COUNTING HERON TRIANGLES WITH CONSTRAINTS

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Abstract

Heron triangles have the property that all three of their sides as well as their area are positive integers. In this paper, we give some estimates for the number of Heron triangles with two of their sides fixed. We provide a general bound on this count $H(a, b)$, where the sides a, b are fixed positive integers, and the estimate here is better than the one of Ionascu, Luca and Stănică for the general situation of fixed sides a, b. In the case of primes sides p, q, there is an additional hypothesis which helps us to drop the upper bounds on $H(p, q)$. In particular, we prove that $H(p, q)$ is less than or equal to 1 when $p - q \equiv 2 \pmod 4$. We also provide a count for the number of Heron triangles with a fixed height (there exists only one such when the height is prime). Moreover, we study the decomposability property of a Heron triangle into two similar ones, and provide some cases when a Heron triangle is not decomposable.

1. Introduction

In the field of two-dimensional Euclidean geometry, a *Heron triangle* has the property that its three sides, as well as its area are positive integers. Many interesting questions can be raised about these triangles, and there has been a plethora of research regarding several properties of the Heron triangles.

Ionascu, Luca and Stănică [10] found an upper bound for the number of Heron triangles with two fixed sides. They also found sharper upper bounds for the number of Heron triangles with two fixed prime sides. We improve upon the general bounds for two fixed integer sides, as well as give tight bounds for the case where the two fixed sides are primes. We also prove certain upper bounds for the number of Heron triangles for special cases involving the fixed sides; namely, fixed sides that are prime squares, twin primes, or Sophie Germain primes. We also find an example of a Heron triangle whose sides are all perfect squares (question raised in [13]), namely, the triangle of sides $[1853^2; 4380^2; 4427^2]$ and of area 32918611718880. We present the first instance of such a triangle in this paper.

Further, we study the decomposability of Heron triangles into two smaller Heron triangles. It is known from [5] that any Heron triangle is radially decomposable. We show that any isosceles Heron triangle is decomposable, and prove a few results regarding the non-decomposability of certain Heron triangles.

Throughout this paper, we denote the three sides of a general triangle by lower case letters, like a, b, c, and the corresponding vertices by capital letters, like A, B, C. By abuse of notation, we use the same capital letters for the angles, as well as for the corresponding vertices of the triangle. The semi-perimeter $(a + b + c)/2$ is denoted by s and hence the area is given by $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ (Heron formula).

2. Heron Triangles with Two Fixed Sides

Let $H(a, b)$ be the number of Heron triangles whose two sides a, b are fixed. In [10], a general upper bound for $H(a, b)$ has been proposed, as follows.

Proposition 1 ([10])**.** *If $a \le b$ are fixed, then $0 \le H(a, b) \le \min\{2a-1, 4\,(\tau(ab))^2\}$, where $\tau(n)$ represents the number of positive divisors of an integer n.*

We first find a better bound on $H(a, b)$. Later in this section, we also study the case with two fixed prime sides, and once again improve the corresponding bounds proved in [10].

2.1. Counting Heron Triangles with Fixed Integer Sides

Since a triangle can be uniquely determined by the length of two sides and the angle between them, we simply find the maximum number of all possible values of the angle between the two fixed sides such that the triangle is a Heron triangle; this number will give an upper bound for $H(a, b)$.

Let the prime factorization of an integer n be

$$n = 2^{a_0} \prod_{i=1}^{s} p_i^{a_i} \prod_{j=1}^{r} q_j^{e_j} \tag{1}$$

with primes $p_i \equiv 3 \pmod 4$ and $q_j \equiv 1 \pmod 4$ for all i, j.

We shall require the following known results (see for example [1, Chapter XIV], with a correction in [15]) on primitive Pythagorean triples involving the prime factorization of n as above.

Lemma 2. *Assume that the prime factorization of a positive integer n is as in* (1). *Then:*

1. *n can be represented as the sum of two positive coprime squares if and only if $a_i = 0$ for all $i = 0, 1, \dots, s$.*
2. *Assuming that $a_i = 0$ for all $i = 0, 1, \dots, s$, the number of representations of n as the sum of two positive coprime squares, ignoring signs and order, is given by $2^{w(n)-1}$, where $w(n)$ denotes the number of distinct primes factors of n which are $\equiv 1 \pmod 4$.*

Lemma 3. *Assume that the prime factorization of a positive integer n is as in* (1). *Then the number of primitive Pythagorean triples (u, w, v) with hypotenuse v dividing n is given by $T(n) = \frac{1}{2}(\prod_{j=1}^{r}(2e_j + 1) - 1)$.*

Proof. For each factor v of n, the number of primitive Pythagorean triples with v as hypotenuse is given by the number of representations of v as a sum of two positive coprime squares $m^2 + n^2$. From Lemma 2, we know that v can be represented as $m^2 + n^2$ with $\gcd(m, n) = 1$ if and only if all of its prime factors are $\equiv 1 \pmod 4$, and the number of such representations is $2^{w(v)-1}$, where $w(v)$ denotes the number of distinct prime factors of v.

