AVOIDING TYPE $(1,2)$ OR $(2,1)$ PATTERNS IN A PARTITION OF A SET

Toufik Mansour
Department of Mathematics, University of Haifa, Haifa, Israel
tmansour@univ.haifa.ac.il

Mark Shattuck
Department of Mathematics, University of Tennessee, Knoxville, Tennessee
shattuck@math.utk.edu

Received: 9/22/10, Accepted: 2/22/12, Published: 3/1/12

Abstract

A *partition* π of the set $[n] = \{1, 2, \ldots, n\}$ is a collection $\{B_1, \ldots, B_k\}$ of nonempty pairwise disjoint subsets of $[n]$ (called *blocks*) whose union equals $[n]$. In this paper, we find exact formulas and/or generating functions for the number of partitions of $[n]$ with k blocks, where k is fixed, which avoid 3-letter patterns of type $x - yz$ or $xy - z$, providing generalizations in several instances. In the particular cases of $23 - 1$, $22 - 1$, and $32 - 1$, we are only able to find recurrences and functional equations satisfied by the generating function, since in these cases there does not appear to be a simple explicit formula for it.

1. Introduction

A *partition* of $[n] = \{1, 2, \ldots, n\}$ is a decomposition of $[n]$ into nonempty pairwise disjoint subsets $B_1, B_2, \ldots, B_k$, called *blocks*, which are listed in increasing order of their least elements ($1 \leq k \leq n$). The set of all partitions of $[n]$ with exactly k blocks will be denoted by $P(n, k)$ and has cardinality given by $S(n, k)$, the well-known Stirling number of the second kind [16]. We will represent a partition $\Pi = B_1, B_2, \ldots, B_k$ in the *canonical sequential form* $\pi = \pi_1\pi_2 \cdots \pi_n$ such that $j \in B_{\pi_j}$, $1 \leq j \leq n$. From now on, we will identify each partition with its canonical sequential form. For example, if $\Pi = \{1, 4\}, \{2, 5, 7\}, \{3\}, \{6, 8\}$ is a partition of $[8]$, then its canonical sequential form is $\pi = 12312424$ and in such a case we write $\Pi = \pi$. Note that $\pi = \pi_1\pi_2 \cdots \pi_n \in P(n, k)$ is a *restricted growth function* from $[n]$ to $[k]$ (see, e.g., [13] for details), meaning that it satisfies (i) $\pi_1 = 1$, (ii) π is onto $[k]$, and (iii) $\pi_{i+1} \leq \max\{\pi_1, \pi_2, \ldots, \pi_i\} + 1$ for all i, $1 \leq i \leq n - 1$.

A *generalized subword pattern* τ is a (possibly hyphenated) word of $[\ell]^m$ which contains all of the letters in $[\ell]$. We say that a word $\sigma \in [k]^n$ *contains* a generalized subword pattern τ if σ contains a subsequence isomorphic to τ in which entries of σ corresponding to consecutive entries of τ not separated by a hyphen must be adjacent. Otherwise, we say that σ *avoids* τ. For example, a word $\sigma = a_1a_2\cdots a_n$ avoids the pattern $1-32$ if it has no subsequence $a_ja_ia_{i+1}$ with $j < i$ and $a_j < a_{i+1} < a_i$ and avoids the pattern $13-2$ if it has no subsequence $a_ia_{i+1}a_j$ with $j > i+1$ and $a_i < a_j < a_{i+1}$. Generalized patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the word were first introduced by Babson and Steingrímsson [1] in identifying Mahonian statistics on the symmetric group S_n.

Pattern avoidance is a classical problem in enumerative combinatorics. The first consideration of pattern avoidance began with that of permutations avoiding a pattern τ of length 3 with distinct letters and no adjacency requirements. Knuth [10] found that for any $\tau \in S_3$, there are C_n members of S_n which avoid τ, where C_n denotes the n-th Catalan number. Later, Simion and Schmidt [15] determined the number of elements of S_n avoiding the patterns in any subset of S_3, which was extended to words in $[k]^n$ by Burstein [2]. More recently, there has been comparable work done on pattern avoidance in set partitions, using various definitions for avoidance. The reader is referred to the papers by Klazar [9], Sagan [14], and Jelínek and Mansour [8] and to the references therein.

In this paper, we consider the problem of enumerating the partitions of $[n]$ with k blocks, where k is fixed, which avoid a single pattern of type $x-yz$ or of type $xy-z$, where partitions are construed as words in their canonical sequential forms. (We shall refer to 3-letter patterns of the forms $x-yz$ and $xy-z$ as type (1,2) and type (2,1), respectively.) This extends earlier work by Claesson and Mansour [5] on permutations, by Burstein and Mansour [3] on words in $[k]^n$, and by Heubach and Mansour [7] on compositions.

Given a generalized pattern τ, let $P_\tau(n,k)$ denote the subset of $P(n,k)$ which avoids τ, where k is fixed. If $a_n := |P_\tau(n,k)|$ for $n \geq 0$, then we define the generating function $F_\tau(x;k)$ by

$$F_\tau(x;k) = \sum_{n\geq 0} a_n x^n.$$

In the next two sections, we find exact formulas and/or generating functions for the cardinality of $P_\tau(n,k)$ for patterns τ of type (1,2) and type (2,1), providing generalizations in several instances. In the particular cases of $23-1$, $22-1$, and $32-1$, we were unable to find explicit formulas for $F_\tau(x;k)$, but were able to derive recurrences as well as functional equations satisfied by $F_\tau(x;k)$ through use of generating functions and generating trees, respectively.

2. Type (1,2) Patterns

We start with a theorem concerning a general class of patterns.

Theorem 1. *Let $\tau = \ell - \tau'$ be a generalized pattern with one dash such that τ' is a subword pattern over the alphabet $[\ell - 1]$. Then*

$$F_\tau(x;k) = \frac{x^k}{\prod_{j=1}^{\ell-1}(1-jx)} \prod_{j=\ell}^{k} \frac{W_{\tau'}(x;j-1)}{1 - xW_{\tau'}(x;j-1)},$$

where $W_{\tau'}(x;m)$ is the generating function for the number of words of length n over the alphabet $[m]$ that avoid the pattern τ'.

Proof. First note that each member π of $P(n,k)$ may be expressed as

$$\pi = 1\pi^{(1)}2\pi^{(2)}\cdots k\pi^{(k)},$$

where $\pi^{(j)}$ is a word over the alphabet $[j]$ which we decompose further as

$$\pi^{(j,1)}j\cdots\pi^{(j,s)}j\pi^{(j,s+1)},$$

where each $\pi^{(j,i)}$ is a word over the alphabet $[j-1]$. Thus, in terms of generating functions, we may write

$$F_\tau(x;k) = x^k \prod_{j=1}^{k} \frac{W_{\tau'}(x;j-1)}{1 - xW_{\tau'}(x;j-1)}.$$

From the definitions, we have $W_{\tau'}(x;j) = \frac{1}{1-jx}$ for all $j = 0, 1, \ldots, \ell - 2$, which completes the proof. □

For example, Theorem 1 for $\tau = 3-12$ or $\tau = 3-21$, together with $W_{12}(x;j) = W_{21}(x;j) = \frac{1}{(1-x)^j}$, yields an explicit formula in these cases.

Example 2. For each positive integer k,

$$\begin{aligned} F_{3-12}(x;k) = F_{3-21}(x;k) &= \frac{x^k}{(1-x)(1-2x)} \prod_{j=3}^{k} \frac{1/(1-x)^{j-1}}{1 - x/(1-x)^{j-1}} \\ &= x^k \prod_{j=1}^{k} \frac{1}{(1-x)^{j-1} - x}. \end{aligned}$$

Writing the $\pi^{(j,i)}$ in reverse order for all i and j shows directly that $F_{3-12}(x;k) = F_{3-21}(x;k)$, where the $\pi^{(j,i)}$ are as defined in the proof above. Also, Theorem 1 for $\tau = 2-11$ together with $W_{11}(x;j) = \frac{1+x}{1-(j-1)x}$ (see [3, Section 2]) yields

