# STURMIAN EXPANSIONS AND ENTROPY

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**Abstract**

Using the irrational rotation transformation $T$ as a map of the interval (2-interval exchange), we study the corresponding $f$-expansions, which turn out to be Sturmian sequences. Contrasting the maps $T$ for more familiar expansions, like $\beta$-expansions and continued fraction expansions, this map $T$ is zero-entropy and invertible. As a result, the corresponding $f$-expansions have some unusual properties. Most significantly, convergence is painfully slow and there are no periodic (or finite) Sturmian expansions. We also generalize this to $n$-interval exchanges $T$, with $2 < n \leq \infty$, discussing the von Neumann adding machine transformation ($n = \infty$) in detail.

## 1. Introduction

Sturmian sequences were introduced by Morse and Hedlund [10] as the sequences that code the orbits of the geodesic flow on a flat 2-torus. In this paper, we restrict our attention to (1-sided) aperiodic Sturmian sequences, which may be defined to be those sequences $\boldsymbol{d} = .d_1d_2d_3\cdots \in \{0,1\}^{\mathbb{N}}$ that have exactly $n+1$ distinct *factors* (subsequences $\boldsymbol{u} = d_jd_{j+1}\ldots d_{j+n-1}$) of length $n$. This property is often expressed by saying that a Sturmian sequence $\boldsymbol{d}$ has *complexity function* $c_{\boldsymbol{d}}(n) = n+1$. If $c_{\boldsymbol{e}}(n)$ is the complexity function of a sequence $\boldsymbol{e} \in \{0,1\}^{\mathbb{N}}$, it is known (see [5], Chapter 6) that $c_{\boldsymbol{e}}(k) = k$ for some $k$ if and only if $\boldsymbol{e}$ is eventually periodic. Thus Sturmian sequences are the least complex among aperiodic sequences.

A sequence $\boldsymbol{d} = .d_1d_2d_3\cdots \in \{0,1\}^{\mathbb{N}}$ is said to be *balanced* if for any $i, j, \ell \geq 1$

$$\left| \sum_{k=i}^{i+\ell-1} d_k - \sum_{k=j}^{j+\ell-1} d_k \right| \leq 1.$$

It can be shown that a sequence is balanced if and only if it is Sturmian, and from this, one can prove (see [5], Chapter 6) that the limit

$$\alpha = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\infty} d_k \tag{1}$$

exists, and that $\alpha$ is irrational. The number $\alpha$ is called the *slope* of $\boldsymbol{d}$.

Morse and Hedlund [10] showed that if $\boldsymbol{d} = .d_1d_2d_3\dots$ is a Sturmian sequence with slope $\alpha \in (0,1)\backslash\mathbb{Q}$, then there is a unique $x \in [0,1)$, called the *intercept*, so that $\boldsymbol{d}$ either has the form

$$d_n = \lfloor \alpha(n+1) + x \rfloor - \lfloor \alpha n + x \rfloor, \tag{2}$$

for all $n \in \mathbb{N}$, or

$$d_n = \lceil \alpha(n+1) + x \rceil - \lceil \alpha n + x \rceil. \tag{3}$$

Note that (2) and (3) are the same unless $n\alpha + x = 0 \bmod 1$ for some $n > 1$, in which case they disagree in exactly one or two adjacent digits.

Given a Sturmian sequence $\boldsymbol{d}$, one can easily determine its slope $\alpha$ using (1). The goal of this paper is to exhibit a similarly simple formula for the intercept $x$. In particular, we show how the intercept $x$ can be obtained using a well know generalization of continued fraction and radix expansions, called an $f$-expansions. Another way to say this is that a Sturmian sequence $\boldsymbol{d} = .d_1d_2d_2\dots \in \{0,1\}^{\mathbb{N}}$ can be regarded as type "binary expansion" of its intercept $x$. We refer to this as the *Sturmian $\alpha$-expansion* of $x$ (and call $\alpha$ the *base*).

After a brief discussion of $f$-expansions in general, we discuss general some properties of Sturmian $\alpha$-expansions. In particular, Sturmian $\alpha$-expansions differ significantly from nearly all other familiar numeration systems, including continued fraction expansions and $\beta$-expansions — two examples we use to draw this contrast. We conclude by mentioning several other examples that have properties similar to Sturmian $\alpha$-expansions.

## 2. $f$-Expansions

Let $f : \mathbb{R} \to [0,1]$ be a continuous monotonic function with $\overline{f(\mathbb{R})} = [0,1]$. An *$f$-expansion* is an expression of the form

$$x = f(d_1 + f(d_2 + f(d_3 + \dots))), \tag{4}$$

where the *digits* $d_k$ are integers. We call $\boldsymbol{d} = .d_1d_2d_3\dots$ the *digit sequence* of the expansion (4). In particular, the expression (4) means that $x_n \to x$, where

$$x_n = f(d_1 + f(d_2 + \dots + f(d_n))). \tag{5}$$

This idea goes back to Kakeya [7], who observed in 1924 that examples of $f$-expansions include[1] both regular continued fractions and base $\beta$ radix expansions, $\beta > 1$. In particular, regular continued fractions

$$x = f(d_1 + f(d_2 + f(d_3 + \dots))) = \cfrac{1}{d_1 + \cfrac{1}{d_2 + \cfrac{1}{d_3 + \cdots}}},$$

correspond to the case $f(x) = 1/x$, whereas base-$\beta$ radix expansions

$$x = f(d_1 + f(d_2 + f(d_3 + \dots))) = \cfrac{d_1 + \cfrac{d_2 + \cfrac{d_3 + \dots}{\beta}}{\beta}}{\beta} = \sum_{k=1}^{\infty} \frac{d_k}{\beta^k}$$

correspond to $f(x) = x/\beta$.