Thus, the number of primitive Pythagorean triples (u, w, v) with hypotenuse v dividing n is given by

$$T(n) = \sum_{j=1}^{r} 2^{j-1} \sigma_j(\alpha_1, \alpha_2, \dots, \alpha_r),$$

where $\sigma_j(\alpha_1, \alpha_2, \dots, \alpha_r)$, the j-th elementary symmetric polynomial, and $0 \le \alpha_j \le e_j$ for $1 \le j \le r$. The term $\sigma_j(\alpha_1, \alpha_2, \dots, \alpha_r)$ denotes the number of ways in which a factor v of n can be constructed using j distinct prime factors of n, each $\equiv 1 \pmod 4$. Using known properties of elementary polynomials, the expression for

$T(n)$ further simplifies to:

$$\begin{aligned} T(n) &= \frac{1}{2}\sum_{j=1}^{r} 2^j \sigma_j(\alpha_1, \alpha_2, \ldots, \alpha_r) \\ &= \frac{1}{2}\left(\sum_{j=0}^{r} 2^j \sigma_j(\alpha_1, \alpha_2, \ldots, \alpha_r) - 1\right) \\ &= \frac{1}{2}\left(\prod_{j=1}^{r}(2e_j + 1) - 1\right), \end{aligned}$$

as we have $0 \leq \alpha_j \leq e_j$ for all $1 \leq j \leq r$. Hence the result. □

Assuming the previous notation, we present our first result, which follows an idea of [10, Theorem 2.3].

Theorem 4. *Let a, b be two fixed integers, and let ab be factored as in* (1)*. Then*

$$H(a, b) \;\leq\; \frac{3 + (-1)^{ab}}{2}\left(\prod_{j=1}^{r}(2e_j + 1) - 1\right).$$

Proof. Let C be the angle between the sides a, b, and so, $\triangle = \frac{1}{2}ab\sin C$. Then, $\sin C = \frac{2\triangle}{ab} = \frac{u}{v}$ for some integers u, v with $\gcd(u, v) = 1$, which implies that the integers u, v are two components from a primitive Pythagorean triple with v as hypotenuse. Thus we have $v = m^2 + n^2$, $u \in \{m^2 - n^2, 2mn\}$ for $\gcd(m, n) = 1$, $m > n$.

Also note that $\frac{2\triangle}{ab} = \frac{u}{v}$ implies $v|ab$, and for each factor v of ab, there are two options $\{m^2 - n^2, 2mn\}$ for u. Thus, the number of possible values of $\sin C$ is bounded above by $2T(ab)$, where $T(ab) = \frac{1}{2}(\prod_{j=1}^{r}(2e_j + 1) - 1)$ denotes the number of primitive Pythagorean triples with hypotenuse v dividing ab, derived from Lemma 3.

In case ab is odd, so is $v = m^2 + n^2$. In this case, $\frac{2\triangle}{ab} = \frac{u}{v}$ implies that $2|u$, and thus for each factor v of ab, the only possible choice for u is $2mn$ (as $m^2 - n^2$ is odd). Therefore, for ab odd, the number of possible values of $\sin C$ is bounded above by $T(ab)$, where $T(ab) = \frac{1}{2}(\prod_{j=1}^{r}(2e_j + 1) - 1)$, derived from Lemma 3.

For each possible value of $\sin C$, there are at most two possible values of C, and thus we have $H(a, b) \leq 2T(ab)$ if ab is odd, and $H(a, b) \leq 4T(ab)$ if ab is even, from which we derive the result. □

Theorem 4 immediately offers us an interesting observation regarding a special class of fixed sides (a, b).

Corollary 5. *If all the prime factors of ab are $\equiv 3 \pmod 4$, then $H(a, b) = 0$.*

Proof. If all prime factors of ab are $\equiv 3 \pmod 4$, the prime factorization of ab is $\prod_{i=1}^{s} p_i^{a_i}$, as in (1), with $e_j = 0$ for all $1 \le j \le r$. In this case, $T(ab) = \frac{1}{2}(\prod_{j=1}^{r}(2e_j+1)-1) = 0$, and hence $H(a,b) = 0$ from Theorem 4. □

At this point, let us refer to the results available in [10]. When two fixed sides are primes, $p, q\ (\neq 2)$ say, one can get the following bounds from [10, Theorem 2.4]:

$$H(p,q) \text{ is } \begin{cases} = 0 & \text{if both } p \text{ and } q \text{ are} \equiv 3 \pmod 4, \\ = 2 & \text{if } p = q \equiv 1 \pmod 4, \\ \le 2 & \text{if } p \neq q \text{ and exactly one of } p \text{ and } q \text{ is} \equiv 3 \pmod 4, \\ \le 5 & \text{if } p \neq q \text{ and both } p \text{ and } q \text{ are} \equiv 1 \pmod 4. \end{cases} \tag{2}$$

From Theorem 4, we immediately obtain the same bounds in two of the cases.

Corollary 6. *Let $p, q\ (\neq 2)$ be two fixed prime sides of a triangle. Then:*

$$H(p,q) \ \textit{is} \ \begin{cases} = 0 & \textit{if both } p \textit{ and } q \textit{ are} \equiv 3 \pmod 4, \\ \le 2 & \textit{if } p \neq q \textit{ and exactly one of } p \textit{ and } q \textit{ is} \equiv 3 \pmod 4. \end{cases}$$

Proof. In the first case, $H(p,q) = 0$ from Corollary 5. In the second case, $T(pq) = \frac{1}{2}((2\times 1+1)-1) = 1$, and thus $H(p,q) \le 2T(p,q) = 2$ from Theorem 4, as pq is obviously odd. □

For the remaining two cases, we obtain the following from Theorem 4.

