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TRIANGULAR NUMBERS IN GEOMETRIC PROGRESSION

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Abstract

In [R. K. Guy, Unsolved Problems in Number Theory, 3rd ed. Springer Verlag, New York, 2004, D23], it is stated that Sierpinski asked the question of whether or not there exist four (distinct) triangular numbers in geometric progression. Szymiczek conjectured that the answer is negative. Recently M. A. Bennett [Integers: Electronic Journal of Combinatorial Number Theory **5(1)** (2005)] proved that there do not exist four distinct triangular numbers in geometric progression with the common ratio being a positive integer. In this paper we prove that there do not exist four distinct triangular numbers in geometric progression. Thus Sierpinski's question is answered and Szymiczek's conjecture is confirmed.

In [4, D23], it is stated that Sierpinski asked the question of whether or not there exist four (distinct) triangular numbers in geometric progression. Szymiczek conjectured that the answer is negative. Recall that a triangular number is one of the form $T_n = \frac{n(n+1)}{2}$ for $n \in \mathbb{N}$. The problem of finding three such triangular numbers is readily reduced to finding solutions to a Pell equation(whereby, an old result of Gerardin[3] (see also[2], [5]) implies that there are infinitely many such triples, the smallest of which is (T_1, T_3, T_8)). Recently M. A. Bennett[1] proved that there do not exist four distinct triangular numbers in geometric progression with the ratio being positive integer. In this paper, we extend Bennett's result to the rational common ratio and prove that there do not exist four distinct triangular numbers in geometric progression. Thus Sierpinski's question is answered and Szymiczek's conjecture is confirmed.

Theorem *There do not exist four distinct triangular numbers in geometric progression.*

Proof. Suppose that there exist four distinct triangular numbers T_{n_1}, T_{n_2}, T_{n_3}, T_{n_4} in geometric progression. Let q be the common ratio. It is obvious that $q > 0$ and $q \neq 1$. Without

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loss of generality, we may assume that $0 < q < 1$. Let $a = 8T_{n_1}$. Then

$$8T_{n_2} = aq, \quad 8T_{n_3} = aq^2, \quad 8T_{n_4} = aq^3.$$

Let $m_i = 2n_i + 1$ ($i = 1, 2, 3, 4$). Then

$$a + 1 = m_1^2, \quad aq + 1 = m_2^2, \quad aq^2 + 1 = m_3^2, \quad aq^3 + 1 = m_4^2. \tag{1}$$

Let

$$q = \frac{b_1}{a_1}, \quad a_1, b_1 \in \mathbb{Z}, \ (a_1, b_1) = 1, \ a_1 \geq 1.$$

Because aq^3 is positive integer, we have $a_1^3 \mid ab_1^3$. Noting that $(a_1, b_1) = 1$, we have $a_1^3 \mid a$. Let $a = a_1^3 a_0$, $a_0 \in \mathbb{N}$. By (1) we have

$$m_1^2 - a_1^3 a_0 = 1, \quad m_3^2 - b_1^2 a_1 a_0 = 1. \tag{2}$$

Because $a = m_1^2 - 1$ and $a = a_1^3 a_0 \in \mathbb{N}$, we have $a_1 a_0$ is not a perfect square.

Let $x_0 + y_0\sqrt{a_0 a_1}$ be the basic solution of Pell equation $x^2 - a_0 a_1 y^2 = 1$. Then by (2) and the theory of Pell equations, we have

$$m_1 + a_1\sqrt{a_0 a_1} = (x_0 + y_0\sqrt{a_0 a_1})^k,$$

$$m_3 + b_1\sqrt{a_0 a_1} = (x_0 + y_0\sqrt{a_0 a_1})^l.$$

where k, l are all positive integers. By $0 < q < 1$ and (1) we have $m_1 > m_3$ and $a_1 > b_1$. So $k > l \geq 1$.

If $k = 2$, then $m_1 + a_1\sqrt{a_0 a_1} = (x_0 + y_0\sqrt{a_0 a_1})^2$. Thus we have $a_1 = 2x_0 y_0$. So $x_0 \mid a_1$. Since $x_0^2 - a_0 a_1 y_0^2 = 1$, we have $x_0 = 1$, a contradiction with $x_0 + y_0\sqrt{a_0 a_1}$ being the basic solution of Pell equation $x^2 - a_0 a_1 y^2 = 1$. If $k \geq 3$, then $m_1 + a_1\sqrt{a_0 a_1} = (x_0 + y_0\sqrt{a_0 a_1})^3$. Thus $a_1 > \binom{k}{3} x_0^{k-3} a_1 a_0 y_0^3$, which is obviously impossible. $\square$

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