Although more than one digit sequence in (4) may yield the same number $x \in [0, 1)$ (just as $0.099\cdots = 0.100\dots$ in base 10), there is a standard algorithm that takes $x$ and produces a particular digit sequence $\boldsymbol{d} = .d_1 d_2 d_3 \dots$ that we call the *proper $f$-expansion of $x$*. As Rényi [14] observed in 1957, this algorithm may be described in terms of a dynamical system. Starting with $f$, we define the *$f$-transformation* $T : [0, 1) \to [0, 1)$ by

$$Tx = f^{-1}(x) \bmod 1. \tag{6}$$

We also define a *labeled interval partition* $\xi$ a.e. on $[0, 1)$, defined to be the positive measure level sets of the function $p_\xi : [0, 1) \to \mathbb{Z}$ defined $p_\xi(x) = \lfloor f^{-1}(x) \rfloor$. In particular, $\xi = \{\Delta(d) : d \in \mathcal{D}\}$, where $\Delta(d) = [a, b) = \xi^{-1}(d)$, and $\mathcal{D} = \{d \in \mathbb{Z} : a \neq b\}$. The sets $\Delta(d)$ are called *fundamental intervals*, and $\mathcal{D} \subseteq \mathbb{Z}$ is called the *digit set*. Assuming $x$ is such that $T^{n-1}x$ exists for all $n \in \mathbb{N}$, the proper digit sequence $\boldsymbol{d} = .d_1 d_2 d_3 \dots$ is defined by

$$d_n = p_\xi(T^{n-1}x), \quad n \in \mathbb{N}. \tag{7}$$

If the proper digit sequence $\boldsymbol{d} = .d_1 d_2 d_3 \dots$ is used in the $f$-expansion (4), we call it the *proper $f$-expansion* of $x$.

For continued fractions, the $f$-transformation is the *Gauss map* $Tx = 1/x \bmod 1$, and $\mathcal{D} = \mathbb{N}$. (Sometimes in cases like this, where some of the digits $d_n$ are multi-digit numbers when written in base 10, it will be convenient to write the digit sequence as $\boldsymbol{d} = [d_1, d_2, d_3, \dots]$ rather than $\boldsymbol{d} = .d_1 d_2 d_3 \dots$.) For base-$\beta$ radix

[1] The same observation was made independently by Bissinger [2] for $f$ increasing, and Everett [4] for $f$ decreasing.

expansions, the $f$-transformation $Tx = \beta x \bmod 1$ is called the *$\beta$-transformation*, and $\mathcal{D} = \{0, 1, \ldots \lceil \beta \rceil - 1\}$. The case $\beta \in \mathbb{N}$ gives the usual radix expansions (e.g., base 2, base 10).

We say $f$-expansions have *unique proper digits* if the proper digit sequence map $x \mapsto \boldsymbol{d}$ is injective, and we say $f$-expansions are *valid* if for each $x$, such that $T^n x$ exists for all $n \geq 0$, the proper $f$-expansion converges to $x$. A typical approach to this problem is the following (see [7] and also [17]).

**Theorem 1 (Kakeya's theorem).** *Assume $f$ is strictly monotone on an interval $(a, b) \subseteq \mathbb{R}$ with $a, b \in \mathbb{Z}$, and*

$$\begin{gathered} -\infty \leq a < a + 1 < b \leq +\infty, \\ \textit{and} \\ \overline{f((a, b))} = [0, 1]. \end{gathered} \tag{8}$$

*If the $f$-transformation $T$ satisfies*

$$|T'(x)| > 1 \quad \text{a.e.}, \tag{9}$$

*then $f$ has unique proper digits and $f$-expansions are valid.*

Note that (9) is equivalent to

$$|f'(x)| < 1 \text{ a.e. on } (a, b). \tag{10}$$

Similar results due to Bissinger [2] and Everett [4] require that $f$ satisfies a Lipshitz condition with constant $K < 1$ instead of (10).

## 3. Sturmian $\alpha$-Expansions

Let us fix an irrational number $\alpha \in [0, 1) \backslash \mathbb{Q}$ and consider as the *irrational rotation transformation* $Tx = x + \alpha \bmod 1$. This can be interpreted as the $f$-transformation (6) for the function

$$f(x) = \begin{cases} 0 & \text{if } x < \alpha \\ x - \alpha & \text{if } \alpha \leq x \leq \alpha + 1 \\ 1 & \text{if } x > \alpha + 1. \end{cases} \tag{11}$$

The corresponding labeled partition is given by

$$p_\xi(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 - \alpha \\ 1 & \text{if } 1 - \alpha < x \leq 1, \end{cases}$$

with digit set $\mathcal{D} = \{0, 1\}$.

It is easy to see that for the function $f$, defined by (11), the proper digit sequence $\boldsymbol{d} = .d_1d_2d_3\ldots$ for $x \in [0,1)$, is equal to the Sturmian sequence $\boldsymbol{d}$ given by (2) with slope $\alpha$ and intercept $x$. These are our *Sturmian $\alpha$-expansions*. The irrational rotation transformation $T$ clearly fails to satisfy Kakeya's hypotheses (9). However, we can still obtain the following result[2].

**Theorem 2.** *Sturmian $\alpha$-expansions have unique proper digit sequences, and are valid.*

*Proof.* Given $x \in [0,1)$, let $x \mapsto \boldsymbol{d} = .d_1d_2d_3\ldots$ be the proper digit sequence map, and let $\boldsymbol{d}_n(x) = d_1d_2\ldots d_n \in \mathcal{D}^n$. Let $\xi^{(n)}$ be the partition of $[0,1)$ into those subintervals $[a_k^n, b_k^n)$, $k = 1, 2, \ldots$, of $[0,1)$, on which $\boldsymbol{d}_n(x)$ is constant. We claim that for each $n = 1, 2, \ldots$, there are $|\xi^{(n)}| = n+1$ such intervals, and if they are arranged so that $b_{k+1}^n = a_k^n$, then the *cut points* $a_1^n, a_2^n, \ldots, a_{n+1}^n$ are the first $n+1$ points orbit $O_{T^{-1}}^+(0) = \{T^{-n+1}0 : n \in \mathbb{N}\}$ of 0 under the irrational rotation transformation $T^{-1}x = x + (1-\alpha) \bmod 1$.