Example 3. For each positive integer k,

$$F_{2-11}(x;k) = x^k \prod_{j=1}^{k} \frac{1+x}{1-(j-1)x-x^2}.$$

Theorem 4. *Let $\tau = 1-1\cdots 1$ be a generalized pattern with one dash having length m, where $m \geq 3$. Then*

$$F_\tau(x;k) = x^k \prod_{j=1}^{k} \frac{1-x^{m-1}}{1-jx+(j-1)x^{m-1}}.$$

Proof. Let $a(n,j) := |P_\tau(n,j)|$ for $n, j \geq 0$. If $j \geq 2$, we have

$$a(n,j) = \sum_{i=1}^{m-1} a(n-i,j-1) + (j-1)\sum_{i=1}^{m-2} a(n-i,j), \quad n \geq j,$$

the first term counting all members of $P_\tau(n,j)$ where the letter j can be followed by no letter other than j, the second term counting those members of $P_\tau(n,j)$ ending in a run of exactly i letters of the same kind for some i, $1 \leq i \leq m-2$, and where the letter j is followed by a letter other than j on at least one occasion. Multiplying both sides of the above recurrence by x^n and summing over all $n \geq j$ implies

$$F_\tau(x;j) = \frac{F_\tau(x;j-1)\sum_{i=1}^{m-1} x^i}{1-(j-1)\sum_{i=1}^{m-2} x^i} = \frac{x(1-x^{m-1})F_\tau(x;j-1)}{1-jx+(j-1)x^{m-1}}, \quad j \geq 2,$$

and iterating this yields our result, upon noting the initial condition $F_\tau(x;1) = \frac{x(1-x^{m-1})}{1-x}$. □

When $m = 3$ in Theorem 4, we get

$$F_{1-11}(x;k) = x(1+x)^k \left(x^{k-1} \prod_{j=1}^{k-1} \frac{1}{1-jx} \right),$$

which implies the following explicit formula.

Corollary 5. *The number of partitions of $[n]$ with k blocks avoiding the subword pattern $1-11$ is given by*

$$\sum_{j=0}^{\min\{k,n-k\}} S(n-j-1,k-1)\binom{k}{j}.$$

We can provide a combinatorial explanation as follows. First note that the only places where a member of $P_{1-11}(n,k)$ can have a *level* (i.e., an occurrence of 11) are at the first appearances of letters. So to form a member of $P_{1-11}(n,k)$ having exactly j levels, first choose a member $\lambda \in P(n-j,k)$ having no levels, which as is well-known can be done in $S(n-j-1,k-1)$ ways (see, e.g., [12]), and then select a subset T of $[k]$ having cardinality j. Now insert a copy of the letter i just after the first appearance of i within λ for each $i \in T$ to obtain a member of $P_{1-11}(n,k)$ having exactly j levels. □

Theorem 6. *Let $\tau = 1{-}2\cdots 2$ be a generalized pattern with one dash having length m, where $m \geq 3$. Then*

$$F_\tau(x;k) = \frac{x^k}{1-x}\prod_{j=2}^{k}\frac{1-x^{m-2}}{1-jx+(j-2)x^{m-1}+x^m}.$$

Proof. Let $a(n,j) := |P_\tau(n,j)|$ for $n,j \geq 0$. If $j \geq 2$, we have

$$a(n,j) = \sum_{i=1}^{m-2} a(n-i,j-1) + (j-1)\sum_{i=1}^{m-2} a(n-i,j) + a(n-m+1,j), \quad n \geq j,$$

the first term counting all members of $P_\tau(n,j)$ where the letter j can be followed by no letter other than j, the second term counting those members of $P_\tau(n,j)$ ending in a run of exactly i letters of the same kind for some i, $1 \leq i \leq m-2$, and where the letter j is followed by a letter other than j on at least one occasion, and the third term counting those members of $P_\tau(n,j)$ ending in a run of the letter 1 having length at least $m-1$. Multiplying both sides of the above recurrence by x^n and summing over all $n \geq j$ implies

$$F_\tau(x;j) = \frac{F_\tau(x;j-1)\sum_{i=1}^{m-2}x^i}{1-x^{m-1}-(j-1)\sum_{i=1}^{m-2}x^i} = \frac{x(1-x^{m-2})F_\tau(x;j-1)}{1-jx+(j-2)x^{m-1}+x^m}, \quad j \geq 2,$$

and iterating this yields our result, upon noting the initial condition $F_\tau(x;1) = \frac{x}{1-x}$. □

Letting $m = 3$ in Theorem 6 gives

Example 7. For each positive integer k,

$$F_{1-22}(x;k) = \frac{x^k}{1-x}\prod_{j=2}^{k}\frac{1}{1-(j-1)x-x^2}.$$

Taking $k = 2$ in this implies that $P_{1-22}(n,2)$ has cardinality $f_n - 1$, where f_n denotes the n-th Fibonacci number with $f_0 = f_1 = 1$. This may be verified directly using the interpretation for f_n in terms of square-and-domino tilings of length n. For there are f_{n-2} members of $P_{1-22}(n,2)$ ending in a 2, upon noting that every 2 (except the last) must be directly followed by a 1, and $f_{n-1} - 1$ members of

$P_{1-22}(n,2)$ ending in a 1, upon noting that there must be at least one occurrence of a 2 followed directly by a 1 (which we treat as a domino, hence the all-square tiling of length $n-1$ is ruled out).

Proposition 8. *The number of partitions of* $[n]$ *with* k *blocks avoiding either* $1-21$ *or* $1-12$ *is given by* $\binom{n-1}{k-1}$.

Proof. First note that a member of $P_{1-21}(n,k)$ cannot have a descent, hence it must be increasing. A member of $P_{1-12}(n,k)$ must be of the form $12\cdots k\alpha$, where α is a word of length $n-k$ in the alphabet $[k]$ and decreasing, which implies that there are $\binom{(n-k)+k-1}{k-1} = \binom{n-1}{k-1}$ members of $P_{1-12}(n,k)$ as well. □

Proposition 9. *The number of partitions of* $[n]$ *with* k *blocks avoiding either* $2-21$ *or* $2-12$ *is given by* $\binom{n+\binom{k}{2}-1}{n-k}$.

Proof. Suppose $\pi \in P_{2-21}(n,k)$ is written in the form $1\pi_1 2\pi_2\cdots k\pi_k$, where each π_i is a word in the alphabet $[i]$ of length a_i and $a_1+a_2+\cdots+a_k = n-k$. Then each π_i must be increasing in order to avoid an occurrence of $2-21$. Thus, there are $\sum\prod_{i=1}^{k}\binom{a_i+i-1}{i-1}$ members of $P_{2-21}(n,k)$, where the sum is over all k-tuples $(a_1,a_2,\ldots,a_k)$ of non-negative integers having sum $n-k$. Note that this quantity is exactly the coefficient of x^{n-k} in the k-fold convolution $\prod_{i=1}^{k}\frac{1}{(1-x)^i} = \frac{1}{(1-x)^{\binom{k+1}{2}}}$, which is $\binom{n-k+\binom{k+1}{2}-1}{n-k} = \binom{n+\binom{k}{2}-1}{n-k}$. A similar argument applies to the pattern $2-12$, with the π_i now required to be decreasing. □

Theorem 10. *Let* $\tau = 1-32\cdots 2$ *be a generalized pattern of length* $m \geq 3$. *For each positive integer* k,

$$F_\tau(x;k) = x^k(1-x^{m-2})^{\binom{k-1}{2}}\prod_{j=1}^{k}\frac{x^{m-3}}{x^{m-3}-x^{m-2}-1+(1-x^{m-2})^{j-1}}.$$

Proof. Let $\tau' = 21\cdots 1$ be a subword pattern of length $m-1$. In [4, Section 2.1], Burstein and Mansour showed that the generating function for the number of words θ of length n over the alphabet $[k]$ such that θ avoids the subword τ' is given by

$$G_{\tau'}(x;k) = \frac{1}{1-x^{3-m}(1-(1-x^{m-2})^k)}.$$

They also showed that the generating function for the number of words θ of length n over the alphabet $[k]$ such that $(k+1)\theta$ avoids the subword τ' is given by

$$H_{\tau'}(x;k) = \frac{(1-x^{m-2})^k}{1-x^{3-m}(1-(1-x^{m-2})^k)}.$$

Since each word $k\theta$ over the alphabet $[k]$ can be decomposed as $k\theta^{(1)}k\theta^{(2)}k\cdots k\theta^{(s)}$, where $\theta^{(j)}$ is a word over the alphabet $[k-1]$, we have that the generating function

$H'_{\tau'}(x;k)$ for the number of words $k\theta$ of length n over the alphabet $[k]$ that avoid the subword τ' is given by

$$H'_{\tau'}(x;k) = \frac{xH_{\tau'}(x;k-1)}{1-xH_{\tau'}(x;k-1)} = \frac{x(1-x^{m-2})^{k-1}}{1-x^{3-m}(1-(1-x^{m-2})^k)}.$$

