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THE DIOPHANTINE EQUATION $F_n^y + F_{n+1}^x = F_m^x$

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Abstract

Here, we find all the solutions of the title Diophantine equation in positive integer variables (m, n, x, y), where F_k is the k–th term of the Fibonacci sequence.

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_0 = 0,\ F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. The Diophantine equation

$$F_n^x + F_{n+1}^x = F_m \tag{1}$$

in positive integers (m, n, x) was studied in [7]. There, it was showed that there exists no solution other than $(m, n) = (3, 1)$ for which $1^x + 1^x = 2$ (valid for all positive integers x), and the solutions for $x = 1$ and $x = 2$ arising via the formulas $F_n + F_{n+1} = F_{n+2}$ and $F_n^2 + F_{n+1}^2 = F_{2n+1}$. Equation (1) was revisited in [6] under the more general form

$$F_n^x + F_{n+1}^x = F_m^y \tag{2}$$

in positive integers (m, n, x, y) and it was shown that the only solutions of equation (2) with $y > 1$ are $(m, n, x, y) = (3, 4, 1, 3),\ (4, 2, 3, 2)$. Here, we reverse the role of two exponents in equation (2) and study the equation

$$F_n^x + F_{n+1}^y = F_m^x \qquad \text{or} \qquad F_n^y + F_{n+1}^x = F_m^x \tag{3}$$

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in positive integers (m, n, x, y). Our result is the following.

Theorem 1. *The only positive integer solution* (m, n, x, y) *of one of equations* (3) *with* $n \geq 3$ *and* $x \neq y$ *is* $(5, 3, 2, 4)$ *for which* $F_3^4 + F_4^2 = F_5^2$.

We note that the solutions of equation (3) either with $n \in \{1, 2\}$ or $x = y$ are contained in the solutions of equation (2) and therefore are of no interest.

Before getting to the proof, we mention that similar looking equations have already been studied. For example, in [4], it was shown that the only solution in positive integers (k, ℓ, n, r) of the equation

$$F_1^k + F_2^k + \cdots + F_{n-1}^k = F_{n+1}^\ell + \cdots + F_{n+r}^\ell$$

is $(k, \ell, n, r) = (8, 2, 4, 3)$, while in [9], T. Miyazaki showed that the only positive integer solutions (x, y, z, n) of the equation

$$F_n^x + F_{n+1}^y = F_{2n+1}^z$$

are for $(x, y, z) = (2, 2, 1)$ (and for all positive integers n).

2. Preliminary Results

We write $(\alpha, \beta) = ((1 + \sqrt{5})/2, (1 - \sqrt{5})/2)$ and use the Binet formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{valid for all} \quad n \geq 0. \tag{4}$$

We also use the inequality

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \quad \text{valid for all} \quad n \geq 1. \tag{5}$$

We will need the following elementary inequality.

Lemma 1. *For* $n \geq 3$, *we have* $F_n^5 \geq F_{n+1}^3$.

Proof. The inequality is clearly true for $n = 3$, so we assume that $n \geq 4$. Observe that $F_{n+1}/F_n \leq 5/3$, since the above inequality is equivalent to $3F_{n+1} \leq 5F_n$, or $3(F_n + F_{n-1}) \leq 5F_n$, or $3F_{n-1} \leq 2F_n$, further with $3F_{n-1} \leq 2(F_{n-1} + F_{n-2})$, or $F_{n-1} \leq 2F_{n-2}$, or $F_{n-2} + F_{n-3} \leq 2F_{n-2}$, or $F_{n-3} \leq F_{n-2}$, which is clearly true for $n \geq 4$. Thus,

$$\left(\frac{F_{n+1}}{F_n}\right)^3 \leq \left(\frac{5}{3}\right)^3 < 3^2 \leq F_n^2$$

for $n \geq 4$, which is equivalent to $F_{n+1}^3 \leq F_n^5$. □

We shall need a couple of results from the theory of lower bounds for nonzero linear forms in complex and p-adic logarithms which we now recall.