The claim is true for $n = 1$, so assume it holds for $n$. Since $T^{-1}$ is an is an irrational rotation, $O_{T^{-1}}^+(0)$ is dense in $[0,1)$ (see [19]) so $T^{-(n+1)}0$ is in the interior of $[a_\ell^n, b_\ell^n)$ for some $\ell$. We then have

$$[a_k^{n+1}, b_k^{n+1}) = \begin{cases} [a_k^n, b_k^n) & \text{if } 1 \le k < \ell, \\ [a_k^n, T^{-(n+1)}0) & \text{if } k = \ell, \\ [T^{-(n+1)}0, b_k^n) & \text{if } k = \ell+1, \\ [a_{k-1}^n, b_{k-1}^n) & \text{if } \ell+1 < k \le n+1, \end{cases}$$

so the claim holds for $n+1$.

By the claim, the cutpoints of $\xi^{(n)}$ satisfy $\{a_1^n, a_2^n, \ldots, a_{n+1}^n\} = \{T^{-k}0 : k = 0, \ldots, n\}$ for all $n$. Define $||\xi^{(n)}|| = \max\{b - a : \Delta = [a,b) \in \xi^{(n)}\}$. Since $O_{T^{-1}}^+(x)$ is dense, it follows that $||\xi^{(n)}|| \to 0$. This shows Sturmain $\alpha$-expansions have unique proper digit sequences.

For $x \in [0,1)$ and $n \in \mathbb{N}$ we have

$$x \in [a^n(x), b^n(x)) := [a_k^n, b_k^n) \in \xi^{(n)} \tag{12}$$

for some unique $k = k(n)$. We claim that

$$a^n(x) = f(d_1 + f(d_2 + \cdots + f(d_n)))$$

and

$$b^n(x) = f(d_1 + f(d_2 + \cdots + f(d_n + 1)))$$

Indeed, $[f(d_1), f(d_1+1)) = [f(0), f(1)) = [0, 1-\alpha) = [a^1(x), b^1(x))$ if $d_1 = 0$ and $[f(d_1), f(d_1+1)) = [f(1), f(2)) = [1-\alpha, 1) = [a^1(x), b^1(x))$ if $d_1 = 1$, so the claim

[2]This fact was noted in passing by Parry in [12].

holds for $n = 1$. We proceed by induction. Note that

$$[a^n(x), b^n(x)) = [a^1(x), b^1(x)) \cap T^{-1}[a^{n-1}(x'), b^{n-1}(x'))$$

where $x' = Tx$. By induction,

$$a^{n-1}(x') = f(d_2 + f(d_3 + \cdots + f(d_n)))$$

and

$$b^{n-1}(x') = f(d_2 + f(d_3 + \cdots + f(d_n + 1))).$$

Thus

$$\begin{aligned} a^n(x) &= T^{-1}(a^{n-1}(x')) \cap [a^1(x), b^1(x)) \\ &= f(d_1 + a^{n-1}(x')) \\ &= f(d_1 + f(d_2 + \cdots + f(d_n))), \end{aligned}$$

and

$$\begin{aligned} b^n(x) &= T^{-1}(b^{n-1}(x')) \cap [a^1(x), b^1(x)) \\ &= f(d_1 + b^{n-1}(x')) \\ &= f(d_1 + f(d_2 + \cdots + f(d_n + 1))). \end{aligned}$$

Finally, since $||\xi^{(n)}|| \to 0$, it follows that $x_n = a^n(x) \to x$, so Sturmian $\alpha$-expansions are valid. □

As an example, let the base be $\alpha = \sqrt{2} - 1$. Then for $x = 1/2$, we have

$$\boldsymbol{d} = .0101001010100101001010100101001010010101001010010101000\ldots,$$

admittedly, not a very intuitive expansion for $1/2$. The first 30 partial convergents (partial $f$-expansions) are shown in Table 1. Note that (in general) the convergents $x_n$ lie in the set $\mathbb{Z} + \alpha\mathbb{Z}$ (in the example, $\mathbb{Z}[\sqrt{2}]$). Table 1 suggests that convergence is very slow. This is reinforced by Figure 1, which shows a plot of the first 1000 convergents. Note that $x_{1000} = 1105 - 781\sqrt{2} \sim .49921$, still correct to only three decimal places. Figure 1 and Table 1 both suggest that there are long intervals of $n$ where the convergents $x_n$ remain constant.

As a second example, again for $\alpha = \sqrt{2} - 1$, let $x = 16 - 11\sqrt{2} \sim .44365$. Then we get

$$\boldsymbol{d} = .0101001010100101001010010101001010010101001010010100101\ldots$$

A calculation shows that $x_1 = 0$, $x_2 = x_3 = 3 - 2\sqrt{2}$ $x_4 = \cdots = x_9 = 6 - 4\sqrt{2}$ and $x_{10} = x_{11} = x_{12} = \cdots = 16 - 11\sqrt{2}$. So the convergents reach $x$ after a finite number of steps (and never change) even though the represenation $\boldsymbol{d}$ is infinite. These facts are explained by the following proposition, which we state without proof.

| $n$ | $x_n$ | $\sim x_n$ | $n$ | $x_n$ | $\sim x_n$ | $n$ | $x_n$ | $\sim x_n$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | .00000 | 11 | $16-11\sqrt{2}$ | .44365 | 21 | $16-11\sqrt{2}$ | .44365 |
| 2 | $3-2\sqrt{2}$ | .17157 | 12 | $16-11\sqrt{2}$ | .44365 | 22 | $16-11\sqrt{2}$ | .44365 |
| 3 | $3-2\sqrt{2}$ | .17157 | 13 | $16-11\sqrt{2}$ | .44365 | 23 | $33-23\sqrt{2}$ | .47309 |
| 4 | $6-4\sqrt{2}$ | .34315 | 14 | $16-11\sqrt{2}$ | .44365 | 24 | $33-23\sqrt{2}$ | .47309 |
| 5 | $6-4\sqrt{2}$ | .34315 | 15 | $16-11\sqrt{2}$ | .44365 | 25 | $33-23\sqrt{2}$ | .47309 |
| 6 | $6-4\sqrt{2}$ | .34315 | 16 | $16-11\sqrt{2}$ | .44365 | 26 | $33-23\sqrt{2}$ | .47309 |
| 7 | $6-4\sqrt{2}$ | .34315 | 17 | $16-11\sqrt{2}$ | .44365 | 27 | $33-23\sqrt{2}$ | .47309 |
| 8 | $6-4\sqrt{2}$ | .34315 | 18 | $16-11\sqrt{2}$ | .44365 | 28 | $33-23\sqrt{2}$ | .47309 |
| 9 | $6-4\sqrt{2}$ | .34315 | 19 | $16-11\sqrt{2}$ | .44365 | 29 | $33-23\sqrt{2}$ | .47309 |
| 10 | $6-4\sqrt{2}$ | .34315 | 20 | $16-11\sqrt{2}$ | .44365 | 30 | $33-23\sqrt{2}$ | .47309 |

Table 1: First 20 convergents of $x = 1/2 = .5$ in the Sturmian expansion, base $\alpha = \sqrt{2}-1$.