We will say that a word $\theta = \theta_1 \cdots \theta_n$ *almost avoids* τ' if there exists no i such that $\theta_i > \theta_{i+1} = \cdots = \theta_{i+m-2} > 1$. Now let us find the generating function $J(x;k)$ for the number of words $k\theta = k\theta_1 \cdots \theta_n$ of length n over the alphabet $[k]$ which almost avoid τ'. Since the word $k\theta$ can be written as $k\theta^{(1)}1\theta^{(2)}1\cdots 1\theta^{(d)}$, where $\theta^{(j)}$ is a word over the alphabet $\{2,3,\ldots,k\}$, we obtain

$$J(x;k) = \frac{H'_{\tau'}(x;k-1)}{1-xG_{\tau'}(x;k-1)},$$

for all $k \geq 2$. Suppose $\pi \in P_\tau(n,k)$ is written as $1\pi^{(1)}2\pi^{(2)}\cdots k\pi^{(k)}$, where each $\pi^{(j)}$ is a word over the alphabet $[j]$. Note π avoids τ if and only if $j\pi^{(j)}$ almost avoids τ' for all j. Hence, the generating function for the number of partitions $\pi \in P_\tau(x,k)$ is given by

$$\begin{aligned} F_\tau(x;k) &= \frac{x}{1-x}\prod_{j=2}^{k} J(x;j) = \frac{x}{1-x}\prod_{j=2}^{k}\frac{H'_{\tau'}(x;j-1)}{1-xG_{\tau'}(x;j-1)} \\ &= \frac{x}{1-x}\prod_{j=2}^{k}\frac{x^{m-2}(1-x^{m-2})^{j-2}}{x^{m-3}-x^{m-2}-1+(1-x^{m-2})^{j-1}}, \end{aligned}$$

which yields the desired result. □

Letting $m=3$ in Theorem 10 gives

Example 11. For each positive integer k,

$$F_{1-32}(x;k) = x^k(1-x)^{\binom{k-1}{2}}\prod_{j=1}^{k}\frac{1}{(1-x)^{j-1}-x}.$$

Theorem 12. *Let $\tau = 1-23\cdots 3$ be a generalized pattern of length $m \geq 3$. For each positive integer $k \geq 2$,*

$$F_\tau(x;k) = x^k(1-x^{m-3}+x^{m-2})^{k-2}\prod_{j=1}^{k}\frac{x^{m-3}}{x^{m-3}-x^{m-2}-1+(1-x^{m-2})^{j-1}},$$

with $F_\tau(x;1) = \frac{x}{1-x}$.

Proof. Let $\tau' = 12\cdots 2$ be a subword pattern of length $m-1$. In [4, Section 2.1], Burstein and Mansour showed that the generating function for the number of words θ of length n over the alphabet $[k]$ such that θ avoids the subword τ' is given by

$$G_{\tau'}(x;k) = \frac{1}{1 - x^{3-m}(1-(1-x^{m-2})^k)}.$$

We will say that a word $\theta = \theta_1 \cdots \theta_n$ *almost avoids* τ' if there exists no i such that $1 < \theta_i < \theta_{i+1} = \cdots = \theta_{i+m-2}$. Now let us find the generating function $H(x;k)$ for the number of words θ of length n over the alphabet $[k]$ that almost avoid τ'. Since the word θ can be written as $\theta^{(1)}1\theta^{(2)}1\cdots 1\theta^{(d)}$, where $\theta^{(j)}$ is a word over the alphabet $\{2,3,\ldots,k\}$, we obtain

$$H(x;k) = \frac{G_{\tau'}(x;k-1)}{1 - xG_{\tau'}(x;k-1)} = \frac{1}{1 - x - x^{3-m}(1-(1-x^2)^{k-1})}.$$

Suppose $\pi = \pi^{(0)}k^{a_1}\pi^{(1)}k^{a_2}\pi^{(2)}\cdots k^{a_s}\pi^{(s)} \in P_\tau(n,k)$, where $\pi^{(0)}$ does not contain the letter k, each a_i is a positive integer, and each $\pi^{(j)}$ is a nonempty word over the alphabet $[k-1]$ if $1 \le j < s$, with $\pi^{(s)}$ possibly empty. Since π starts with the letter 1, it avoids τ if and only if (1) $\pi^{(0)}$ is a partition with exactly $k-1$ blocks that avoids τ, (2) $\pi^{(j)}$ almost avoids τ' for all $j = 1,2,\ldots,s$, and (3) if $a_j > m-3$, then the rightmost letter of $\pi^{(j-1)}$ is 1 for all $j = 1,2,\ldots,s$. Hence, the generating function for the number of partitions $\pi \in P_\tau(x,k)$ is given by

$$F_\tau(x;k) = \frac{H(x;k-1)(\frac{x-x^{m-2}}{1-x} + \frac{x^{m-1}}{1-x})F_\tau(x;k-1)}{1 - \frac{x-x^{m-2}}{1-x}(H(x;k-1)-1) - \frac{x^{m-1}}{1-x}H(x;k-1)},$$

which, by using the expression above for $H(x;k)$, implies that

$$\begin{aligned} F_\tau(x;k) &= \frac{\frac{x-x^{m-2}+x^{m-1}}{1-x}F_\tau(x;k-1)}{1 - x - x^{3-m}(1-(1-x^{m-2})^{k-2}) - \frac{(x-x^{m-2})\left(1-\frac{1}{H(x;k-1)}\right)+x^{m-1}}{1-x}} \\ &= \frac{x^{m-2}(1-x^{m-3}+x^{m-2})F_\tau(x;k-1)}{x^{m-3}-x^{m-2}-1+(1-x^{m-2})^{k-1}}, \end{aligned}$$

for all $k \ge 3$. Iterating the above recurrence relation and using the fact that $F_\tau(x;2) = \frac{x^2}{(1-x)(1-2x)}$ yields the desired result. □

Letting $m = 3$ in Theorem 12 gives

Example 13. For each positive integer $k \ge 2$,

$$F_{1-23}(x;k) = x^{2k-2}\prod_{j=1}^{k}\frac{1}{(1-x)^{j-1} - x},$$

with $F_{1-23}(x;1) = \frac{x}{1-x}$.

Comparing Examples 13 and 2, we see that the cardinality of $P_{1-23}(n+k-2,k)$ is the same as that for $P_{3-21}(n,k)$ for $k \geq 2$. For a direct bijection, first write $\pi \in P_{1-23}(n+k-2,k)$ as $1\pi^{(1)}2\pi^{(2)}(13)\pi^{(3)}\cdots(1k)\pi^{(k)}$, where a 1 directly precedes the first occurrence of each $i \in \{3,4,\ldots,k\}$ and $\pi^{(i)}$ is a word in the alphabet $[i]$. Then remove each of these $k-2$ 1's and replace each word $\pi^{(i)} = a_1a_2\cdots a_r$ with the word $\hat{\pi}^{(i)} = (i+1-a_1)(i+1-a_2)\cdots(i+1-a_r)$ to obtain $\hat{\pi} = 1\hat{\pi}^{(1)}2\hat{\pi}^{(2)}3\hat{\pi}^{(3)}\cdots k\hat{\pi}^{(k)}$ belonging to $P_{3-21}(n,k)$, as may be verified.

The results of this section are summarized in Table 1 below.

τ	Reference	τ	Reference	τ	Reference
$1-11$	Corollary 5	$1-32$	Example 11	$2-21$	Proposition 9
$1-12$	Proposition 8	$2-11$	Example 3	$2-31$	[6]
$1-21$	Proposition 8	$2-12$	Proposition 9	$3-12$	Example 2
$1-22$	Example 7	$2-13$	[11]	$3-21$	Example 2
$1-23$	Example 13				

Table 1: Three letter generalized patterns of type $(1,2)$

3. Type (2,1) Patterns

We open with a general theorem which follows immediately from the fact that each partition π with exactly k blocks may be uniquely expressed as $\pi' kw$ where π' is a partition with exactly $k-1$ blocks and w is a word over the alphabet $[k]$.

Theorem 14. *Let $\tau = \tau' - \ell$ be a generalized pattern with one dash such that τ' is a subword pattern over the alphabet $[\ell-1]$. Then*

$$F_\tau(x;k) = xW_\tau(x;k)F_{\tau'}(x;k-1),$$

where $W_\tau(x;r)$ is the generating function for the number of words of length n over the alphabet $[r]$ which avoid the pattern τ.

The following formula for $F_{12-3}(x;k)$ follows from taking $\tau = 12-3$ in Theorem 14 and is also obvious from the definitions.