Corollary 7. *Let $p, q\ (\neq 2)$ be two fixed prime sides of a triangle. Then:*

$$H(p,q) \ \textit{is} \ \begin{cases} \le 4 & \textit{if } p = q \equiv 1 \pmod 4, \\ \le 8 & \textit{if } p \neq q \textit{ and both } p \textit{ and } q \textit{ are} \equiv 1 \pmod 4. \end{cases}$$

Proof. In both the cases, pq is odd, and thus we have $H(p,q) \le 2T(pq)$ from Theorem 4. In the first case, $T(pq) = T(p^2) = \frac{1}{2}((2\times 2+1)-1) = 2$, and thus $H(p,q) \le 2T(p,q) = 4$. In the second case, $T(pq) = \frac{1}{2}((2\times 1+1)(2\times 1+1)-1) = 4$, and thus $H(p,q) \le 2T(p,q) = 8$. □

Corollary 7 is slightly weaker than [10], but it has the advantage that it follows easily from a more general result (Theorem 4), whereas the result from [10] had to be dealt with as a separate problem, just for the primes. In the following section, we discuss further improvements in the bounds for prime sides.

Let us compare our result with that of [10] by displaying some examples in Table 1. Note that in the work of Ionascu, Luca and Stănică [10], $H(a,b) \le \min\{2a-1, 4\,(\tau(ab))^2\}$ (column 7 of Table 1), as stated in Proposition 1, as well. The bound on $H(a,b)$ that we propose in this section (column 4 in Table 1) is much closer to the actual value (column 3 in Table 1), and it is clear that our result is sharper than the existing bound, when a, b are composite integers.

a	b	$H(a,b)$ (actual)	Upper bound from Theorem 4	Upper bound of [10]		
				$2a-1$	$4\,(\tau(ab))^2$	$\min\{2a-1, 4(\tau(ab))^2\}$
17	27	0	2	33	256	33
51	52	5	16	101	2304	101
65	87	4	26	129	1024	129
125	125	6	12	249	196	196
305	377	4	80	609	1024	609
714	728	5	16	1427	57600	1427
1189	1275	8	134	2377	9216	2377
3034	3434	7	160	6067	9216	6067
7089	7228	5	16	14177	20736	14177
81713	49274	0	16	98547	256	256

Table 1: Comparison of the bounds on $H(a,b)$ when a,b are composite integers.

2.2. Counting Heron Triangles with Fixed Prime Sides

In this section, we present certain results, which under an additional hypothesis slightly improves the bounds of $H(a,b)$ from [10, Theorem 2.4].

Theorem 8. *Let p,q $(\neq 2)$ be two fixed prime sides of a triangle. Then:*

$$H(p,q)\ \textit{is}\ \begin{cases} =\ 0 & \textit{if both } p \textit{ and } q \textit{ are} \equiv 3 \pmod 4,\\ =\ 2 & \textit{if } p=q\equiv 1 \pmod 4,\\ =\ 0 & \textit{if } p<q \textit{ with } p\equiv 1 \pmod 4 \textit{ and } q\equiv 3 \pmod 4,\\ \leq\ 1 & \textit{if } p>q \textit{ with } p\equiv 1 \pmod 4 \textit{ and } q\equiv 3 \pmod 4,\\ \leq\ 5 & \textit{if } p\neq q \textit{ and both } p \textit{ and } q \textit{ are} \equiv 1 \pmod 4.\end{cases}$$

Moreover, when both p and q are $\equiv 1 \pmod 4$ with $q>p$ and $(t+1)(q-tp)$ is not a perfect square, where $t=\lfloor q/p \rfloor$, then $H(p,q)\leq 4$.

Proof. We consider each case, separately (some are included in [10], certainly, but we include them nonetheless, since we provide simpler proofs).

`Case I.` Both p and q are $\equiv 3 \pmod 4$. This follows from Corollary 5.

`Case II.` $p=q\equiv 1 \pmod 4$. When $p=q\equiv 1 \pmod 4$ (and third side $2w$), we get $H(p,q)\leq 4$ as an immediate consequence from Theorem 4, but an exact count is easy to obtain. The area is $\Delta = w\sqrt{p^2-w^2}$, and so, p is part of a Pythagorean triple, that is, there exist integers m,n with $p=m^2+n^2$. By Lemma 2, since $p\equiv 1 \pmod 4$, there is only one such representation (excluding order, say $m>n$), and so, either $w=2mn$, or $w=m^2-n^2$, which implies that $H(p,q)=2$, in this case.

For the next three cases, recall that the third side of the triangle is even, say $2w$. Using Heron's formula,

$$\begin{aligned} 4\Delta &= \sqrt{(p+q+2w)(p+q-2w)(2w+p-q)(2w-p+q)}\\ &= \sqrt{((p+q)^2-4w^2)(4w^2-(p-q)^2)}. \end{aligned} \tag{3}$$

`Case III.` $p < q$ with $p \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$. As $p < q$, let us write $q - p = 4k + 2$, and suppose that $2w$ is the third side. Certainly, by the triangle inequality, $(q - p)/2 = 2k + 1 < w < (q + p)/2 = p + 2k + 1$. From (3), we obtain

$$\begin{aligned} \Delta &= \sqrt{(p + 2k + 1 + w)(p + 2k + 1 - w)(w + 2k + 1)(w - 2k - 1)} \\ &= \sqrt{((p + 2k + 1)^2 - w^2)(w^2 - (2k + 1)^2)}. \end{aligned}$$

We consider several cases. If there exists a prime $r| \gcd(p+2k+1-w, w-2k-1)$, then $r = p$, and so, $p|w - 2k - 1 < p$, a contradiction. If there exists a prime $r| \gcd(p+2k+1-w, w+2k+1)$, then $r = p+4k+2 = q$, and so, $q|p+2k+1-w < p < q$, a contradiction. Moreover, if there exists a prime $r| \gcd(p+2k+1+w, w-2k-1)$, then $r = p + 4k + 2 = q$, and so $q|w - 2k - 1 < p$, a contradiction. It remains to consider the case of $r| \gcd(p + 2k + 1 + w, w + 2k + 1)$, which implies that $r = p$. Combining all cases, we obtain that, for some integers m, n, we have

$$(p + 2k + 1)^2 - w^2 = p^\ell m^2 \text{ and } w^2 - (2k + 1)^2 = p^\ell n^2.$$

Add the previous two equations and obtain $pq = p^\ell(m^2 + n^2)$, and so $\ell \in \{0, 1\}$. If $\ell = 0$, then $pq = m^2 + n^2$, and by Lemma 2, there are no representations of pq as a sum of squares, as $pq \equiv 3 \pmod 4$. If $\ell = 1$, then we get that $q = m^2 + n^2$, which is also a contradiction by Lemma 2. Thus, we get that $H(p, q) = 0$.