**Proposition 3.** *Fix $\alpha \in [0,1)\backslash\mathbb{Q}$, and for $x \in [0,1)$, define*

$$\underline{x}_n = \max\left(\{T^{-k}0 : k = 0,\dots,n\} \cap [0,x)\right)$$

*and*

$$\overline{x}_n = \min\left((\{T^{-k}0 : k = 1,\dots,n\} \cup \{1\}) \cap (x,1]\right),$$

*so that $x \in [\underline{x}_n, \overline{x}_n)$ for all $n \in \mathbb{N}$. Then there exist strictly increasing sequences $\underline{n}_k, \overline{n}_k \in \mathbb{N}$ so that*

$$\underline{x}_n = T^{-\underline{n}_k}0 \text{ for } n \in [\underline{n}_k, \underline{n}_{k+1})$$

*and*

$$\overline{x}_n = T^{-\overline{n}_k}0 \text{ for } n \in [\overline{n}_k, \overline{n}_{k+1}).$$

*Moreover, if $a^n(x)$ and $b^n(x)$ are as in (12), then $[a^n(x), b^n(x)) = [\underline{x}_n, \overline{x}_n)$.* □

Note that $x_n = \underline{x}_n$. We are now in a position to give a qualitative description of the long intervals in $\mathbb{N}$ on which the convergents are constant. By Proposition 3, these are the intervals $[\underline{n}_k, \underline{n}_{k+1})$. At step $\underline{n}_k$ there are $\underline{n}_k + 1$ intervals in $\xi^{(\underline{n}_k)}$, one of which, $[a^{\underline{n}_k}(x), b^{\underline{n}_k}(x))$ contains $x$. Thus $O^+_{T^{-1}}(x)$ will visit every other interval in $\xi^{(\underline{n}_k)}$ at least once before its first return to $[a^{\underline{n}_k}(x), b^{\underline{n}_k}(x))$. Thus, $\underline{n}_{k+1}$ will be a least twice $\underline{n}_k$. We will discuss the relation between Sturmian $\alpha$-expansions and Ostrowski numeration (see [5], Chapter 5) in a later paper [15].

Now let us consider Sturmian $\alpha$-expansions for digit sequences $\boldsymbol{c} \in \mathcal{D}^{\mathbb{N}}$ that are not necessarily proper. In particular, given any $\boldsymbol{c} = .c_1c_2c_3\cdots \in \{0,1\}^{\mathbb{N}}$ let

$$\varepsilon(\boldsymbol{c}) = f(c_1 + f(c_2 + f(c_3 + \dots))).$$

Let $\prec$ denote lexicographic order on $\{0,1\}^{\mathbb{N}}$. That is, $\boldsymbol{c} \prec \boldsymbol{e}$, $\boldsymbol{e} = .e_1e_2e_3\dots$, if and only if for some $n \geq 1$, $c_1\dots c_n = e_1\dots e_n$, $c_{n+1} = 0$ and $e_{n+1} = 1$.

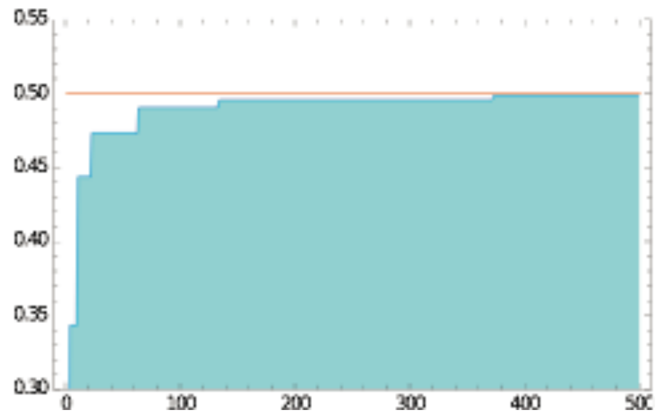


Figure 1: A bar graph of the first 500 convergents of $x = 1/2$ in Sturmian base $\alpha = \sqrt{2} - 1$.

**Lemma 4.** *If $f$ is given by* (11)*, then for any $\boldsymbol{c} = .c_1c_2c_3 \cdots \in \{0,1\}^{\mathbb{N}}$, the $f$-expansion $\varepsilon(\boldsymbol{c}) = f(c_1 + f(c_2 + f(c_3 + \dots)))$ converges. Moreover, if $\boldsymbol{c} \prec \boldsymbol{e}$ then $\varepsilon(\boldsymbol{c}) \leq \varepsilon(\boldsymbol{e})$.*

*Proof.* Since $f$ is nondecreasing, $x_{n+1} \geq x_n$, and moreover, $x_n \leq 1$ since $f(x) \leq 1$. Thus $\varepsilon(\boldsymbol{c})$ converges.

Suppose $c_1 = 0$ and $e_1 = 1$ so that $\boldsymbol{c} \prec \boldsymbol{e}$. Since $f(c_2 + f(c_3 + \dots)) \leq 1$ and $f(e_2 + f(e_3 + \dots)) \geq 0$, it follows that $c_1 + f(c_2 + \dots) \leq e_1 + f(e_2 + \dots)$, and so $f(c_1 + f(c_2 + \dots)) \leq f(e_1 + f(e_2 + \dots))$.