Example 15. $F_{12-3}(x;1) = \frac{x}{1-x}$, $F_{12-3}(x;2) = \frac{x^2}{(1-x)(1-2x)}$, and $F_{12-3}(x;k) = 0$ for all $k \geq 3$.

Taking $\tau = 21-3$ in the prior theorem together with

$$W_{21-3}(x;k) = \prod_{j=1}^{k} \frac{(1-x)^{j-1}}{(1-x)^{j-1}-x}$$

(see Theorem 3.6 of [3]) and $F_{21}(x;k-1) = \frac{x^{k-1}}{(1-x)^{k-1}}$ yields

Example 16. For each positive integer k,

$$F_{21-3}(x;k) = \frac{x^k}{1-x} \prod_{j=2}^{k} \frac{(1-x)^{j-2}}{(1-x)^{j-1}-x}.$$

Taking $\tau = 11-2$ in the prior theorem together with

$$W_{11-2}(x;k) = \prod_{j=0}^{k-1} \frac{1-(j-1)x}{1-jx-x^2}$$

(see Theorem 3.2 of [3]) and

$$F_{11}(x;k-1) = \frac{x^{k-1}}{\prod_{j=1}^{k-2}(1-jx)}$$

(see, e.g., [12]) yields

Example 17. For each positive integer k,

$$F_{11-2}(x;k) = x^k(1+x) \prod_{j=0}^{k-1} \frac{1}{1-jx-x^2}.$$

Letting $k=2$ in the last formula, we see that $P_{11-2}(n,2)$ has cardinality

$$\sum_{j=0}^{n-2} f_j = f_n - 1,$$

where f_m denotes the m-th Fibonacci number. Note that the sum counts all members of $P_{11-2}(n,2)$ according to the number, $n-2-j$, of trailing 1's.

Theorem 18. *For each positive integer k,*

$$F_{1\cdots1-1}(x;k) = F_{1-1\cdots1}(x;k),$$

where the generalized patterns have the same length $m \geq 3$.

Proof. We first express a member $\lambda \in P_{1-1\cdots1}(n,k)$ as $\lambda = x_1w_1x_2w_2\cdots x_rw_r$, where each x_i is a (maximal) sequence of consecutive 1's and each w_i is a word in the alphabet $[k]-\{1\}$ (with w_r possibly empty). Let $\lambda_1 = x_rw_1x_{r-1}w_2\cdots x_1w_r$ be the partition gotten by reversing the order of the runs of consecutive 1's. Now reverse the order of the runs of consecutive 2's in λ_1, and, likewise, reverse the order of the runs for each subsequent member of $[k]$. The resulting partition α will belong to $P_{1\cdots1-1}(n,k)$ since all runs (except possibly the last) of a given letter will have length at most $m-2$, and the mapping $\lambda \mapsto \alpha$ is readily seen to be a bijection. For example, if $n=12$, $k=3$, $m=3$, and $\lambda = 111213323132 \in P_{1-11}(12,3)$, then $\alpha = 121323111332 \in P_{11-1}(12,3)$. $\square$

The bijection of the preceding proof shows, more generally, that the statistics recording the number of occurrences of $1\cdots 1-1$ and $1-1\cdots 1$ are identically distributed on $P(n,k)$. We next look at the patterns $12-1$ and $12-2$.

Proposition 19. *The number of partitions of* $[n]$ *with* k *blocks avoiding either* $12-1$ *or* $12-2$ *is given by* $\binom{n-1}{k-1}$.

Proof. A member of $P_{12-1}(n,k)$ cannot have a descent since the first descent in a partition always produces an occurrence of $12-1$. If a member of $P_{12-2}(n,k)$ is expressed in the form $1\pi_1 2\pi_2\cdots k\pi_k$, where each π_i is a (possibly empty) word in the alphabet $[i]$, then it must be the case that each π_i consists only of 1's, whence there are $\binom{(n-k)+k-1}{k-1}=\binom{n-1}{k-1}$ members of $P_{12-2}(n,k)$ in all. To see this, note first that π_2 in the above decomposition must consist only of 1's in order to avoid an occurrence of $12-2$. This in turn implies that π_3 must consist of all 1's, for if it had a 2 or a 3, then there would be an occurrence of $12-2$ when taken with the first occurrence of 2 or 3. And so on, inductively, each π_i must consist only of 1's. □

Theorem 20. *The number of partitions of* $[n]$ *with* k *blocks avoiding* $21-2$ *is given by* $\binom{n+\binom{k}{2}-1}{n-k}$.

Proof. Within a partition, we'll call a group of consecutive letters of the same type i an *i-block*. First note that within any member of $P_{21-2}(n,k)$, all of the letters k must occur within a single block. The letters $k-1$ either occur as a single block to the left of the k-block or occur as two blocks, one to the left and another to the right of the k-block. In general, suppose one has chosen the relative positions of the j blocks for each $j\in\{2,3,\ldots,k\}$ and one wishes to insert 1-blocks. One must place a 1-block at the beginning. Once this is done, you may then place a 1-block directly to the right of the *last* j-block for any $j>1$.

Let $z=\frac{x}{1-x}$. Upon conditioning on the number, i, of 1-blocks to be inserted following j-blocks for $j\in\{2,3,\ldots,k\}$ (note that for each of the $\binom{k-1}{i}$ choices regarding the insertion sites for the i additional 1-blocks, there is a contribution of z^{i+1} towards the generating function), we see that

$$\begin{aligned} F_{21-2}(x;k) &= \left(z\sum_{i=0}^{k-1}\binom{k-1}{i}z^i\right)F_{21-2}(x;k-1)\\ &= z(1+z)^{k-1}F_{21-2}(x;k-1), \qquad\qquad k\geq 2. \end{aligned}$$

Iterating this and noting the initial condition $F_{21-2}(x;1)=z$, we get

$$F_{21-2}(x;k)=z^k(1+z)^{\binom{k}{2}}=\frac{x^k}{(1-x)^{\binom{k+1}{2}}},$$

which implies that $P_{21-2}(n,k)$ has cardinality $\binom{n-k+\binom{k+1}{2}-1}{n-k}=\binom{n+\binom{k}{2}-1}{n-k}$. □

Theorem 21. *The number of partitions of* $[n]$ *with* k *blocks avoiding* $21-1$ *is given by* $\binom{n+\binom{k}{2}-1}{n-k}$.

Proof. We construct an explicit bijection between the set $P_{21-2}(n,k)$ and the set $P_{21-1}(n,k)$ and use the prior theorem. Suppose $\lambda = \lambda_1 = a_1a_2\cdots a_n \in P_{21-2}(n,k)$ and that the first (maximal) descent occurs at index i and has length t, where $t \geq 2$. Then there are no occurrences of the letters $a_i, a_{i+1}, \ldots, a_{i+t-2}$ past position $i+t-1$. Replace any occurrences of the letter a_{i+t-1} past position $i+t-1$ with the letter a_i, letting λ_2 denote the resulting member of $P(n,k)$. (By maximal descent, we mean a descent which is not strictly contained within any other.)

Now suppose the *second* (maximal) descent of λ_2 occurs at index k and has length s (note that the first maximal descent of λ_2 occurs at index i, since the first $i+t-1$ positions of λ_1 were not changed by the above replacement, and hence $k > i+t-1$). Then replace any occurrences of the letter a_{k+s-1} past the $(k+s-1)$-st position with the letter a_{k+s-2}. Then look at the new partition and replace any occurrences of the letter a_{k+s-2} past the $(k+s-1)$-st position with the letter a_{k+s-3} and so on until one makes the replacement of a_k for a_{k+1} in all positions past the $(k+s-1)$-st position. (Note that the second and subsequent steps must be defined in this fashion since, for instance, the element a_i might be part of the second maximal descent of λ_2 and therefore create a $21-1$ that needs to be removed). Similarly, define λ_3 using λ_2, and so on. After a finite number of steps, say m, all occurrences of $21-1$ will have been removed and thus the resulting partition λ_m belongs to $P_{21-1}(n,k)$. The mapping $\lambda \mapsto \lambda_m$ may be reversed by starting on the right and considering the $(m-1)$-st (maximal) descent in the partition λ_m, the $(m-2)$-nd (maximal) descent in λ_{m-1}, and so on (note that, by construction, the partition λ_i has at least $i-1$ maximal descents for all i). □

In what follows, we let $F_\tau(x;k|a_s\cdots a_1)$ denote the generating function for the number of partitions $\pi = \pi_1\pi_2\cdots\pi_n$ of $[n]$ with exactly k blocks that avoid the pattern τ and have $\pi_{n+1-j} = a_j$ for all $j = 1,2,\ldots,s$. The next proposition gives a recursive way of finding the generating functions $F_{23-1}(x;k|a)$ for all positive integers a and k.