For an algebraic number η we write $h(\eta)$ for its logarithmic height whose formula is

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\eta^{(i)}|, 1\}\right)\right),$$

with d being the degree of η over $\mathbb{Q}$ and

$$f(X) = a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X] \tag{6}$$

being the minimal primitive polynomial over the integers having positive leading coefficient a_0 and η as a root.

With this notation, Matveev (see [8] or Theorem 9.4 in [1]) proved the following deep theorem:

Theorem 2. *Let $\mathbb{K}$ be a real number field of degree D over $\mathbb{Q}$, $\gamma_1, \ldots, \gamma_t$ be nonzero elements of $\mathbb{K}$, and $b_1, \ldots, b_t$ be nonzero integers. Put*

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$\Lambda = \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1.$$

Let $A_1, \ldots, A_t$ be real numbers such that

$$A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}, \quad i = 1, \ldots, t.$$

Then, assuming that $\Lambda \neq 0$, we have

$$|\Lambda| > \exp\left(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_t\right).$$

We shall also need the rational case version of a linear form in p-adic logarithms proved by Kunrui Yu [10]. For a nonzero rational number r and a prime number p put $\operatorname{ord}_p(r)$ for the exponent of p in the factorization of r.

Theorem 3. *Let $\gamma_1, \ldots, \gamma_t$ be nonzero rational numbers and $b_1, \ldots, b_t$ be nonzero integers. Put*

$$B \geq \max\{|b_1|, \ldots, |b_t|, 3\},$$

and

$$\Lambda = \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1.$$

Let $A_1, \ldots, A_t$ be real numbers such that

$$A_i \geq \max\{h(\gamma_i), \log p\}, \quad i = 1, \ldots, t.$$

Then, assuming that $\Lambda \neq 0$, we have

$$\operatorname{ord}_p(\Lambda) < 19(20\sqrt{t+1})^{2t+2} \frac{p}{(\log p)^2} \log(e^5 t) A_1 \cdots A_t \log B.$$

3. The Proof of Theorem 1

3.1. Inequalities Among the Variables m, n and x, y

We start with the following lemma.

Lemma 2. *In any positive integer solution (m, n, x, y) of either one of equations (3) with $n \geq 3$ and $x \neq y$, we have:*

(i) $m > n$;

(ii) $x < y$;

(iii) $m \geq 5$;

(iv) $y(n+1) > (m-2)x$ *and* $(n-2)y < (m-1)x$.

Proof. (i) Either one of equations (3) implies that $F_m^x > F_n^x$, therefore $m > n$.

(ii) Let us now show that $x < y$. Assuming otherwise, we have that

$$F_m^x < F_n^x + F_{n+1}^x < (F_n + F_{n+1})^x = F_{n+2}^x,$$

therefore $m < n+2$. The case $m \in \{n, n+1\}$ is impossible because F_n and F_{n+1} are coprime, so we get $m < n$, contradicting (i). (iii) Since $m > n$ by (i) and the fact that F_n is coprime to F_{n+1}, we deduce in fact that $m > n+1$, and since $n \geq 3$, we get that $m \geq 5$.

(iv) This follows from (ii) and inequalities (5). More precisely,

$$\begin{aligned} \alpha^{(n+1)y} &> F_{n+2}^y = (F_n + F_{n+1})^y > \max\{F_n^x + F_{n+1}^y, F_n^y + F_{n+1}^x\} \\ &\geq F_m^x > \alpha^{(m-2)x}, \end{aligned}$$

implying the first inequality (iv), and

$$\alpha^{(n-2)y} < F_n^y < \min\{F_n^x + F_{n+1}^y, F_n^y + F_{n+1}^x\} \leq F_m^x < \alpha^{(m-1)x},$$

implying the second inequality (iv). □

3.2. Bounding y in Terms of m

Lemma 3. *Any positive integer solution (m, n, x, y) with $n \geq 3$ and $x \neq y$ of equation (3) satisfies one of the following inequalities*

(i) $y < 2 \times 10^{13} m^2 \log m$ *if* $y \leq 2x$;

(ii) $y < 10^{13} m \log m$ *if* $y > 2x$.