Now suppose $\boldsymbol{c} \prec \boldsymbol{e}$. Let $n$ be such that $c_1c_2 \dots c_{n-1} = e_1e_2 \dots e_{n-1}$, with $c_n = 0$ and $e_n = 1$. By the previous paragraph $f(c_n + f(c_{n+1} + \dots)) \leq f(e_n + f(e_{n+1} + \dots))$. Since $f$ is increasing it follows that $f(c_1 + \cdots + f(c_n + f(c_{n+1} + \dots))) \leq f(e_1 + \cdots + f(e_n + f(e_{n+1} + \dots))$. □

## 4. Ergodic Properties of Sturmian $\alpha$-Expansions

As Rényi observed in his landmark paper [14], many properties of $f$-expansions reflect the "ergodic" properties of the corresponding $f$-transformation $T$. In this section, we compare the irrational rotation map $Tx = x + \alpha \bmod 1$ to the Gauss map $Tx = 1/x \bmod 1$ and to the $\beta$-transformations $Tx = \beta x \bmod 1$. A $T$-*invariant probability measure* $\mu$ on $[0, 1)$ is a Borel measure so that $\mu([0, 1)) = 1$ and $\mu(T^{-1}E) = \mu(E)$ for every Borel set $E$. A measure $\mu$ is an *absolutely continuous* if there is a density $\rho(x) \geq 0$ on $[0, 1)$ with $\mu(E) = \int_E \rho(x)\, dx$, and "*Lebesgue-equivalent*" if $\rho(x) > 0$ a.e. (i.e., $\rho(x) = d\mu/dx$ is the Radon-Nikodym derivative of

$\mu$). A measure is *ergodic* if $\mu(T^{-1}E\Delta E) = 0$ implies $\mu(A) = 0$ or 1.

All of the transformations we are discussing have an ergodic Lebesgue-equivalent invariant probability measure ("ELEM" for short). For the Gauss map, the density for this measure, called the *Gauss measure*, is given by $\rho(x) = \frac{1}{\log 2}\frac{1}{1+x}$. For $\beta$-transformations $T$ with $\beta \in \mathbb{N}$, Lebesgue measure itself is invariant (i.e., $\rho(x) = 1$). When $\beta \notin \mathbb{N}$, $\rho(x)$ is a step function, and $\mu$ is called the *Parry measure* (see [14] and [11]). Finally, if $T$ is an irrational rotation transformation the ELEM is, again, Lebesgue measure.

For all three of these transformations, the existence of a ELEM implies that for the corresponding proper $f$-representation $\boldsymbol{d}$, almost every $x \in [0, 1)$ is a *normal number.* In particular, for $\boldsymbol{u} \in \mathcal{D}^n$ let

$$L_n(\boldsymbol{u}, \boldsymbol{d}) = |\{j \in [1, \ldots, n] : \boldsymbol{d}|_{[j, \ldots j+|\boldsymbol{u}|-1]} = \boldsymbol{u}\}|$$

denote the number of *occurrences* of $\boldsymbol{u}$ in the first $n$ places in $\boldsymbol{d}$. A standard argument using Birkhoff ergodic theorem (see e.g.[19]) shows that for almost every $x \in [0, 1)$, for any $\boldsymbol{u} \in \mathcal{D}^n$,

$$\lim_{n\to\infty} \frac{1}{n} L_n(\boldsymbol{u}, \boldsymbol{d}) = \int_{\Delta(d_1)\cap T^{-1}\Delta(d_2)\cap\cdots\cap T^{-n+1}\Delta(d_n)} \rho(x)\, dx. \tag{13}$$

Thus, in a typical proper $f$-expansion, every finite sequence of digits occurs with a well defined frequency (which may sometimes be zero).

Beyond these simple facts, however, an irrational rotation transformations $T$ is very different from either the Gauss map or any $\beta$-transformation, and this this leads to some unusual properties for Sturmian $\alpha$-expansions. To begin with, the Gauss map or any $\beta$-transformation has other ergodic invariant measures besides its ELEM. On the other hand, Lebesgue measure is the unique invariant measure for an irrational rotation $T$, a property known as *unique ergodicity.* One corollary of unique ergodicity is that the ergodic theorem (13) converges for all $x$ (see [19]) rather than just almost everywhere. Thus *every* $x \in [0, 1)$ is a normal number for a Sturmian $\alpha$-expansion.

Another consequence of unique ergodicity is *minimality*, which means that $O_T^+(x)$ is dense for every $x$. A minimal map $T$ has no periodic or eventually periodic points. This means that there are no periodic or eventually periodic proper Sturmian $\alpha$-expansions. But periodic points are dense for both the Gauss map and $\beta$-transformations. In both of these cases, periodic expansions have important number theoretic consequences. However, even though there are no periodic expansions for Sturmian $\alpha$-expansions, minimality implies that all proper digit sequences have the following "almost periodicity" property. Suppose a finite sequence $\boldsymbol{u} \in \{0, 1\}^n$ occurs in the proper Sturmian $\alpha$-expansion $\boldsymbol{c}$ of some $y$. Then there is a constant $K > 0$ ($K = K(\boldsymbol{u}, \alpha)$) so that $\boldsymbol{u}$ occurs within $K$ of an arbitrary location in the proper Sturmian $\alpha$-expansion $\boldsymbol{d}$ of any $x$. Qualitatively, all proper Sturmian $\alpha$-expansions look pretty much alike.

A special case of eventual periodicity for $\beta$-expansions occurs when the proper expansion of $x$ ends in zeros. In base $\beta = 2$, for example, such numbers are the dyadic rationals. We say the $\beta$-expansion of $x$ is *finite*. Note that the $\beta$-expansion of $x$ is a finite sum in this case. A similar situation can be imposed on continued fractions by defining $T0 = p_\xi(0) = f(0) = 0$ and allowing $0 \in \mathcal{D}$. In this case, a number $x \in [0, 1)$ has a finite continued fraction expansion if and only if it is rational.

At first it appears that there is no analogous situation for Sturmian expansions, since no proper Sturmian expansions end in infinitely many zeros. On the other hand, let $x \in O_+(0)$ and let $\boldsymbol{d} = .d_1d_2d_3\ldots$ be the proper Sturmian expansion of $x$. Then there exists $n_0 \in \mathbb{N}$ so that $x = f(d_1 + f(d_2 + \cdots + f(d_n)))$ for $n \geq n_0$ (namely, $T^{n_0}0 = x$). Moreover, it is easy to see that the digit sequence $\boldsymbol{d}' = .d_1d_2\ldots d_{n_0}0000\ldots$ gives an improper Sturmian $\alpha$-expansion of $x$. It follows that there are uncountably many $\boldsymbol{d}'' \in \{0,1\}^{\mathbb{N}}$ satisfying $\boldsymbol{d}' \prec \boldsymbol{d}'' \prec \boldsymbol{d}$ that are all Sturmian $\alpha$-expansions of $x$. So in this sense Sturmian $\alpha$-expansions can be highly non-unique.