`Case IV.` $p > q$ with $p \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$. In this case, we write $p - q = 4k + 2$, and proceed almost identically as in the previous case to obtain that, for some integers m, n, we have

$$(q + 2k + 1)^2 - w^2 = q^\ell m^2 \text{ and } w^2 - (2k + 1)^2 = q^\ell n^2.$$

Add the previous two equations and obtain $pq = q^\ell(m^2 + n^2)$, and so $\ell \in \{0, 1\}$. If $\ell = 0$, then $pq = m^2 + n^2$, and by Lemma 2, there are no representations of pq as a sum of squares, as $pq \equiv 3 \pmod 4$. If $\ell = 1$, then we get that $p = m^2 + n^2$, which can be written in only one way, according to Lemma 2.

Now since $w^2 - (2k + 1)^2 = qn^2$, we get

$$w = \pm(2k + 1) + tq = \pm\frac{p - q}{2} + tq,$$

for some integer t. We also know that $2w < p + q$ and $p < 2w + q$, and thus w can not be of the form $\frac{p-q}{2} + tq$. Hence w must be of the form $-\frac{p-q}{2} + tq$. Moreover, since $2w < p + q$ and $p < 2w + q$, the only option for t is $\lfloor p/q \rfloor$. Thus, considering all cases discussed till now, we get that $H(p, q) \leq 1$.

`Case V.` $p \neq q$ and both p and q are $\equiv 1 \pmod 4$. Assume $p < q$ and write $q - p = 4k$, and $2w$ for the third side of the Heron triangle. Certainly, by the triangle inequality, $(q-p)/2 = 2k < w < (q+p)/2 = p+2k$. From (3), we obtain

$$\begin{aligned}\Delta &= \sqrt{(p+2k+w)(p+2k-w)(w+2k)(w-2k)} \\ &= \sqrt{((p+2k)^2-w^2)(w^2-(2k)^2)}.\end{aligned}$$

We consider several subcases. If there exists a prime $r|\gcd(p+2k-w, w-2k)$, then $r=p$, and so, $p|w-2k<p$, a contradiction. If there exists a prime $r|\gcd(p+2k-w, w+2k)$, then $r=p+4k=q$, and so, $q|p+2k-w<p<q$, a contradiction. Moreover, if there exists a prime $r|\gcd(p+2k+w, w-2k)$, then $r=p+4k=q$, and so $q|w-2k<p$, a contradiction. It remains to consider the case of $r|\gcd(p+2k+w, w+2k)$, which implies that $r=p$. Combining all cases, we obtain that, for some integers m, n, we have

$$(p+2k)^2-w^2 = p^\ell m^2 \text{ and } w^2-(2k)^2 = p^\ell n^2.$$

Add the previous two equations and obtain $pq = p^\ell(m^2+n^2)$, and so $\ell \in \{0,1\}$. If $\ell = 0$, then $pq = m^2+n^2$, and by Lemma 2, there are only two representations of pq as a sum of squares.

If $\ell = 1$, then we get that q can be written as $q = m^2+n^2$ in only one way, disregarding order. In this situation $w^2 \equiv (2k)^2 \pmod p$, and thus w is of the form $w = \pm 2k + tp$ for some integer t. Since we know $2w < p+q$ and $q < 2w+p$, it can not be of the form $w = 2k+tp$. Hence we must have $w = -2k+tp$, and as we know $2w < p+q$ and $q < 2w+p$, the only option for t is $\lfloor q/p \rfloor$. Considering all cases, we get that $H(p,q) \le 2\times 2+1 = 5$.

The last claim follows easily, since if we assume that $(t+1)(q-tp)$ is not a perfect square, for $t = \lfloor q/p \rfloor$, then the two displayed identities from above (for $\ell = 1$) would imply that $(p+2k)^2-(tp-2k)^2 = pm^2$, which is equivalent to $(t+1)(4k-(t-1)p) = m^2$, that is, $(t+1)(q-tp) = m^2$, a contradiction. Thus the count for the $l = 1$ case will not appear in this case and the bound will be $2\times 2 = 4$. $\square$

A natural question is whether the range of values in the last three inequalities of Theorem 8 is completely covered. Given the extensive computations we performed, *we conjecture that* $H(p,q)$ *never attains the values* $4, 5$ (under $p \ne q$ with $p \equiv 1 \pmod 4$ and $q \equiv 1 \pmod 4$). However, we can certainly show that the following values are attained:

1. $H(p,q)$ is 0 or 1, when $p > q$ with $p \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$ and

2. $H(p,q)$ is 0 or 1 or 2 or 3, when $p \ne q$ with $p \equiv 1 \pmod 4$ and $q \equiv 1 \pmod 4$.

In Table 2, we provide examples that indeed all such cases are possible. It does require further study for properties of the primes to exactly identify the corresponding values of $H(p,q)$, which we leave as an open problem.

In the following section, we take a look at the bounds on $H(a,b)$ where the sides a, b are fixed primes with special properties.