Proof. We distinguish the following two cases.

Case 1. $y \leq 2x$. In this case, we apply a linear form in 2-adic logarithms upon observing that exactly of $F_n,\ F_{n+1},\ F_m$ is even. The linear form is of the form

$$\Lambda = F_a^u F_b^{-v} - 1,$$

where a and b are distinct in $\{n, n+1, m\}$ such that F_a and F_b are odd, and u and v are in $\{2x, 2y\}$. In any case, if c is such that $\{a, b, c\} = \{n, n+1, m\}$ then it is always the case that F_c is even and $F_c^x \mid F_a^u - F_b^v$, therefore

$$\mathrm{ord}_2(\Lambda) \geq \mathrm{ord}_2(F_c^x) \geq x \geq y/2. \tag{7}$$

To get an upper bound on $\mathrm{ord}_2(\Lambda)$, we use Theorem 3. We take the parameters $t = 2,\ \gamma_1 = F_a,\ \gamma_2 = F_b,\ b_1 = u,\ b_2 = -v$. We can take $B = 2y$. Since $n + 1 < m$, by inequalities (5), we can take

$$A_1 = A_2 = m \log \alpha > \max\{\log F_a, \log F_b, \log 2\}.$$

Theorem 3 now gives

$$\mathrm{ord}_2(\Lambda) \leq 19(20\sqrt{3})^6 \left(\frac{2}{(\log 2)^2}\right) \log(2e^5)(m \log \alpha)^2 \log(2y), \tag{8}$$

which compared with (7) gives

$$\begin{aligned} 2y &\leq 4 \times 19 \times (20\sqrt{3})^6 \left(\frac{2}{(\log 2)^2}\right) \log(2e^5)(\log \alpha)^2 \log(2y) \\ &< 8 \times 10^{11} m^2 \log(2y). \end{aligned}$$

Using the fact that for $A > 3$ the inequality

$$t < A \log t \qquad \text{implies} \qquad t < 2A \log A$$

(with $A = 8 \times 10^{11} m^2$), we have

$$2y < 2 \times 8 \times 10^{11} m^2 (\log(8 \times 10^{11}) + 2 \log m) < 2 \times 8 \times 10^{11} m^2 (20 \log m),$$

therefore

$$y < 2 \times 10^{13} m^2 \log m, \tag{9}$$

which takes care of (i). In the above inequalities we also used the fact that

$$\log(8 \times 10^{11}) + 2 \log m < 20 \log m,$$

which holds because $m \geq 5$.

Case 2. $y > 2x$. In this case, we use a linear form in complex logarithms. This linear form is one of

$$\Lambda = F_m^x F_{n+1}^{-y} - 1 \qquad \text{or} \qquad F_m^x F_n^{-y} - 1$$

depending on whether we work with the left equation (3) or with the right equation (3), respectively. Clearly, $\Lambda > 0$. We first find an upper bound on Λ which follows from equation (3). In case of the left equation (3), we have

$$\Lambda = \frac{F_n^x}{F_{n+1}^y} < \frac{F_{n+1}^x}{F_{n+1}^y} < \frac{1}{F_{n+1}^{y/2}}. \tag{10}$$

In case of the right equation (3), we have, by Lemma 1,

$$\Lambda = \frac{F_{n+1}^x}{F_n^y} < \frac{F_{n+1}^x}{F_{n+1}^{3y/5}} < \frac{1}{F_{n+1}^{3y/5-y/2}} = \frac{1}{F_{n+1}^{y/10}}. \tag{11}$$