## 5. Entropy and Generators

Let $T$ be a measure-preserving transformation of $[0, 1)$, with $\mu$ the invariant Borel probability measure. Let $\xi$ be a finite or countable partition of $[0, 1)$ into positive measure Borel sets $C$. In general, we do not assume $\mu$ is an ELEM or that $\xi$ is a labeled interval partition. We say a Borel set $A$ satisfies $A \leq \xi$ if $A$ is a union of elements $C \in \xi$. Let $\xi \vee \xi' := \{C \cap C' : C \in \xi, C' \in \xi', \mu(C \cap C') > 0\}$. Define $\xi^{(n)} = \xi \vee T\xi \vee \cdots \vee T^{-n+1}\xi$, and if $T$ is invertible, also define $\xi^{(-n,n)} := T^{-n}\xi \vee \cdots \vee \xi \vee \cdots \vee T^n\xi$. A partition $\xi$ is called a 1-*sided generator* $T$ if for any Borel set $A$, and $n \in \mathbb{N}$, there exists $A_n \in \xi^{(n)}$ so that $\mu(A_n \Delta A) \to 0$. If $T$ is invertible, $\xi$ is a 2-*sided generator* if there is an $A_n \leq \xi^{(-n,n)}$ such that $\mu(A_n \Delta A) \to 0$.

Let $T$ be an $f$-transformation with an ELEM $\mu$ and let $\xi$ be the corresponding partition into fundamental intervals. It follows from the Lebesgue Density Theorem that $\xi$ is a 1-sided generator if and only if $||\xi^{(n)}|| \to 0$. This is equivalent to the unique proper digits property, and it holds for all three transformations under consideration.

The entropy of a finite partition $\xi$ is given by $H(\xi) = -\sum_{C \in \xi} \mu(C) \log(C)$. Note that $H(\xi) \leq \log(|\xi|)$. The entropy of $T$ with respect to $\xi$ is defined by $h_\mu(T, \xi) = \lim_{n\to\infty} \frac{1}{n} H(\xi^{(n)})$, and *entropy* of $T$ is defined by $h_\mu(T) = \sup_{H(\xi)<\infty} h_\mu(T, \xi)$. In practice, the supremum in the definition of entropy often makes it difficult to apply directly, but the Kolmogorov-Sinai theorem, says the supremum is achieved, $h_\mu(T) = h_\mu(T, \xi)$, provided $\xi$ is a (1- or 2-sided) generator.

In the case of an irrational rotation transformation $T$, we have $|\xi^{(n)}| = c_{\boldsymbol{d}}(n) =$

$n+1$ (for any $x$), so $H(\xi^{(n)}) \leq \log(n+1)$. Since $\xi$ is a (1-sided) generator for $T$ (by Theorem 2), the Kolmogorov-Sinai theorem implies $h_\mu(T) = 0$. When $\beta \in \mathbb{N}$, the Kolmogorov-Sinai theorem shows the $\beta$-transformation $T$ has $h_\mu(T) = \log \beta$. It is not so easy to apply this when $\beta \notin \mathbb{N}$, or to the Gauss map $T$. However, for a lot of $f$-transformations $T$, the entropy is given by *Rohlin's entropy formula*:

$$h_\mu(T) = \int_0^1 \log |T'(x)| \, d\mu. \tag{14}$$

In particular (14) gives the well known result $h_\mu(T) = \pi^2/(6 \log 2)$ for the the Gauss map (with Gauss measure) and gives $h_\mu(T) = \log \beta$ for all $\beta$-transformations $T$. Note that the entropy is positive in both of these cases.

The validity of Rohlin's formula (14) can be deduced under various hypotheses (see e.g., [16],[13]), which always seem, at least implicitly, to include Kakeya's hypothesis (9). This suggests that (14) is valid only in the case $h_\mu(T) > 0$ We note, however, that for Rohlin's entropy formula gives the correct answer $h_\mu(T) = 0$ for irrational rotation transformations $T$, if only by coincidence, since they satisfy $T'(x) \equiv 1$.

The fact that irrational rotation transformations $T$ have zero entropy contributes to the strangeness of Sturmian $\alpha$-expansions. The Kolmogorov-Sinai theorem shows that entropy zero comes from the low complexity $c_{\boldsymbol{d}}(n) = n+1$ of Sturmian sequences, and is thus directly related to the slow convergence Sturmian $\alpha$-expansions. Heuristically, most additional digits in a Sturmian $\alpha$-expansion contribute no new information about the number $x$.

Even more unusual is the fact that irrational rotation transformations $T$ are invertible, whereas $\beta$-transformations and the Gauss map are not. It follows from the invertibility that the Sturmian $\alpha$-expansion of any $x$ extends to a two-sided sequence

$$\boldsymbol{d} = \ldots d_{-2} d_{-1} d_0 . d_1 d_2 \ldots,$$

where $d_n = p_\xi(T^{n-1}x)$. Since the digits to the right of the "radix point" completely determine $x$, the digits to the left contribute no new information. Equivalently, a typical one-sided Sturmian sequence has a unique two-sided extension. The only exceptions to this (which are countable in number) occur when $x = n\alpha \bmod 1$ for $n > 1$. In such a case there are exactly two left-extensions that differ on exactly two adjacent digits.

For $\beta$-expansions, allowing finitely many (nonzero) digits to the left of the radix point gives an expansion of any $x \in \mathbb{R}$. In particular, the $\beta$-expansion of the digit sequence $\boldsymbol{d} = d_{-N} d_{-N+1} \ldots d_0 . d_1 d_2 \ldots$ is $x = \sum_{k=-N}^{\infty} d_k \beta^{-k}$. For continued fractions, expansions of all $x \in \mathbb{R}$ are obtained with a single non-zero digit to the left of the radix-point. The digits for the expansion

$$x = d_0 + \cfrac{1}{d_1 + \cfrac{1}{d_2 + \cfrac{1}{d_3 + \cdots}}}.$$

are usually written $\boldsymbol{d} = [d_0; d_1, d_2, d_3, \dots]$. In both cases, this works because the corresponding $f$-transformation $T$ is not invertible.

