of (2.3) of $S^+_{r,n,d}$ is available; the expressions of the lower bounds $S^-_{r,n,d}$ are simple in examples (A), (D) and more complicated in examples (B), (C), where the integral $I_{r,n,d}$ is not expressed in terms of elementary functions, for arbitrary r.

Each example is concluded by a table of numerical values of $S^\pm_{r,n,d}$ (computed with the MATHEMATICA package), which are seen to be fairly close; the relative uncertainty $(S^+_{r,n,d} - S^-_{r,n,d})/S^-_{r,n,d}$ is also evaluated. In cases (A), (C), (D) the space dimension is $d=1,2,3$, respectively, and we take for n the smallest integer $> d/2$: this choice of n is the most interesting in many applications to PDE's. In case (B) where n is larger, the uncertainty is even smaller. Whenever we give numerical values, we round from above the digits of $S^{+}_{r,n,d}$, and from below the digits of $S^{-}_{r,n,d}$.

(A) Case $n=1$, $d=1$ Equations (2.3), (2.4) give $S^{+}_{r,1,1}$ for all $r\in[2,\infty]$; the values at the extremes are

$$S^{+}_{2,1,1}=1, \quad S^{+}_{\infty,1,1}=1/\sqrt{2}\simeq 0.7072 \tag{4.1}$$

(coinciding with the sharp imbedding constants due to Prop. 3.1). Let us pass to the lower bounds. The function $\varphi_{r,1,1}$ is given by (3.19) and attains its minimum at a point $\lambda=\lambda_{r,1,1}$; the integral $I_{r,1,1}$ is provided by (3.21), and these objects must be inserted into (3.13). Explicitly,

$$\varphi_{r,1,1}(\lambda)=\frac{\pi(\lambda^2+1)}{4\lambda^{1-2/r}}, \quad \lambda_{r,1,1}=\sqrt{\frac{1-2/r}{1+2/r}}; \ I_{r,1,1}=\left(\frac{\pi}{2}\right)^{r/2}\frac{1}{r}; \tag{4.2}$$

$$S^{-}_{r,1,1}=\frac{I^{1/r}_{r,1,1}}{2^{1/2-1/r}\sqrt{\varphi_{r,1,1}(\lambda_{r,1,1})}}= \\ =\frac{E(1/r)}{2^{1/2-1/r}}E\left(1+\frac{2}{r}\right)^{1/4}E\left(1-\frac{2}{r}\right)^{1/4}. \tag{4.3}$$

Computing numerically the bounds (2.3), (4.3) for many values of $r\in(2,+\infty)$, we always found $(S^{+}_{r,1,1}-S^{-}_{r,1,1})/S^{-}_{r,1,1}<0.05$, the maximum of this relative uncertainty being attained for $r\simeq 6$. Here are some numerical values:

r	2.2	3	4	6	50	1000
$S^{+}_{r,1,1}$	0.8832	0.7212	0.6624	0.6345	0.6782	0.7046.
$S^{-}_{r,1,1}$	0.8730	0.6973	0.6347	0.6057	0.6632	0.7027

(4.4)

(B) Case $n=3$, $d=1$ Equations (2.3), (2.4) give $S^{+}_{r,3,1}$ for all r; in particular

$$S^{+}_{2,3,1}=1, \quad S^{+}_{\infty,3,1}=\sqrt{3}/4\simeq 0.4331. \tag{4.5}$$

We pass to the lower bounds. Equations (3.19), (3.14), (1.6) give

$$\varphi_{r,3,1}(\lambda) = \frac{3\pi(\lambda^6 + 3\lambda^4 + 7\lambda^2 + 21)}{512\lambda^{1-2/r}};$$

$$I_{r,3,1} = \left(\frac{\pi}{2}\right)^{r/2} \int_0^{+\infty} dt(t^2 + 3t + 3)^r e^{-rt}. \tag{4.6}$$