So, from (10) and (11), we get that the inequality

$$\Lambda < \frac{1}{F_{n+1}^{y/10}} \tag{12}$$

holds in all instances. We now find a lower bound on Λ by using Theorem 2. We take $t = 2$, $\gamma_1 = F_m$, $\gamma_2 = F_u$ with $u \in \{n, n+1\}$, $b_1 = x$, $b_2 = -y$. We take $\mathbb{K} = \mathbb{Q}$, so $D = 1$. We take $B = y$. By inequality (5), we can take $A_1 = m \log \alpha$ and $A_2 = \log F_{n+1}$. We then get that

$$\Lambda > \exp\left(-1.4 \times 30^5 \times 2^{4.5} \times (m \log \alpha) \times \log F_{n+1} \times (1 + \log y)\right),$$

which together with (12) gives

$$(y/10) \log F_{n+1} < 1.4 \times 30^5 \times 2^{4.5} \times (m \log \alpha) \times \log F_{n+1} \times (1 + \log y),$$

or

$$y < 14 \times 30^5 \times 2^{4.5} \times \log \alpha \times m \times (3 \log y) < 2 \times 10^{11} m \log y,$$

where we used the inequality $1 + \log y < 3 \log y$, which holds for all $y \geq 2$. Thus,

$$y < 4 \times 10^{11} m (\log(2 \times 10^{11}) + \log m) < 4 \times 10^{11} (20 \log m) < 10^{13} m \log m,$$

where we used the fact that $\log(2 \times 10^{11}) + \log m < 20 \log m$ for all $m \geq 5$. This takes care of (ii). $\square$

3.3. Small m

Lemma 4. *If* $(m,n,x,y) \neq (5,3,2,4)$ *is a positive integer solution of equation* (3) *with* $n \geq 3$ *and* $x \neq y$*, then* $m \geq 1000$.

Proof. Assume that we work with the left equation (3). Then

$$F_{n+1}^{y} = F_m^x - F_n^x. \tag{13}$$

Assume first that $y \geq 20$. Observe that from the above equation we get that $F_m - F_n$ is a divisor of F_{n+1}. Let $D_{m,n} = \gcd(F_m - F_n, F_{n+1})$. We first checked computationally that there is no pair (m,n) with $6 \leq n+3 < m \leq 999$, such that $p^{20} \mid F_m - F_n$ for some prime factor p of $D_{m,n}$. It follows that all prime factors of $F_m - F_n$ appear in its factorization at powers smaller than 20. But if that is so, it should be the case that $D_{m,n}^{20}$ is divisible by $F_m - F_n$. We checked computationally that this is not the case for any such pair (m,n). The conclusion of this computation is that $m \in \{n+2, n+3\}$. Now

$$F_{n+2} - F_n = F_{n+1} \qquad \text{and} \qquad F_{n+3} - F_n = 2F_{n+1}.$$

Together with formula (8), we get that $F_m^x - F_n^x = F_{n+1}^y$ is divisible by exactly the same primes as $F_m - F_n$. By Carmichael's Primitive Divisor Theorem (see [3]) for Lucas sequences with coprime integer roots, we get that $x \leq 6$. So,

$$F_m^x - F_n^x < F_m^x \leq F_{n+3}^6 < (2F_{n+2})^6 < (4F_{n+1})^6 = 2^{12} F_{n+1}^6 < F_{n+1}^{20} \leq F_{n+1}^y,$$

a contradiction. This calculation shows that $1 \leq x < y \leq 19$. We tested the remaining range $1 \leq x < y \leq 19$ and $3 \leq n < m \leq 999$ by brute force and no solution came up. A similar argument works for the right equation (3) with one exception. Namely, in the case when $5 \leq n+2 < m \leq 999$, by putting $D_{m,n} = \gcd(F_m - F_{n+1}, F_n)$ computations revealed that, as before, $p^{20} \nmid F_m - F_n$ for any prime $p \mid D_{m,n}$ and any such pair (m,n), but the pair $(m,n) = (14,8)$ has the property that D_{mn}^{20}, is a multiple of $F_m - F_n$ and is the only such pair. Namely, in this case $F_m - F_n = F_{14} - F_9 = 343 = 7^3$, and $F_8 = 21 = 3 \times 7$. In this last case however, again by Carmichael's Primitive Divisor Theorem, $F_{14}^x - F_9^x$ should have a prime factor $p \equiv 1 \pmod{x}$ if $x > 6$ which does not divide $F_{14} - F_8$, but this is not the case if $x > 6$ since $F_{14}^x - F_9^x = F_8^y = 3^y \times 7^y$. Hence, again $x \leq 6$, and we get a contradiction because $y \geq 20$. This shows that, as for the case of the left equation (3), we must have $1 \leq x < y \leq 19$. Again we tested this remaining range by brute force and only the solution $(5,3,2,4)$ of the right equation (3) showed up. The lemma is therefore proved. ∎