Case	p	q	$H(p,q)$	Third side
1	13	3	0	-
	5	3	1	4
2	17	5	0	-
	13	5	1	12
	37	13	2	30,40
	4241	2729	3	1530,1850, 6888

Table 2: $H(p,q)$ for different values of p and q.

2.3. Counting Heron Triangles with Special Prime Sides

The case where the two sides a, b are Sophie Germain primes comes directly from Theorem 8, as follows.

Corollary 9. *Let $p, q = 2p+1$ be Sophie Germain primes. Then $H(p,q) = 0$.*

Proof. We consider two cases. First suppose that $p \equiv 3 \pmod 4$. Then $q = 2p+1 \equiv 3 \pmod 4$, and by Case I of Theorem 8, we have $H(p,q) = 0$. Now suppose $p \equiv 1 \pmod 4$. Then we have $q = 2p+1 \equiv 3 \pmod 4$, and by Case III of Theorem 8, we immediately obtain $H(p,q) = 0$. □

We obtain a similar result for Mersenne primes.

Corollary 10. *Let p, q be two Mersenne prime numbers. Then $H(p,q) = 0$.*

Proof. Suppose that the sides p, q are Mersenne primes, $p = 2^x - 1, q = 2^y - 1$ (for some primes $x \leq y$). Then, both p and q are $\equiv 3 \pmod 4$ and so, by Case I of Theorem 8, we have $H(a,b) = 0$. □

In case of twin primes, we not only obtain a bound on $H(p,q)$, but can also estimate the third side in each case. The result is as follows.

Theorem 11. *Let $p, q = p+2$ be twin primes. Then $H(p,q) \leq 1$. Moreover, $H(p,q) = 1$ if and only if $p-2$ is a perfect square, and if that is so, the third side of the triangle must be $2p-2$, and one must have $p \equiv 11 \pmod{12}$, $p \not\equiv 7 \pmod 8$.*

Proof. Recall that the third side of the triangle is even, say $2w$. Also, by the triangle inequality, we have $w < p+1$. Using Heron's formula and (3),

$$\Delta = \sqrt{(p+1+w)(p+1-w)(w+1)(w-1)} = \sqrt{((p+1)^2 - w^2)(w^2-1)}.$$

Suppose that $r \neq 2$ is a prime. As $\gcd(w+1, w-1)|2$, one cannot have $r|\gcd(w+1, w-1)$. If $r|\gcd(p+1-w, w-1)$, then $r = p$, which implies that $p|w-1 < p$, a contradiction. If $r|\gcd(p+1+w, w-1)$, then $r = p+2 = q$, which implies that $q|w-1 < p$, a contradiction. If $r|\gcd(p+1-w, w+1)$, then $r = p+2 = q$, which implies that $q|w+1 < p+2 = q$, again a contradiction. If $r|\gcd(p+1+w, w+1)$, then $r = p$, which implies that $p|w + 1 \le p + 1$, that is $w + 1 = p$. To summarize, if there exists a prime $r \neq 2$ such that $r|w^2 - 1$, then either $w+1 = p$, or either of $w+1$ and $w - 1$ is a whole square (note that $w + 1$ and $w - 1$ cannot be simultaneously integer squares).

In the first case, where $w + 1 = p$, we have $H(a, b) \le 1$. In fact, we can say more. Since $w + 1 = p$, then $\Delta = 2p\sqrt{p-2}$. Writing $\Delta = 2p\Delta'$, we get $p = \Delta'^2 + 2$. Thus, $p \equiv 3 \pmod 4$ and since p is the smaller prime in a twin prime pair, it must also satisfy $p \equiv 5 \pmod 6$, and so $p \equiv 11 \pmod{12}$. From Gauss' sum of three squares function formula applied to $p = \Delta'^2 + 1^2 + 1^2$ (or simply by looking at $\Delta' \pmod 4$), we see that we also need $p \not\equiv 7 \pmod 8$.

In the second case, assume that the odd prime r divides $w - 1$, and $w + 1 = 2^e$ for some positive integer e. From our discussion, we must have $w - 1 = m^2$, for some integer m. It easily follows that $e = 1$, and so $w = 1$. In that case, $4\Delta = \sqrt{2p(p+2)} \notin \mathbb{Z}$, since p is odd. Next, assume that $w - 1 = 2^f$ for some positive integer f, and there is an odd prime $r|w + 1$, that is, $w + 1 = n^2$ (for some integer n), as per our earlier discussion. Then, we easily obtain $w = 3$, and so, $2\Delta = \sqrt{2(p-2)(p+4)} \notin \mathbb{Z}$, since the prime p is odd. Thus, $H(p, q) = 0$ in both these cases.

The only remaining case is when $w + 1 = 2^e$ and $w - 1 = 2^f$ simultaneously for positive integer e, f. In this case, one obtains $w = 3$ once again, and thus, $H(p, q) = 0$ in this case. $\square$

Example 12. We can give several examples of Heron triangles based on twin primes with parameters $(p, q, 2p-2, \Delta)$, satisfying the conditions of our theorem: $(3, 5, 4, 6)$, $(11, 13, 20, 66)$, $(227, 229, 452, 6810)$.

It may be interesting to investigate primes with other special properties as well.

2.4. Heron Triangles with Square Sides

In this section, we first answer affirmatively an open question of Sastry [13], that there exists a primitive (co-prime sides) Heron triangle with square sides. For that, we ran experiments (using a "bounded" approach) in GNU/Linux environment using C with GMP, and stopped when we obtained the following example.

Example 13. There exists a Heron triangle with square sides, namely

$$[a, b, c, \triangle] = [1853^2, 4380^2, 4427^2, 32918611718880].$$

Now, we turn our attention to general Heron triangles with square sides, and prove the following.