Proposition 22. *The generating function* $F_{23-1}(x;k)$ *is given by*

$$\sum_{a=1}^{k} F_{23-1}(x;k|a),$$

where the generating functions $F_{23-1}(x;k|a)$ *satisfy the recurrence relation*

$$F_{23-1}(x;k|a) = \frac{x}{(1-x)^{k-a}-x}\sum_{j=1}^{a-1} F_{23-1}(x;k|j) + \frac{x^2}{(1-x)^{k-a}-x}\sum_{j=1}^{a} F_{23-1}(x;k-1|j),$$

for $1 \le a \le k-2$, *with the initial conditions*

$$F_{23-1}(x;k|k) = xF_{23-1}(x;k) + xF_{23-1}(x;k-1)$$

and

$$F_{23-1}(x;k|k-1) = xF_{23-1}(x;k)$$

if $k \ge 3$, *where* $F_{23-1}(x;2|1) = \frac{x^3}{(1-x)(1-2x)}$ *and* $F_{23-1}(x;2|2) = \frac{x^2}{1-2x}$.

Proof. From the definitions, the initial conditions are obvious and $F_{23-1}(x;k) = \sum_{a=1}^{k} F_{23-1}(x;k|a)$. Also, for all $1 \le a \le k-2$,

$$\begin{aligned} &F_{23-1}(x;k|a) \\ &\quad = \sum_{b=1}^{a} F_{23-1}(x;k|b,a) + F_{23-1}(x;k|a+1,a) + \sum_{b=a+2}^{k} F_{23-1}(x;k|b,a) \\ &\quad = x\sum_{j=1}^{a} F_{23-1}(x;k|j) + xF_{23-1}(x;k|a) + \sum_{j=a+2}^{k} F_{23-1}(x;k|j,a). \end{aligned} \tag{1}$$

We now find a formula for $F_{23-1}(x;k|b,a)$, where $k > b > a+1 \ge 2$:

$$\begin{aligned} &F_{23-1}(x;k|b,a) \\ &\quad = \sum_{j=1}^{k} F_{23-1}(x;k|j,b,a) \\ &\quad = \sum_{j=1}^{a} F_{23-1}(x;k|j,b,a) + \sum_{j=a+1}^{b-1} F_{23-1}(x;k|j,b,a) + \sum_{j=b}^{k} F_{23-1}(x;k|j,b,a) \\ &\quad = x^2\sum_{j=1}^{a} F_{23-1}(x;k|j) + x\sum_{j=b}^{k} F_{23-1}(x;k|j,a), \end{aligned} \tag{2}$$

with

$$\begin{aligned} &F_{23-1}(x;k|k,a) \\ &\quad = \sum_{j=1}^{a} F_{23-1}(x;k|j,k,a) + F_{23-1}(x;k|k,k,a) \\ &\quad = x^2\sum_{j=1}^{a} F_{23-1}(x;k|j) + x^2\sum_{j=1}^{a} F_{23-1}(x;k-1|j) + xF_{23-1}(x;k|k,a). \end{aligned} \tag{3}$$

Equations (1)-(2) imply

$$F_{23-1}(x;k|b,a) - xF_{23-1}(x;k|a) = -x^2F_{23-1}(x;k|a) - x\sum_{j=a+2}^{b-1} F_{23-1}(x;k|j,a),$$

with the initial condition $F_{23-1}(x;k|a+1,a) = xF_{23-1}(x;k|a)$ (which is obvious from the definitions). Thus, by induction on b, we obtain

$$F_{23-1}(x;k|b,a) = x(1-x)^{b-a-1}F_{23-1}(x;k|a), \quad b = a+1, a+2, \ldots, k-1. \tag{4}$$

Therefore, equations (1), (3) and (4) imply

$$\begin{aligned} F_{23-1}(x;k|a) &= x\sum_{j=1}^{a} F_{23-1}(x;k|j) + xF_{23-1}(x;k|a)\sum_{b=a+1}^{k-1}(1-x)^{b-a-1} \\ &+\tfrac{x^2}{1-x}\sum_{j=1}^{a}(F_{23-1}(x;k|j)+F_{23-1}(x;k-1|j)), \end{aligned}$$

which is equivalent to

$$\begin{aligned} &F_{23-1}(x;k|a) \\ &= \frac{x}{(1-x)^{k-a}-x}\sum_{j=1}^{a-1}F_{23-1}(x;k|j) + \frac{x^2}{(1-x)^{k-a}-x}\sum_{j=1}^{a}F_{23-1}(x;k-1|j), \end{aligned}$$

as claimed. □

Note that $F_{23-1}(x;k|1)$ is determined by the recurrence in Proposition 3.6 above for all $k \geq 3$ by simple iteration. Comparing with Example 2.6 above, we see that $F_{23-1}(x;k|1) = xF_{1-23}(x;k)$ for all $k \geq 2$, which is easily seen directly. These values determine the $F_{23-1}(x;k|2)$ for all k, which may then be used to determine the $F_{23-1}(x;k|3)$ and so on.

Remark. By induction on a, there is also the following direct relation between $F_{23-1}(x;k|a)$ and $F_{23-1}(x;k-1|a)$ for all $k \geq 3$ and $1 \leq a \leq k-2$ which results from Proposition 22:

$$\begin{aligned} &F_{23-1}(x;k|a) \\ &= \sum_{i=1}^{a-1}\left(\frac{x^2\prod_{j=i+1}^{a-1}\left(1+\frac{x^2}{(1-x)^{k-j}-x}\right)}{(1-x)^{k-a}-x}\frac{x}{(1-x)^{k-i}-x}\sum_{j=1}^{i}F_{23-1}(x;k-1|j)\right) \\ &\quad + \left(\frac{x^2}{(1-x)^{k-a}-x}\sum_{j=1}^{a}F_{23-1}(x;k-1|j)\right). \end{aligned}$$

The above proposition may also be used to find an explicit formula for the generating function $F_{23-1}(x;k)$ for any given k. For $k=2$, we have $F_{23-1}(x;2|1) = \frac{x^3}{(1-x)(1-2x)}$, $F_{23-1}(x;2|2) = \frac{x^2}{1-2x}$ and $F_{23-1}(x;2) = \frac{x^2}{(1-x)(1-2x)}$. For $k=3$, we have

$$\begin{aligned} &F_{23-1}(x;3|1) = \frac{x^2}{1-3x+x^2}F_{23-1}(x;2|1) = \frac{x^5}{(1-x)(1-2x)(1-3x+x^2)}, \\ &F_{23-1}(x;3|2) = xF_{23-1}(x;3), \\ &F_{23-1}(x;3|3) = xF_{23-1}(x;3) + \frac{x^3}{(1-x)(1-2x)}. \end{aligned}$$

By summing the above equations, we obtain $(1-2x)F_{23-1}(x;3) = \frac{x^3}{1-3x+x^2}$, which implies

$$F_{23-1}(x;3) = \frac{x^3}{(1-3x+x^2)(1-2x)}.$$

Likewise, one has

$$F_{23-1}(x;4) = \frac{x^4(1-5x+8x^2-4x^3+x^4)}{(1-x)(1-2x)^2(1-3x+x^2)(1-4x+3x^2-x^3)}.$$

Similarly, there are comparable recurrences satisfied by the generating functions $F_{22-1}(x;k|a)$.

Proposition 23. *The generating function $F_{22-1}(x;k)$ is given by*

$$\sum_{a=1}^{k} F_{22-1}(x;k|a),$$

where the generating functions $F_{22-1}(x;k|a)$ satisfy the recurrence relation

$$\begin{aligned} &F_{22-1}(x;k|a)\\ &= \frac{x(1+x)}{1-(k-a)x-x^2}\sum_{j=1}^{a-1} F_{22-1}(x;k|j) + \frac{x}{1-(k-a)x-x^2}F_{22-1}(x;k-1|a), \end{aligned}$$

for all $1 \le a \le k-2$, with the initial conditions

$$F_{22-1}(x;k|k) = xF_{22-1}(x;k) + xF_{22-1}(x;k-1)$$

and

$$F_{22-1}(x;k|k-1) = x(F_{22-1}(x;k) - x^2F_{22-1}(x;k) - x^2F_{22-1}(x;k-1))$$

if $k \ge 2$, where $F_{22-1}(x;1|1) = F_{22-1}(x;1) = \frac{x}{1-x}$.