3.4. Approximating F_m^x

From now on, we assume that $m \geq 1000$.

Lemma 5. *If (m,n,x,y) is a positive integer solution of equation (3) with $n \geq 3$ and $x \neq y$, then*

$$F_m^x = \frac{\alpha^{mx}}{5^{x/2}}\left(1+\zeta_{m,x}\right), \qquad \textit{where} \qquad |\zeta_{m,x}| < \frac{2}{\alpha^m}.$$

Proof. We use the Binet formula (4) to get

$$F_m^x = \frac{\alpha^{mx}}{5^{x/2}}\left(1-\left(\frac{\beta}{\alpha}\right)^m\right)^x = \frac{\alpha^{mx}}{5^{x/2}}\left(1-\frac{(-1)^m}{\alpha^{2m}}\right)^x. \tag{14}$$

Observe that, by Lemmas 2 and 3, we have

$$\frac{x}{\alpha^{2m}} < \frac{y}{\alpha^{2m}} < \frac{2\times 10^{13} m^2 \log m}{\alpha^{2m}} < \frac{1}{\alpha^m},$$

where the last inequality holds for all $m \geq 86$. Thus, if m is odd, then

$$\begin{aligned} 1 \quad &< \quad \left(1-\frac{(-1)^m}{\alpha^{2m}}\right)^x = \left(1+\frac{1}{\alpha^{2m}}\right)^x < \exp\left(\frac{x}{\alpha^{2m}}\right) \\ &< \quad \exp\left(\frac{1}{\alpha^m}\right) < 1+\frac{2}{\alpha^m}, \end{aligned} \tag{15}$$

where we also used the fact that $\exp(t) < 1+2t$ if $t \in (0,1)$. Similarly, when m is even, using the fact that $1-t > \exp(-2t)$ holds for $t \in (0,1/2)$, we have

$$\begin{aligned} 1 \quad &> \quad \left(1-\frac{(-1)^m}{\alpha^{2m}}\right)^x = \left(1-\frac{1}{\alpha^{2m}}\right)^x > \exp\left(-\frac{2x}{\alpha^{2m}}\right) \\ &> \quad \exp\left(-\frac{2}{\alpha^m}\right) > 1-\frac{2}{\alpha^m}. \end{aligned} \tag{16}$$

From estimates (15) and (16), we deduce that in both cases m odd and m even we have

$$\left(1+\frac{(-1)^m}{\alpha^{2m}}\right)^x = 1+\zeta_{m,x}, \qquad \text{with} \qquad |\zeta_{m,x}| < \frac{2}{\alpha^m},$$

which together with formula (14) finishes the proof of this lemma. □

3.5. Approximating F_a^u for $a \in \{n, n+1\}$, $u \in \{x,y\}$ and Large n

Lemma 6. *If (m,n,x,y) is a positive integer solution of equation (3) with $n \geq 3$, $x \neq y$ and $2x \geq y$, then the estimates*

$$F_a^u = \frac{\alpha^{au}}{5^{u/2}}\left(1+\zeta_{a,u}\right), \qquad \textit{where} \qquad |\zeta_{a,u}| < \frac{2}{\alpha^n}, \tag{17}$$

hold for $a \in \{n, n+1\}$ and $u \in \{x,y\}$.