Proposition 14. *There is no isosceles Heron triangles with square sides.*

Proof. Let the sides of the triangle be a^2, a^2, b^2, and the height h corresponding to b^2. Since the semi-perimeter of a Heron triangle is an integer, then b is even, say $b = 2b_1$. By Pythagoras' theorem, $a^4 = h^2 + 4b_1^2$, and since in a Heron triangle, the heights are always rational numbers, we obtain that h is an integer. As $a^2, h, 2b_1^2$ form a Pythagorean triple, we can have the following two cases.

`Case 1:` $2b_1^2 = 2mn, h = m^2 - n^2, a^2 = m^2 + n^2$ for some integers m, n with $\gcd(m, n) = 1$. So, $b_1^2 = mn$. Now since m, n are co-prime to each other, $m = x^2, n = y^2$ for some integers x, y. Therefore we get $a^2 = x^4 + y^4$, which does not have any integer solution, by a known consequence of Fermat's Last Theorem.

`Case 2:` $2b_1^2 = m^2 - n^2, h = 2mn, a^2 = m^2 + n^2$ for some integers m, n with $\gcd(m, n) = 1$. As, $m^2 - n^2$ is even and $\gcd(m, n) = 1$, m, n are both odd. So, $m^2 + n^2 \equiv 2 \pmod 8$. But, for any integer a, $a^2 \not\equiv 2 \pmod 8$, and hence we get a contradiction. □

3. Heron Triangles with Other Constraints

In this section, rather than fixing the sides, we impose constraints on other properties of the triangle, namely one of the heights of the triangle, and the property of decomposability.

3.1. Counting Heron Triangles with Fixed Height

We first fix one of the heights of a Heron triangle, which we assume integer, unless otherwise specified. It is an easy exercise to show that if the Heron triangle contains more than one integer height it cannot be primitive (that is, $\gcd(a, b, c) = 1$). We consider non-Pythagorean Heron triangles with fixed height h (of corresponding vertex A) to obtain the following results.

Theorem 15. *The following statements are true:*

1. *Let h be an integer. Then h is the height of a Heron triangle if and only if $h > 2$.*

2. *For a fixed prime height h, there exists only one non-Pythagorean Heron triangle which has $b = c = \frac{h^2+1}{2}$ and $a = h^2 - 1$.*

3. *For a fixed height $h = 2^{\alpha_0} \prod_{i=1}^{k} p_i^{\alpha_i}$, where $p_i > 2$ for all $i \geq 1$, there exist $\frac{1}{4}\left(|2\alpha_0 - 1| \prod_{i=1}^{k}(2\alpha_i + 1) - 1\right)^2$ many non-Pythagorean Heron triangles.*

Proof. Assume that the height of length h is the one corresponding to the side a and to the angle A.

Claim 1: Assume that h is rational and $h \leq 2$; recall that the semi-perimeter $s = (a+b+c)/2$. Square the area and impose the condition on the height to get $s(s-a)(s-b)(s-c) = \frac{a^2h^2}{4} \leq a^2$. Label $x = s-a, y = s-b, z = s-c$. Using the triangle inequality, we get $x+y+z = s > a$ and so, $xyz < a$. Therefore, $x+y+z > xyz$, which we rewrite as $\frac{1}{xy} + \frac{1}{xz} + \frac{1}{yz} > 1$. If $\min\{x,y,z\} \geq 2$, then $1 < \frac{1}{xy} + \frac{1}{xz} + \frac{1}{yz} \leq \frac{3}{4}$, which is a contradiction. Thus, $\min\{x,y,z\} = 1$. Assume $x = 1$ (all the other cases are similar). Then, $\frac{1}{y} + \frac{1}{z} + \frac{1}{yz} > 1$, and so, $\min\{y,z\} \leq 2$.
`Case 1` Let $\min\{y,z\} = 1$, say $y = 1$. Since $x = y = 1$, then $s = a+1 = b+1$, and $a = b$. It follows from $s = a + \frac{c}{2} = a+1$ that $c = 2$. We get a triangle of sides $(a,a,2)$, which cannot be Heron for any integer $a \geq 2$.
`Case 2:` Let $\min\{y,z\} = 2$, say $y = 2$. Then, $\frac{1}{z} + \frac{1}{2z} > \frac{1}{2}$, and so, $z \leq 2$. We have two possibilities, namely $(x,y,z) \in \{(1,1,1),(1,1,2)\}$, and $(a,b,c) \in \{(2,2,2),(3,3,2)\}$. However, neither of these triangles has integer area.

We show the converse under the assumption that the height h is an integer. Certainly, a Heron triangle of height 2 does not exist, since 2 must be part of a Pythagorean triple and that is impossible. If $h = 2k$, then $h^2 = (k^2+1)^2 - (k^2-1)^2$; if $h = 2k+1$, then $h^2 = \left((k+1)^2 + k^2\right)^2 - (2k(k+1))^2$. Thus, we can always write $h^2 = x^2 - y^2$ (for some integers x, y), and so, one may construct an isosceles Heron triangle of height h by taking $b = c = x$, $a = 2y$ (since h is integer and the base is an even integer, the area is an integer).

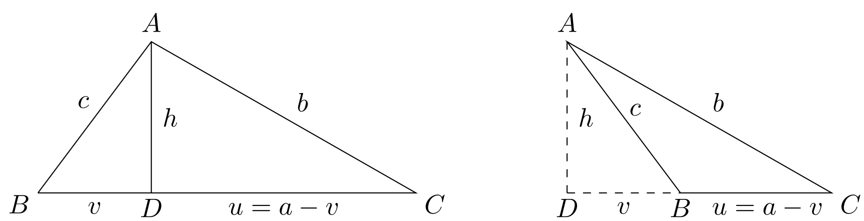

Figure 1: Heron triangle with fixed height h.