Proof. From the definitions, the first initial condition is obvious and

$$F_{22-1}(x;k) = \sum_{a=1}^{k} F_{22-1}(x;k|a).$$

For the second condition, note that all members of $P_{22-1}(n,k)$ ending in the letter $k-1$ may be obtained from the members of $P_{22-1}(n-1,k)$ not ending in two or more k's by adding a $k-1$. Also, for all $1 \le a \le k-2$,

$$\begin{aligned} F_{22-1}(x;k|a) &= \sum_{b=1}^{a} F_{22-1}(x;k|b,a) + \sum_{b=a+1}^{k} F_{22-1}(x;k|b,a)\\ &= x\sum_{j=1}^{a} F_{22-1}(x;k|j) + \sum_{j=a+1}^{k} F_{22-1}(x;k|j,a). \end{aligned} \tag{5}$$

We now find a formula for $F_{22-1}(x;k|b,a)$, with $k>b>a\geq 1$:

$$\begin{aligned} F_{22-1}(x;k|b,a) &= \sum_{j=1}^{k} F_{22-1}(x;k|j,b,a) \\ &= \sum_{j=1}^{a} F_{22-1}(x;k|j,b,a) + \sum_{j=a+1, j\neq b}^{k} F_{22-1}(x;k|j,b,a) \\ &= x^2 \sum_{j=1}^{a} F_{22-1}(x;k|j) + x \sum_{j=a+1, j\neq b}^{k} F_{22-1}(x;k|j,a), \end{aligned} \tag{6}$$

which, by (5), implies

$$F_{22-1}(x;k|b,a) = \frac{x}{1+x} F_{22-1}(x;k|a). \tag{7}$$

Also, $F_{22-1}(x;k|k,a) = xF_{22-1}(x;k-1|a)+xF_{22-1}(x;k|a)-xF_{22-1}(x;k|k,a)$, which gives

$$F_{22-1}(x;k|k,a) = \frac{x}{1+x} F_{22-1}(x;k-1|a) + \frac{x}{1+x} F_{22-1}(x;k|a). \tag{8}$$

Therefore, equations (5), (7) and (8) imply

$$F_{22-1}(x;k|a) = x\sum_{j=1}^{a} F_{22-1}(x;k|j) + \frac{(k-a)x}{1+x} F_{22-1}(x;k|a) + \frac{x}{1+x} F_{22-1}(x;k-1|a),$$

for all $1\leq a\leq k-2$, which gives

$$\begin{aligned} &F_{22-1}(x;k|a) \\ &= \frac{x(1+x)}{1-(k-a)x-x^2} \sum_{j=1}^{a-1} F_{22-1}(x;k|j) + \frac{x}{1-(k-a)x-x^2} F_{22-1}(x;k-1|a), \end{aligned}$$

as claimed. □

Remark. By induction on a, there is the following direct relation between the $F_{22-1}(x;k|a)$ and the $F_{22-1}(x;k-1|a)$ for all $k\geq 3$ and $1\leq a\leq k-2$ which results from Proposition 23:

$$\begin{aligned} F_{22-1}(x;k|a) = (1+x)x^2 \sum_{i=1}^{a-1} \left(\frac{\prod_{j=i+1}^{a-1}(1-(k-1-j)x)}{\prod_{j=i}^{a}(1-(k-j)x-x^2)} F_{22-1}(x;k-1|i) \right) \\ + \frac{x}{1-(k-a)x-x^2} F_{22-1}(x;k-1|a). \end{aligned}$$

The above proposition may also be used to find an explicit formula for the generating function $F_{22-1}(x;k)$ for any given k. For instance, the above proposition for $k=2$ yields

$$\begin{aligned} F_{22-1}(x;2|1) &= (x-x^3)F_{22-1}(x;2) - \frac{x^4}{1-x}, \\ F_{22-1}(x;2|2) &= xF_{22-1}(x;2) + \frac{x^2}{1-x}. \end{aligned}$$

By summing the above equations, we get

$$F_{22-1}(x;2) = (2x - x^3)F_{22-1}(x;2) + x^2(1+x),$$

which implies

$$F_{22-1}(x;2) = \frac{x^2(1+x)}{(1-x)(1-x-x^2)}.$$

Likewise, we have

$$F_{22-1}(x;3) = \frac{x^3(1+x-2x^2-x^3)}{(1-x)^2(1-x-x^2)(1-2x-x^2)}.$$

Finally, there are similar relations involving the generating functions $F_{32-1}(x;k|a)$.

Proposition 24. *The generating function* $F_{32-1}(x;k)$ *is given by*

$$\sum_{a=1}^{k} F_{32-1}(x;k|a),$$

where the generating functions $F_{32-1}(x;k|a)$ *satisfy the recurrence relation*

$$\begin{aligned} &F_{32-1}(x;k|a) \\ &= \frac{x}{(1-x)^{k-a}-x}\sum_{j=1}^{a-1} F_{32-1}(x;k|j) + \frac{x(1-x)^{k-1-a}}{(1-x)^{k-a}-x}F_{32-1}(x;k-1|a), \end{aligned}$$

for all $1 \le a \le k-2$, *with the initial conditions*

$$F_{32-1}(x;k|k) = xF_{32-1}(x;k) + xF_{32-1}(x;k-1)$$

and

$$F_{32-1}(x;k|k-1) = xF_{32-1}(x;k)$$

if $k \ge 3$, *where* $F_{32-1}(x;2|1) = \frac{x^3}{(1-x)(1-2x)}$ *and* $F_{32-1}(x;2|2) = \frac{x^2}{1-2x}$.

Proof. From the definitions, the initial conditions are obvious and $F_{32-1}(x;k) = \sum_{a=1}^{k} F_{32-1}(x;k|a)$. Also, for all $1 \le a \le k-2$,

$$\begin{aligned} F_{32-1}(x;k|a) &= \sum_{b=1}^{a} F_{32-1}(x;k|b,a) + \sum_{b=a+1}^{k-1} F_{32-1}(x;k|b,a) + F_{32-1}(x;k|k,a) \\ &= x\sum_{j=1}^{a} F_{32-1}(x;k|j) + \sum_{b=a+1}^{k-1} F_{32-1}(x;k|b,a) + xF_{32-1}(x;k|a) \\ &\quad + xF_{32-1}(x;k-1|a). \end{aligned} \tag{9}$$

We now find a formula for $F_{32-1}(x;k|b,a)$, with $k>b>a\geq 1$:

$$\begin{aligned}F_{32-1}(x;k|b,a) &= \sum_{j=1}^{k} F_{32-1}(x;k|j,b,a)\\ &= \sum_{j=1}^{a} F_{32-1}(x;k|j,b,a)+\sum_{j=a+1}^{b} F_{32-1}(x;k|j,b,a)\\ &= x^2\sum_{j=1}^{a} F_{32-1}(x;k|j)+x\sum_{j=a+1}^{b} F_{32-1}(x;k|j,a),\end{aligned} \tag{10}$$

where

$$F_{32-1}(x;k|k,a)=xF_{32-1}(x;k|a)+xF_{32-1}(x;k-1|a). \tag{11}$$

Equations (9)-(10) imply

$$F_{32-1}(x;k|b,a)-xF_{32-1}(x;k|a)=-x\sum_{j=b+1}^{k} F_{32-1}(x;k|j,a),$$

with the initial condition (11). Thus, induction on b yields

$$F_{32-1}(x;k|b,a)=x(1-x)^{k-b}F_{32-1}(x;k|a)-x^2(1-x)^{k-1-b}F_{32-1}(x;k-1|a), \tag{12}$$

for $a+1\leq b\leq k-1$. Therefore, equations (9), (11) and (12) imply

$$\begin{aligned}F_{32-1}(x;k|a) &= x\sum_{j=1}^{a} F_{32-1}(x;k|j)+xF_{32-1}(x;k|a)\sum_{b=a+1}^{k}(1-x)^{k-b}\\ &\quad +x\left(1-x\sum_{b=a+1}^{k-1}(1-x)^{k-1-b}\right)F_{32-1}(x;k-1|a),\end{aligned}$$

which is equivalent to

$$F_{32-1}(x;k|a)=\frac{x}{(1-x)^{k-a}-x}\sum_{j=1}^{a-1}F_{32-1}(x;k|j)+\frac{x(1-x)^{k-1-a}}{(1-x)^{k-a}-x}F_{32-1}(x;k-1|a),$$

for all $1\leq a\leq k-2$, as claimed. □

Remark. By induction on a, there is the following direct relation between $F_{32-1}(x;k|a)$ and $F_{32-1}(x;k-1|a)$ for all $k\geq 3$ and $1\leq a\leq k-2$ which results from Proposition 24:

$$\begin{aligned}F_{32-1}(x;k|a) &= x^2\sum_{i=1}^{a-1}\left(\frac{(1-x)^{k-1-i}\prod_{j=i+1}^{a-1}\frac{(1-x)^{k-j}}{(1-x)^{k-j}-x}}{((1-x)^{k-a}-x)((1-x)^{k-i}-x)}\right)F_{32-1}(x;k-1|i)\\ &\quad +x\frac{(1-x)^{k-1-a}}{(1-x)^{k-a}-x}F_{32-1}(x;k-1|a).\end{aligned}$$