Proof. The proof is based on inequality (iv) of Lemma 2, which in the particular case $y \le 2x$ implies

$$y(n+1) > (m-2)x \ge \frac{(m-2)y}{2}, \qquad \text{so} \qquad n > \frac{m-2}{2} - 1 = \frac{m}{2} - 2.$$

Now for $a \in \{n, n+1\}$ and $u \in \{x, y\}$, we have, by the Binet formula (4),

$$F_a^u = \frac{\alpha^{au}}{5^{u/2}} \left(1 - \frac{(-1)^a}{\alpha^{2a}}\right)^u.$$

Observe that, by Lemma 4,

$$\frac{u}{\alpha^{2a}} \le \frac{y}{\alpha^{2n}} \le \frac{2 \times 10^{13} m^2 \log m}{\alpha^{2n}} \le \frac{1}{\alpha^n},$$

where the last inequality is implied by

$$\alpha^n \ge \alpha^{m/2-2} \ge 2 \times 10^{13} m^2 \log m,$$

which holds for all $m \ge 182$. The conclusion of the lemma follows as in the proof of Lemma 5. □

3.6. A Small Linear Form in α and $\sqrt{5}$

Lemma 7. *If* (m, n, x, y) *is a positive integer solution to equation* (3) *with* $n \ge 3$ *and* $x \ne y$ *such that inequalities* (17) *hold, then putting* $\lambda = \min\{n, (m-n-1)y\}$, *we have*

$$\left|1 - \alpha^{ay-mx} 5^{(x-y)/2}\right| < \frac{13}{\alpha^\lambda} \qquad \textit{for some} \qquad a \in \{n, n+1\}. \tag{18}$$

Proof. By Lemma 5, we have

$$\left|F_m^x - \frac{\alpha^{mx}}{5^{x/2}}\right| < \frac{2}{\alpha^m}\left(\frac{\alpha^{mx}}{5^{x/2}}\right).$$

Since $m > n \ge 3$ and $\alpha^m > \alpha^n = 2\alpha + 1 = 2 + \sqrt{5} > 4$, it follows that $2/\alpha^m < 1/2$, therefore the above estimate implies that

$$\left|F_m^x - \frac{\alpha^{mx}}{5^{x/2}}\right| < \frac{1}{2}\left(\frac{\alpha^{mx}}{5^{x/2}}\right), \qquad \text{so} \qquad \frac{1}{2}\left(\frac{\alpha^{mx}}{5^{x/2}}\right) < F_m^x < \frac{3}{2}\left(\frac{\alpha^{mx}}{5^{x/2}}\right). \tag{19}$$

In particular,

$$\left|F_m^x - \frac{\alpha^{mx}}{5^{x/2}}\right| < \left(\frac{4}{\alpha^m}\right) F_m^x.$$

Since we are assuming that estimates (17) hold, we get, by a similar argument, that the estimates

$$\left|F_a^u - \frac{\alpha^{au}}{5^{u/2}}\right| < \left(\frac{4}{\alpha^n}\right) F_a^u \qquad \text{hold with} \qquad a \in \{n, n+1\},\ u \in \{x, y\}.$$

Thus, in the case of the left equation (3), we get

$$\begin{aligned}\left|\frac{\alpha^{mx}}{5^{x/2}}-\frac{\alpha^{nx}}{5^{x/2}}-\frac{\alpha^{(n+1)y}}{5^{y/2}}\right| &\le \left|\frac{\alpha^{mx}}{5^{x/2}}-F_m^x\right|+\left|\frac{\alpha^{nx}}{5^{x/2}}-F_n^x\right|+\left|\frac{\alpha^{(n+1)y}}{5^{y/2}}-F_{n+1}^y\right| \\ &\le \left(\frac{4}{\alpha^m}\right)F_m^x+\left(\frac{4}{\alpha^n}\right)(F_n^x+F_{n+1}^y) \\ &< \left(\frac{8}{\alpha^n}\right)F_m^x<\left(\frac{12}{\alpha^n}\right)\left(\frac{\alpha^{mx}}{5^{x/2}}\right),\end{aligned}$$

so that

$$\left|1-\alpha^{(n+1)y-mx}5^{(x-y)/2}\right|<\frac{1}{\alpha^{(m-n)x}}+\frac{12}{\alpha^n}\le\frac{13}{\alpha^{\min\{n,(m-n)x\}}}. \tag{20}$$