Claim 2: Suppose that a Heron triangle (as in Figure 1) has sides of lengths a, b, c and the prime number height h corresponding to A. In $\triangle ABD$, we have $h^2+v^2 = c^2$, and so, $h^2 = (c+v)(c-v)$. As h is a prime and $v \neq 0$, we get $c+v = h^2$ and $c - v = 1$. Therefore $c = \frac{h^2+1}{2}$ and $v = \frac{h^2-1}{2}$. Similarly, in $\triangle ACD$, we have $b+(a-v) = h^2, b-(a-v) = 1$, and so $b = \frac{h^2+1}{2}$ and $a = h^2-1$. Hence the claim.

Claim 3: Now, let us consider the general case $h = 2^{\alpha_0}\prod_{i=1}^{k} p_i^{\alpha_i}$, where $p_i > 2$ for all $i \geq 1$. In $\triangle ABD$, by Pythagoras' rule, we have $h^2 + v^2 = c^2$, and so, $(c+v)(c-v) = h^2 = 2^{2\alpha_0}\prod_{i=1}^{k} p_i^{2\alpha_i}$. The choices for the pair $(c+v, c-v)$ for which c, v are positive integers must have both $(c+v), (c-v)$ odd or both even.

The number of such distinct choices is

$$X = \frac{1}{2}\left(|2\alpha_0 - 1|\prod_{i=1}^{k}(2\alpha_i + 1) - 1\right).$$

Once v is determined, in $\triangle ACD$, the number of choices for the pair $(b+a-v, b-a+v)$ is, as before, $X = \frac{1}{2}\left(|2\alpha_0 - 1|\prod_{i=1}^{k}(2\alpha_i + 1) - 1\right)$, which will determine a, b.

Note that the choices of sides (c, v) in $\triangle ABD$ and that of sides $(b, a-v)$ of $\triangle ACD$ come from the same list, and each choice generates two non-Pythagorean Heron triangles with height h, as shown in Figure 1, if and only if $v \neq a - v$. In case the choices are the same, i.e., $v = a - v$, then it will generate only one triangle, namely, the first one shown in Figure 1. Thus, for $X = \frac{1}{2}\left(|2\alpha_0 - 1|\prod_{i=1}^{k}(2\alpha_i + 1) - 1\right)$ many distinct choices in the list, we will have $2 \times \binom{X}{2} + 1 \times X = (X^2 - X) + X = X^2$ many distinct non-Pythagorean Heron triangles with fixed height h. Hence the total number of Heron triangles having fixed height h is

$$X^2 = \frac{1}{4}\left(|2\alpha_0 - 1|\prod_{i=1}^{k}(2\alpha_i + 1) - 1\right)^2.$$

Hence the result. $\square$

In view of Claim 1 in Theorem 15, it might be tempting to propose that, for all rationals $h > 2$, a Heron triangle having height h exists. However, that is not true. Take for example, $h = 5/2$. Assume that such a Heron triangle exists, with sides a, b, c ($u + v = a$, with $u, v \in \mathbb{Q}$, since $h \in \mathbb{Q}$), as in Figure 1. Then we get $5^2 = (2c)^2 - (2u)^2$, with $u' := 2u \in \mathbb{Z}$. Thus, $2c - u' \in \{1, 5, 25\}$, respectively, $2c + u' \in \{25, 5, 1\}$. However, none of the obtained systems have integers c, u' as solutions.

In case of Claim 3, to take an example, one may consider the case $h = 10$. From Theorem 15, we obtain the number of distinct non-Pythagorean Heron triangles as

$$\frac{1}{4}\left(|2 \times 1 - 1|(2 \times 1 + 1) - 1\right)^2 = \frac{1}{4} \times 2^2 = 1.$$

It can be verified that the only non-Pythagorean Heron triangle with height $h = 10$ has sides $(26, 48, 26)$.

3.2. Decomposable Heron Triangles

It is known that any Heron triangle is radially decomposable, that is, it can be subdivided into n isosceles Heron triangles each composed of two circum-radii and one side of the n-gon [5]. In this section we investigate the decomposability of a Heron triangle into two Heron triangles as in Figure 2 (we say, on the side BC). We first consider the case when the Heron triangle is isosceles. In this setting, we can prove that a Heron triangle is always decomposable.

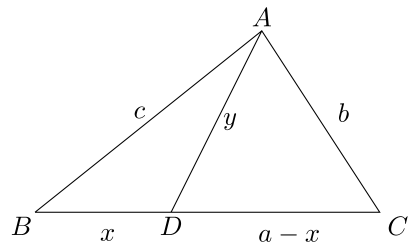

Figure 2: Heron triangle decomposed into two Heron triangles.

Proposition 16. *An isosceles Heron triangle of sides a, a, b can always be decomposed along the height corresponding to b.*

Proof. Since our triangle is a Heron triangle, its semi-perimeter $s = \frac{2a+b}{2}$ is an integer. Hence, b must be even. So, $b_1 = \frac{b}{2}$ is an integer. As $\triangle = \frac{bh}{2}$ is the area of the triangle, which is an integer, the height h must be a rational number. Also $h^2 = a^2 - b_1^2$ is an integer. Hence, h must be an integer.