The above proposition may also be used to find an explicit formula for the generating function $F_{32-1}(x;k)$ for any given k. For $k=2$, we have $F_{32-1}(x;2|1) = \frac{x^3}{(1-x)(1-2x)}$, $F_{32-1}(x;2|2) = \frac{x^2}{1-2x}$ and $F_{32-1}(x;2) = \frac{x^2}{(1-x)(1-2x)}$. For $k=3$, we have

$$\begin{aligned} F_{32-1}(x;3|1) &= \frac{x(1-x)}{1-3x+x^2}F_{32-1}(x;2|1) = \frac{x^4}{(1-2x)(1-3x+x^2)}, \\ F_{32-1}(x;3|2) &= xF_{32-1}(x;3), \\ F_{32-1}(x;3|3) &= xF_{32-1}(x;3) + \frac{x^3}{(1-x)(1-2x)}. \end{aligned}$$

By summing the above equations, we obtain $(1-2x)F_{32-1}(x;3) = \frac{x^3}{(1-x)(1-3x+x^2)}$, which implies

$$F_{32-1}(x;3) = \frac{x^3}{(1-3x+x^2)(1-2x)(1-x)}.$$

The results of this section are summarized in Table 2 below.

τ	Reference	τ	Reference
$11-1$	Theorem 18	$21-2$	Theorem 20
$11-2$	Example 17	$21-3$	Example 16
$12-1$	Proposition 19	$22-1$	No explicit formula
$12-2$	Proposition 19	$23-1$	No explicit formula
$12-3$	Example 15	$31-2$	Open
$13-2$	Open	$32-1$	No explicit formula
$21-1$	Theorem 21		

Table 2: Three letter generalized patterns of type $(2,1)$

4. Generating Trees

In this section, we use the methodology of generating trees to count set partitions avoiding a pattern. A generating tree is an infinite rooted tree, which essentially is a process that generates labels from a single label of the root by successively applying certain rules. Formally speaking, a generating tree consists of the single label of its root along with its succession rules, see [17].

For example, we can count the words in $[k]^n$ by inserting a letter from $[k]$ to the right of the rightmost letter of a word. For the purpose of enumeration, we only keep track of labels on words in the case when members of $[k]^n$ are formed from the empty word. For example, the generating tree for the words in $[k]^n$ is given by

$$\begin{cases} \textbf{Root}: (k) \\ \textbf{Rule}: (k) \rightsquigarrow (k)^k. \end{cases}$$

In this setting, partitions of $[n]$ will be regarded as words of the form πj over the alphabet $\{1, 2, \ldots\}$, where the letter j belongs to the set $\{1, 2, \ldots, 1 + \max \pi\}$. Thus, if we label each partition by a label (a), where a is the maximum element of the partition, we see that the generating tree for the partitions of $[n]$ is given by

$$\begin{cases} \textbf{Root}: (1) \\ \textbf{Rule}: (a) \rightsquigarrow (a)^a(a+1). \end{cases}$$

We now present the generating trees for partitions of $[n]$ avoiding the patterns $23-1$, $22-1$ and $32-1$.

4.1. The Pattern 23 − 1

Theorem 25. *The generating tree $\mathcal{T}_{23-1}$ for the partitions of $[n]$ that avoid $23-1$ is given by*

$$\begin{cases} \textbf{Root}: (1,1,1) \\ \textbf{Rule}: (a,b,c) \rightsquigarrow (a,b,a)\cdots(a,b,c)(c,b,c+1)\cdots(c,b,b)(c,b+1,b+1). \end{cases}$$

Proof. We label each partition $\pi = \pi_1 \cdots \pi_n$ of $[n]$ by (a, b, c), where

$$c = \pi_n, \quad b = \max_{1 \le i \le n} \pi_i, \quad \text{and} \quad a = \begin{cases} 1, & \text{if } \pi = 11\cdots 1; \\ \max\{\pi_i : \pi_i < \pi_{i+1}\}, & \text{otherwise.} \end{cases}$$

Clearly, the partition $\{1\}$ of $[1]$ is labelled by $(1, 1, 1)$ and $c \ge a$ (for otherwise, π would contain the pattern $23 - 1$). If we have a partition π associated with a label (a, b, c), then each child of π is a partition of the form $\pi' = \pi c'$, where $c' = a, a+1, \ldots, b+1$, for if $c' < a$, then π' would contain $23 - 1$, which is not allowed. Thus, we have the three cases:

- if $b \ge c' > c$, then π' is labelled by (c, b, c'),
- if $c' = b + 1$, then π' is labelled by $(c, b + 1, b + 1)$,
- if $c \ge c' \ge a$, then π' is labelled by (a, b, c').

Combining the above cases yields our generating tree. □

Let $H_{23-1}(t; a, b, c)$ be the generating function for the number of partitions of level n that are labelled by (a, b, c) in the generating tree $\mathcal{T}_{23-1}$, as described in

Theorem 25 above. Define $H_{23-1}(t,u,v,w)=\sum_{a,b,c}H_{23-1}(t;a,b,c)u^av^bw^c$. Then Theorem 25 implies

$$\begin{aligned}
&H_{23-1}(t,u,v,w)\\
&=tuvw\\
&\quad+t\sum_{a,b,c}H_{23-1}(t;a,b,c)v^b(u^a(w^a+\cdots+w^c)+u^c(w^{c+1}+\cdots+w^b)+u^cv^1w^{b+1})\\
&=tuvw\\
&\quad+t\sum_{a,b,c}H_{23-1}(t;a,b,c)v^b\left(u^a\frac{w^a-w^{c+1}}{1-w}+u^c\frac{w^{c+1}-w^{b+1}}{1-w}+u^cv^1w^{b+1}\right)\\
&=tuvw+\frac{t}{1-w}(H_{23-1}(t,uw,v,1)-wH_{23-1}(t,u,v,w))\\
&\quad+\frac{tw}{1-w}(H_{23-1}(t,1,v,uw)-H_{23-1}(t,1,vw,u))+tvwH_{23-1}(t,1,vw,u),
\end{aligned}$$

which gives the following result.

Theorem 26. *The generating function $H_{23-1}(t,u,v,w)$ satisfies*

$$\begin{aligned}
H_{23-1}(t,u,v,w)=tuvw&+\frac{t}{1-w}(H_{23-1}(t,uw,v,1)-wH_{23-1}(t,u,v,w))\\
&+\frac{tw}{1-w}(H_{23-1}(t,1,v,uw)-H_{23-1}(t,1,vw,u))\\
&+tvwH_{23-1}(t,1,vw,u).
\end{aligned}$$

Using the above theorem, we see that the first fifteen values of the sequence recording the number of partitions of $[n]$, $n\geq 1$, which avoid the pattern $23-1$ are 1, 2, 5, 14, 42, 132, 430, 1444, 4983, 17634, 63906, 236940, 898123, 3478623 and 13761820.

4.2. The Pattern $22-1$

Theorem 27. *The generating tree $\mathcal{T}_{22-1}$ for the partitions of $[n]$ that avoid $22-1$ is given by*

$$\begin{cases}
\textbf{Root}: (1,1,1)\\
\textbf{Rule}: (a,b,c)\rightsquigarrow (a,b,a)\cdots(a,b,c-1)(c,b,c)\\
\qquad\qquad\qquad\qquad (a,b,c+1)\cdots(a,b,b)(a,b+1,b+1).
\end{cases}$$

Proof. We label each partition $\pi=\pi_1\cdots\pi_n$ of $[n]$ by (a,b,c), where $c=\pi_n$, $b=\max_{1\leq i\leq n}\pi_i$, and

$$a=\begin{cases}1, & \text{if there is no } i \text{ such that } \pi_i=\pi_{i+1};\\ \max\{\pi_i:\pi_i=\pi_{i+1}\}, & \text{otherwise.}\end{cases}$$

Clearly, the partition $\{1\}$ of $[1]$ is labelled by $(1,1,1)$ and $c \geq a$ (for otherwise, π would contain the pattern $22-1$). If we have a partition π associated with a label (a,b,c), then each child of π is a partition of the form $\pi' = \pi c'$, where $c' = a, a+1, \ldots, b+1$, for if $c' < a$, then π' would contain $22-1$, which is not allowed. Thus, we have the four cases:

- if $c > c' \geq a$, then π' is labelled by (a,b,c'),
- if $c' = c$, then π' is labelled by (c,b,c),
- if $b \geq c' > c$, then π' is labelled by (a,b,c'),
- if $c' = b+1$, then π' is labelled by $(a,b+1,b+1)$.