The minimum point $\lambda_{r,3,1}$ of $\varphi_{r,3,1}$ is the positive solution of the equation

$$\left(5 + \frac{2}{r}\right)\lambda^6 + \left(9 + \frac{6}{r}\right)\lambda^4 + \left(7 + \frac{14}{r}\right)\lambda^2 - \left(21 - \frac{42}{r}\right) = 0; \tag{4.7}$$

the integral $I_{r,3,1}$ can be computed analytically for integer r, and numerically otherwise. The final lower bounds, and some numerical values for them and for the upper bounds (2.3) are

$$S^-_{r,3,1} = \frac{I^{1/r}_{r,3,1}}{2^{7/2-1/r}\sqrt{\varphi_{r,3,1}(\lambda_{r,3,1})}}. \tag{4.8}$$

r	2.2	3	6	10	20
$S^+_{r,3,1}$	0.8605	0.6475	0.4888	0.4519	0.4341 .
$S^-_{r,3,1}$	0.8597	0.6458	0.4872	0.4507	0.4333

(4.9)

For each r in this table $(S^+_{r,3,1} - S^-_{r,3,1})/S^-_{r,1,1} < 0.004$, with a maximum uncertainty for $r = 6$.

(C) Case $n = 2, d = 2$ Equations (2.3), (2.4) give $S^+_{r,2,2}$ for all r, and in particular

$$S^+_{2,2,2} = 1, \quad S^+_{\infty,2,2} = 1/\sqrt{4\pi} \simeq 0.2821. \tag{4.10}$$

The function $\varphi_{r,2,2}$ computed *via* Eq. (3.21), its minimum point $\lambda_{r,2,2}$ and the integral $I_{r,2,2}$, defined by (3.14), are given by

$$\varphi_{r,2,2}(\lambda) = \frac{\lambda^4 + \lambda^2 + 1}{6\lambda^{2-4/r}}, \quad \lambda_{r,2,2} = \sqrt{\frac{-1/r + \sqrt{1 - 3/r^2}}{1 + 2/r}};$$

$$I_{r,2,2} = \int_0^{+\infty} dt\ t\Big(tK_1(t)\Big)^r. \tag{4.11}$$

The above integral must be computed numerically. The final expression for the lower bounds, and some numerical values for them and for the upper bounds (2.3) are

$$S^-_{r,2,2} = \frac{I^{1/r}_{r,2,2}}{2^{3/2-1/r}\pi^{1/2-1/r}\sqrt{\varphi_{r,2,2}(\lambda_{r,2,2})}}, \tag{4.12}$$

r	2.1	3	6	18	50	100
$S^+_{r,2,2}$	0.8494	0.4557	0.2949	0.2644	0.2694	0.2737.
$S^-_{r,2,2}$	0.8465	0.4455	0.2854	0.2582	0.2659	0.2715

(4.13)

It is $(S^+_{r,2,2} - S^-_{r,2,2})/S^-_{r,2,2} < 0.04$ for all r in this table, with a maximum uncertainty for $r = 6$.

(D) Case $n = 2$, $d = 3$ Equations (2.3), (2.4) give $S^+_{r,2,3}$ for all r, and in particular

$$S^+_{2,2,3} = 1, \quad S^+_{\infty,2,3} = 1/\sqrt{8\pi} \simeq 0.1995. \tag{4.14}$$

The function $\varphi_{r,2,3}$ computed from Eq. (3.19), its minimum point $\lambda_{r,2,3}$ and the integral $I_{r,2,3}$ provided by (3.21) are

$$\varphi_{r,2,3}(\lambda) = \frac{\pi(5\lambda^4 + 2\lambda^2 + 1)}{32\lambda^{3-6/r}}, \quad \lambda_{r,2,3} = \sqrt{\frac{1 - 6/r + 4\sqrt{1 + 3/r - 9/r^2}}{5(1 + 6/r)}};$$
$$I_{r,2,3} = \left(\frac{\pi}{2}\right)^{r/2} \frac{2}{r^3}. \tag{4.15}$$

The final expression for the lower bounds, and some numerical values for them and for the upper bounds are

$$S^-_{r,2,3} = \frac{\pi^{1/r} E(1/r)^3}{2^{5/2-3/r}\sqrt{\varphi_{r,2,3}(\lambda_{r,2,3})}}. \tag{4.16}$$

r	2.1	3	4	7	11	20	100	1000
$S^+_{r,2,3}$	0.7830	0.3118	0.2183	0.1657	0.1594	0.1647	0.1864	0.1975.
$S^-_{r,2,3}$	0.7762	0.2912	0.1986	0.1486	0.1437	0.1511	0.1795	0.1960

(4.17)

For these and other values of r in $(2, \infty)$, we always found $(S^+_{r,2,3} - S^-_{r,2,3})/S^-_{r,2,3} < 0.12$, the maximum uncertainty occurring for $r \simeq 7$.

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