To interpret continued fraction and $\beta$-expansions with more non-zero digits to the left of the radix point, however, one needs to consider the natural extension of $T$. This is the smallest invertible measure-preserving transformation $\tilde{T}$ having $T$ as a factor. For example, if $Tx = 2x \bmod 1$ on $[0, 1)$ (the $f$-transformation for ordinary base 2 expansions), the natural extension is the Lebesgue measure-preserving map $\tilde{T} : [0, 1)^2 \to [0, 1)^2$, defined $\tilde{T}(x, y) = (2x \bmod 1, (\lfloor 2x \rfloor + y)/2)$. This Lebesgue measure-preserving mapping, called the *baker's transformation*, is isomorphic to the 2-sided Bernoulli shift with entropy $\log 2$. It is known that the natural extensions for any $\beta$-transformations is isomorphic to a Bernoulli shift, as is the natural extension of the Gauss map (see [3]).

Since an irrational rotation transformation $T$ is already invertible, it is its own natural extension. There is no new information to be obtained by an extension to a bijection. Entropy theory provides another way to understand this phenomenon. A well-known theorem says that any invertible map $T$ with a 1-sided generator (like the irrational rotation transformation) must have entropy zero (see [19]). Thus, no finite partition can be a 1-sided generator for any invertible transformation $\tilde{T}$ with positive entropy. It is easy to see that the partition $\tilde{\xi} = \{[0, 1/2) \times [0, 1), [1/2, 1) \times [0, 1)\}$ is a 2-sided generator for the baker's transformation $\tilde{T}$, since $\tilde{\xi}^{(-n,n)}$ is the partition of $[0, 1)^2$ into $2^{-n} \times 2^{-n}$ squares). But $\tilde{\xi}^{(n)}$ is the partition of $[0, 1)^2$ into $2^{-n} \times 1$ squares, and the factor corresponding to this partition is just $Tx = 2x \bmod 1$. Thus $\tilde{\xi}$ is not a 1-sided generator for $\tilde{T}$.

## 6. Generalizations

Let $\xi$ and $\xi'$ be partitions of $[0, 1)$ into finitely or countably many intervals of the form $\Delta = [a, b)$. Assume, moreover, that there is a nondecreasing function $p_\xi : [0, 1) \to \mathbb{Z}$ that is constant on each $\Delta \in \xi$, and is unequal on different $\Delta, \Delta' \in \xi$. The existence of such a function is automatic if $|\xi| = d < \infty$, in which case we usually take $\mathcal{D} := p_\xi([0, 1)) = \{0, 1, \dots, d - 1\}$. It is a more substantial restriction if $|\xi| = \infty$. In particular, the only limit points of the set of endpoints of $\Delta \in \xi$ can be 0 and 1 (and at least one must be a limit point).

Let $\mu$ denote Lebesgue measure, and suppose $\tau : \xi \to \xi'$ is such that $\mu(\tau(\Delta)) =$

$\mu(\Delta)$ for every $\Delta \in \xi$. Let $T : [0, 1) \to [0, 1)$ be the mapping so that $T$ maps each $\Delta \in \xi$ by translation to $\tau(\Delta)$. We call $T$ an *interval exchange transformation* (IET) if $|\xi| < \infty$, or an *infinite* interval exchange transformation (IIET) if $|\xi| = \infty$. In either case, $T$ preserves Lebesgue measure.

Let $F(x) = T(x) + p_\xi(x)$ and note that $F : [0, 1) \to \mathbb{R}$ is increasing, continuous on each $\Delta \in \xi$ and continuous from the right on $[0, 1)$. We define $f(x) = F^{-1}(x)$, extended to continuous non-decreasing $f : \mathbb{R} \to [0, 1)$ with $\overline{f(\mathbb{R})} = [0, 1]$.

Let $T$ be an IET or IIET. We call $a \in [0, 1)$ a *cut-point* of $\xi$ if it is the left endpoint of some $\Delta \in \xi$. Given a cutpoint $a$, we define $\xi_a$ to be the set of all intervals in $\Delta \in \xi$ so that $x < a$ for $x \in \Delta$. In particular, $\xi_a$ is a partition of $[0, a)$ into intervals. We say $T$ is *reducible* if there is an $a \in [0, 1)$ that is a cut-point for both $\xi$ and $\xi'$, and such that $\tau(\xi_a) = \xi'_a$. If there are no such $a$, we say $T$ is *irreducible.* If $T$ is reducible, $T([0, a)) = [0, a)$. Such a $T$ cannot be minimal or ergodic.

If $T$ is an irreducible IET, Keane [8] showed that $T$ is minimal if and only if the left endpoints of all the intervals $\Delta \in \xi$ have *infinite* and *distinct orbits* (this is abbreviated IDOC). He proved that if the lengths $\ell_0, \ell_1, \ldots \ell_{d-1}$ of the intervals in $\xi$ are rationally independent then IDOC follows. The case $|\xi| = 2$ is just an irrational rotation transformation $T$. If $T$ is an IET, we call the $f$-expansions IET-expansions.

**Proposition 5.** *If $T$ is an irreducible IET with rationally independent interval lengths (or that satisfies IDOC) then the corresponding IET-expansions are valid.*

The proof of Proposition 5 is almost exactly the same as the proof of Theorem 2. It depends on the fact that IDOC implies $||\xi^{(n)}|| \to 0$.

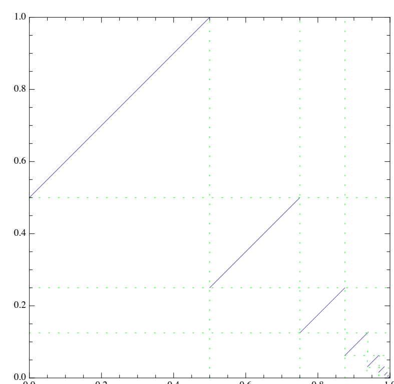


Figure 2: The von Neumann adding machine $T : [0, 1) \to [0, 1)$.