Combining the above cases yields our generating tree. □

Let $H_{22-1}(t;a,b,c)$ be the generating function for the number of partitions of level n that are labelled by (a,b,c) in the generating tree $\mathcal{T}_{22-1}$, as described in Theorem 27. Define $H_{22-1}(t,u,v,w) = \sum_{a,b,c} H_{22-1}(t;a,b,c)u^a v^b w^c$. Then, using similar arguments as in the proof of Theorem 26 above, we obtain

Theorem 28. *The generating function $H_{22-1}(t,u,v,w)$ satisfies*

$$\begin{aligned} &H_{22-1}(t,u,v,w) \\ &\quad = tuvw + \frac{t}{1-w}(H_{22-1}(t,uw,v,1) - H_{22-1}(t,u,v,w)) + tH_{22-1}(t,1,v,uw) \\ &\quad + \frac{tw}{1-w}(H_{22-1}(t,u,v,w) - H_{22-1}(t,u,vw,1)) + tvwH_{22-1}(t,u,vw,1). \end{aligned}$$

Using the above theorem, we see that the first fifteen values of the sequence recording the number of partitions of $[n]$, $n \geq 1$, which avoid the pattern $22-1$ are 1, 2, 5, 14, 44, 153, 585, 2445, 11109, 54570, 288235, 1628429, 9792196, 623191991 and 419527536.

4.3. The Pattern 32 − 1

Theorem 29. *The generating tree $\mathcal{T}_{32-1}$ for the partitions of $[n]$ that avoid $32-1$ is given by*

$$\begin{cases} \textbf{Root}: (1,1,1) \\ \textbf{Rule}: (a,b,c) \rightsquigarrow (a,b,a)\cdots(c-1,b,c-1)(a,b,c)\cdots(a,b,b)(a,b+1,b+1). \end{cases}$$

Proof. We label each partition $\pi = \pi_1 \cdots \pi_n$ of $[n]$ by (a,b,c), where $c = \pi_n$, $b = \max_{1\leq i\leq n} \pi_i$, and

$$a = \begin{cases} 1, & \text{if there is no } i \text{ such that } \pi_i > \pi_{i+1}; \\ \max\{\pi_{i+1} : \pi_i > \pi_{i+1}\}, & \text{otherwise.} \end{cases}$$

We label the partition $\{1\}$ of $[1]$ by $(1,1,1)$, and clearly $c \geq a$ (for otherwise, π would contain $32-1$). If we have a partition π associated with a label (a,b,c), then each child of π is a partition of the form $\pi' = \pi c'$, where $c' = a, a+1, \ldots, b+1$, for if $c' < a$, then π' would contain $32-1$, which is not allowed. Thus, we have the four cases:

- if $c > c' \geq a$, then π' is labelled by (c', b, c'),
- if $c' = c$, then π' is labelled by (a, b, c),
- if $b \geq c' \geq c+1$, then π' is labelled by (a, b, c'),
- if $c' = b+1$, then π' is labelled by $(a, b+1, b+1)$.

Combining the above cases yields our generating tree. □

Let $H_{32-1}(t; a, b, c)$ be the generating function for the number of partitions of level n that are labelled by (a,b,c) in the generating tree $\mathcal{T}_{32-1}$, as described in Theorem 29. Define $H_{32-1}(t,u,v,w) = \sum_{a,b,c} H_{32-1}(t;a,b,c)u^a v^b w^c$. Then, using similar arguments as in the proof of Theorem 26 above, we obtain

Theorem 30. *The generating function* $H_{32-1}(t,u,v,w)$ *satisfies*

$$
\begin{aligned}
&H_{32-1}(t,u,v,w) \\
&\quad = tuvw + \frac{t}{1-uw}(H_{32-1}(t,uw,v,1) - H_{32-1}(t,1,v,uw)) \\
&\quad + \frac{t}{1-w}(H_{32-1}(t,u,v,w) - wH_{32-1}(t,u,vw,1)) + tvwH_{32-1}(t,u,vw,1).
\end{aligned}
$$

Using the above theorem, we see that the first fifteen values of the sequence recording the number of partitions of $[n]$, $n \geq 1$, which avoid the pattern $32-1$ are 1, 2, 5, 15, 51, 189, 747, 3110, 13532, 61198, 286493, 1383969, 6881634, 35150498 and 184127828.

5. Concluding Remarks

The enumeration problem of finding the generating function $F_\tau(x;k)$, where τ is a pattern of type $(1,2)$, was completed in the first section above. However, the problem of finding the generating function $F_\tau(x;k)$, where τ is a pattern of type $(2,1)$, is not complete and we have the following remarks: (1) By using generating functions, we obtained recurrence relations satisfied by $F_\tau(x;k)$ in the cases when τ equals $23-1$, $22-1$ or $32-1$, and by using generating trees, we obtained functional equations satisfied by $F_\tau(x;k)$ in these cases. However, we were unable to find explicit formulas for $F_\tau(x;k)$. (2) We also failed to find explicit formulas for the generating function $F_\tau(x;k)$ in the cases when τ equals either $13-2$ or $31-2$. On the other hand, we can write recurrence relations for these cases which are analogous to the case $23-1$ above, for example, but the recurrence relations here are more complicated and require two indices.

References

[1] E. Babson and E. Steingrímsson, Generalized permutation patterns and a classification of the Mahonian statistics, *Sém. Lothar. Combin.* (2000) **B44b**.

[2] A. Burstein, Enumeration of words with forbidden patterns, Ph.D. Thesis, University of Pennsylvania, 1998.

[3] A. Burstein and T. Mansour, Words restricted by 3-letter generalized multipermutation patterns, *Ann. Comb.* **7:1** (2003) 1-14.

[4] A. Burstein and T. Mansour, Counting occurrences of some subword patterns, *Discrete Math. Theor. Comput. Sci.* **6:1** (2003) 1–12.

[5] A. Claesson and T. Mansour, Counting occurrences of a pattern of type $(1,2)$ or $(2,1)$ in permutations, *Adv. in Appl. Math.* **29:2** (2002) 293–310.

[6] E. P. Deng, T. Mansour, and N. Mbarieky, Restricted set partitions, preprint.

[7] S. Heubach and T. Mansour, Enumeration of 3-letter patterns in compositions, *Proceedings of the 2005 Integers Conference in Honor of Ron Graham's 70th Birthday,* (2007) Article #17.

[8] V. Jelínek and T. Mansour, On pattern-avoiding partitions, *Electron. J. Combin.* **15:1** (2008) #R39.

[9] M. Klazar, On *abab*-free and *abba*-free set partitions, *European J. Combin.* **17** (1996) 53–68.

[10] D. E. Knuth, *The Art of Computer Programming, Vol's. 1 and 3,* Addison-Wesley, NY, 1968, 1973.

[11] T. Mansour and N. Mbarieky, Partitions of a set satisfying a certain set of conditions, *Discrete Math.* **309:13** (2009) 4481–4488.

[12] T. Mansour and A. Munagi, Enumeration of partitions by long rises, levels, and descents, *J. Integer Seq.* **12** (2009) Article 09.1.8.

[13] S. Milne, A q-analog of restricted growth functions, Dobinski's equality, and Charlier polynomials, *Trans. Amer. Math. Soc.* **245** (1978) 89–118.

[14] B. E. Sagan, Pattern avoidance in set partitions, *Ars Combin.* **94** (2010) 79–96.

[15] R. Simion and F. W. Schmidt, Restricted permutations, *European J. Combin.* **6** (1985) 383–406.

[16] R. P. Stanley, *Enumerative Combinatorics, Vol. 1*, Cambridge University Press, Cambridge, UK, 1997.

[17] J. West, Generating trees and the Catalan and Schröder numbers, *Discrete Math.* **146** (1995) 247–262.