Now, to prove the proposed statement, we need to show that $\frac{b_1 h}{2}$ is an integer. Note that we have the equation $a^2 = h^2 + b_1^2$ and hence one of h, b_1 must be even (being part of a Pythagorean triple). So, $\frac{b_1 h}{2}$ is indeed an integer, hence the result. $\square$

We can give many examples of such triangles, by simply concatenating two copies of the same Pythagorean triangle. The parameters of the new triangle, which can be decomposed along the height $2mn$, will then be $(a, b, c, \Delta) = (m^2 + n^2, m^2 + n^2, 2(m^2 - n^2), 2mn(m^2 - n^2))$, for any integers $m > n$.

Regarding indecomposability, we can prove the following results.

Proposition 17. *Let a, b, c be the lengths of three sides of a Heron triangle, such that a is prime and $a \nmid b + c$. Then one cannot decompose $\triangle ABC$ into two Heron triangles $\triangle ABD$ and $\triangle ADC$.*

Proof. Let the side BC of $\triangle ABC$ be decomposed into BD and DC where the length of BD is the integer x. Let the length of AD be y. We refer to Figure 2 for an illustration of such a case.

Now, from $\triangle ABC$, we have $\cos B = \frac{c^2+a^2-b^2}{2ac}$, and from $\triangle ABD$, we have $\cos B = \frac{c^2+x^2-y^2}{2cx}$. So, we get $\frac{c^2+a^2-b^2}{2ac} = \frac{c^2+x^2-y^2}{2cx}$, from which we obtain

$$y^2 = c^2 + x^2 - ax - \frac{x(c-b)(c+b)}{a}.$$

Since a is prime, $a \nmid (b+c)$, and $x < a$, we obtain that a must be a divisor of $c - b$. If $b \neq c$, then we must have $a \leq c - b$, which contradicts the triangle inequality. If

$b = c$, then a would have to be even (and prime), that is, $a = 2$. Then $x = 2 - x = 1$, and y would be a non integer height. The proof is done. $\square$

An example of such a Heron triangle has parameters $(a, b, c, \Delta) = (13, 14, 15, 84)$ (where 13 does not divide 14+15), which cannot be decomposed along the base of length 13 (but in this case, it can be decomposed along the height corresponding to the side 14). Another example has parameters $(5, 29, 30, 72)$, and it satisfies our conditions on two sides $5 \nmid 29 + 30,\ 29 \nmid 5 + 30$).

From Proposition 17, we can derive the following two observations.

Remark 18. The above proof shows in fact that if the Heron triangle is isosceles, with $b = c$, then, we only need the base a to be prime, and not the divisibility condition.

Corollary 19. *Let $\triangle ABC$ be a Heron triangle of sides a, b, c and area Δ, such that a is a prime number and $a \nmid \Delta$. Then one cannot decompose $\triangle ABC$ into two Heron triangles $\triangle ABD$ and $\triangle ADC$.*

Proof. Suppose that one can decompose $\triangle ABC$ into two Heron triangles $\triangle ABD$ and $\triangle ADC$. Then, by Proposition 17, we must have $a|b + c$, and hence $a^2|(b + c)^2 - a^2 = (b + c + a)(b + c - a)$. Now, the area Δ satisfies

$$16\Delta^2 = (a + b + c)(a + c - b)(a + b - c)(b + c - a),$$

and hence $a^2|16\Delta^2$, which implies $a|\Delta$, since $a > 2$ is prime (no side of a Heron triangle can be 2). That is a contradiction and the proof is done. $\square$

From Proposition 17, it is clear that $a|b+c$ is a necessary condition for decomposability of $\triangle ABC$ on the side BC. However, this condition is not at all a sufficient condition for such a decomposition. In this direction, we prove the following result.

Proposition 20. *Suppose that a Heron triangle $\triangle ABC$ (as in Figure 2), of sides a, b, c is decomposed into two triangles $\triangle ABD$ and $\triangle ADC$, with $BD = x$. Assume that c, x are odd. If $a|(b + c)$ and $a + (c - b)\frac{c+b}{a} \equiv 4 \pmod 8$, then $\triangle ABD$ can not be a Heron triangle.*

Proof. Suppose that $\triangle ABC$ could be decomposed into $\triangle ABD$ and $\triangle ADC$, both of which are Heron triangles. In such a case, the relation

$$y^2 = c^2 + x^2 - \frac{x}{a} \cdot (a^2 + c^2 - b^2) \tag{4}$$

holds with x, y integers. Note that $\triangle ABD$ is a Heron triangle and two of its sides $AB = c, BD = x$ are odd. Hence, $AD = y$ must be even, and thus $y^2 \equiv 0, 4 \pmod 8$ in the LHS of (4). On the RHS of (4), $c^2 + x^2 \equiv 2 \pmod 8$ as a, x are odd. Thus, for the relation to be true, we require $\frac{x}{a}(a^2 + c^2 - b^2)$ to be equal to

$\pm 2 \pmod 8$. Since x is odd, one requires $\frac{1}{a}(a^2+c^2-b^2) \equiv \pm 2 \pmod 8$, which contradicts the given condition $\frac{1}{a}(a^2+c^2-b^2) \equiv 4 \pmod 8$. Hence $\triangle ABD$ can not be a Heron triangle. □

4. Conclusion

In this paper, we estimate the number $H(a,b)$ of Heron triangles with two fixed sides a, b. We also investigate $H(p,q)$ when the sides p, q are fixed primes, and provide slightly better results compared to [10]. In particular, we prove that $H(p,q)$ is less than or equal to 1 when $p - q \equiv 2 \pmod 4$. We also count Heron triangles with a fixed height and provide an estimate of the number of Heron triangles with a fixed prime height. Moreover, we study the decomposability property of a Heron triangle into two similar ones, and provide some cases where a Heron triangle is not decomposable.

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