Unique ergodicity for an irreducible IET $T$ is a bit stronger (and more difficult to prove) than minimality, but (for appropriate choices of $\tau$) it holds for almost every choice of lengths $\ell_0, \ell_1, \ldots, \ell_{d-1}$ of the intervals in $\xi$ (see [18], [9]), as does weak (but never strong) mixing, (see [1]). The entropy of an IET $T$ is always zero. In summary, IET-expansions have many of the same properties as Sturmian $\alpha$-expansions, with at least one notable difference. An interval exchange transformation $T$ can be minimal but not uniquely ergodic. In such a case there will be non-normal numbers $x$ for the expansions, as well as up to $d$ different kinds of normal numbers (corresponding to, possibly, $d$ different ergodic invariant measures).

We conclude by considering expansions based on the well-known *von Neumann adding machine* (or *odometer*) $T$. Let $a_n = 1 - 1/2^n$, $b_n = 1/2^n$, $\xi = \{[a_n, a_{n+1}) : n = 0, 1, 2, \ldots\}$, $\xi' = \{[b_{n+1}, b_n) : n = 0, 1, 2, \ldots\}$, and $\tau([a_n, a_{n+1})) = [b_{n+1}, b_n)$. Let $T$ be the corresponding IIET (see Figure 2), and let $p_\xi([a_n, a_{n+1})) = n$, noting that $|\xi| = \infty$ and $\mathcal{D} = p_\xi([0,1)) = \mathbb{N} \cup \{0\}$. Define $f : \mathbb{R} \to [0,1)$ as the extension of $F^{-1}$, where $F(x) = T(x) + p_\xi(x)$, so that $\overline{f(\mathbb{R})} = [0,1]$ (see Figures 3 and 4). We call the corresponding $f$-expansions of $x \in [0,1)$ *von Neumann expansions*. The fact that von Neumann expansions are valid follows from the unique ergodicity of $T$, which is well known. In particular, the endpoints of the $\Delta \in \xi$ have dense orbits, and this can be used to show that $||\xi^{(n)}|| \to 0$. The entropy of $T$ is zero.

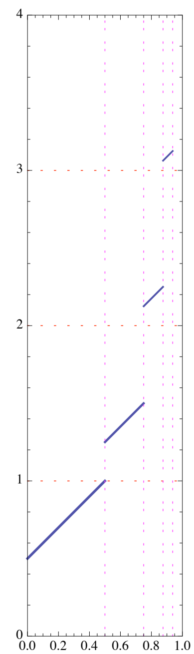


Figure 3: The function $F(x) = T(x) + p_\xi(x)$ for the von Neumann adding machine $T$.

To find the von Neumann expansion of $x \in [0,1)$, we first identify $x$ with its ordinary binary expansion, i.e., $x = .x_1x_2x_3\ldots$ means $x = \sum_{k=1}^{\infty} x_k 2^{-k}$. It is easy to see that

$$T(.x_1x_2x_3\ldots) = \begin{cases} .1x_2x_3x_4\ldots & \text{if } x_1 = 0, \\ .00\ldots01x_{n+1}x_{n+2} & \text{if } x_1x_2\ldots x_{n-1} = 11\ldots1 \text{ and } x_n = 0. \end{cases}$$

So $T$ adds .1 to $.x_1x_2x_3\ldots$ with right carry, which is why $T$ is called an "adding machine". Moreover, $p_\xi(.1^n0x_{n+2}x_{n+3}\ldots) = n$, where $n \geq 0$.

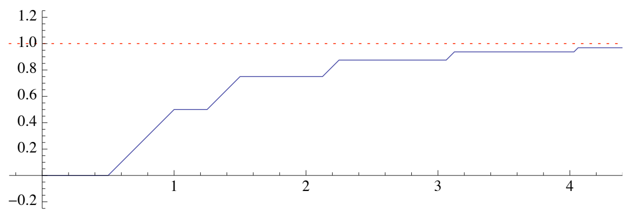


Figure 4: The function $f$ for von Neumann expansions.

As an example, if $x = 1/3 = .01010101\ldots$, then

$$\begin{array}{rll} x = & .010101010101010101010101\ldots & 0 \\ Tx = & .110101010101010101010101\ldots & 2 \\ T^2x = & .001101010101010101010101\ldots & 0 \\ T^3x = & .101101010101010101010101\ldots & 1 \\ T^4x = & .011101010101010101010101\ldots & 0 \\ T^5x = & .111101010101010101010101\ldots & 4 \\ T^6x = & .000011010101010101010101\ldots & 0 \\ T^7x = & .100011010101010101010101\ldots & 1 \\ \ldots & & \end{array}$$

where the numbers in the right column are $d_n$ for $n = 1, 2, 3, \ldots$. Thus we have the digit sequence $\boldsymbol{d} = [0, 2, 0, 1, 0, 4, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 6, \ldots]$.

Notice that in the list $x, Tx, T^2x\ldots$, the first column alternates 0 and 1, the second 00 and 11, third 0000 and 1111 (the first 0000 being truncated to 00), etc. Moreover, 0s in earlier columns mask 1s in later columns. This implies that in any von Neumann expansion, every 2nd digit in $\boldsymbol{d} = [d_1, d_2, d_3, \ldots]$ is a 0, every 4th digit is a 1, every 8th digit is a 2, ..., every $2^{n+1}$st digit an $n$. About $2^{n+1}$ digits of $\boldsymbol{d}$ are needed to determine $n$ binary digits of $x$. So like Sturmain $\alpha$-expansions, von Neumann expansions converge slowly.

As a final remark, we note that if we define $\boldsymbol{e} = .e_1e_2e_3\ldots$, by $e_n = d_n \bmod 2$, then the resulting sequence is a *Toeplitz sequence* (see [6]). For example, the Toeplitz sequence corresponding to $1/3$ is $\boldsymbol{c} = .00010001000101010001000\ldots$. Since it is possible to recover the von Neumann sequence from the Toeplitz sequence, the map $x \mapsto \boldsymbol{c}$ is injective. However, we don't know if it is possible to recover $x$ from $\boldsymbol{c}$ by a simple formula like an $f$-expansion.

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