In the case of the right equation (3), a similar argument gives

$$\left|\frac{\alpha^{mx}}{5^{x/2}}-\frac{\alpha^{ny}}{5^{y/2}}-\frac{\alpha^{(n+1)x}}{5^{x/2}}\right|<\left(\frac{12}{\alpha^n}\right)\left(\frac{\alpha^{mx}}{5^{x/2}}\right),$$

leading to the similar looking inequality as (20), namely

$$\left|1-\alpha^{ny-mx}5^{(x-y)/2}\right|<\frac{13}{\alpha^{\min\{n,(m-n-1)x\}}}, \tag{21}$$

which together with (20) finishes the proof of this lemma. □

3.7. An Upper Bound for λ

Lemma 8. *If (m,n,x,y) is a positive integer solution of equation (3) with $n\ge 3$ and $x\ne y$ such that inequality (18) holds, then*

$$\lambda<5\times 10^{10}\log y. \tag{22}$$

Proof. We put

$$\Lambda=1-\alpha^{ay-mx}5^{(x-y)/2},\qquad \text{where}\qquad a\in\{n,n+1\}$$

is the expression appearing in the left hand side of the inequality (18) from Lemma 7. Since α and 5 are multiplicatively independent, and $x\ne y$, it follows that $\Lambda\ne 0$. Lemmas 6 and 7 show that

$$\log|\Lambda|\le\log 13-\lambda\log\alpha. \tag{23}$$

We next find a lower bound on $\log|\Lambda|$. For this, we use Theorem 2 with the choices $t=2$, $\mathbb{K}=\mathbb{Q}(\sqrt{5})$, $\gamma_1=\alpha$, $\gamma_2=\sqrt{5}$, $b_1=ay-mx$ and $b_2=x-y$. We have $D=2$. Since $b_2<0$ and

$$|\alpha^{b_1}5^{(x-y)/2}-1|<\frac{13}{\alpha^\lambda}\le\frac{13}{\alpha}<\alpha^5-1,$$

we have

$$\alpha^{b_1} 5^{(x-y)/2} < \alpha^5, \quad \text{so} \quad b_1 \le (y-x)\frac{\log\sqrt{5}}{\log\alpha} + 5 < 2(y-x) + 5 \le 2y + 3.$$

Hence, we can take $B = 2y + 3$. We can also take $A_1 = \log\alpha$ and $A_2 = \log 5$. Theorem 2 now tells us that

$$\log|\Lambda| \ge -1.4 \times 30^5 \times 2^{4.5} \times 2^2 (1 + \log 2)(1 + \log(2y+3))(\log\alpha)(\log 5). \tag{24}$$

Putting together inequalities (23) and (24), we get

$$\lambda \log\alpha - \log 13 < 1.4 \times 30^5 \times 2^{4.5} \times 2^2 (1 + \log 2)(1 + \log(2y+3))(\log\alpha)(\log 5),$$

or

$$\lambda < \frac{\log 13}{\log\alpha} + 1.4 \times 30^5 \times 2^{4.5} \times 2^2 (1 + \log 2)(1 + \log(2y+3))(\log 5).$$

Since $1 + \log(2y+3) \le 1 + \log(4y) \le 5\log y$ for all $y \ge 2$, the above inequality gives

$$\lambda < \frac{\log 13}{\log\alpha} + 1.4 \times 30^5 \times 2^{4.5} \times 2^2 (1 + \log 2)(\log 5)(5\log y) < 5 \times 10^{10} \log y, \tag{25}$$

which finishes the proof of this lemma. □

3.8. The Case When $x < y \le 2x$

Lemma 9. *Equation* (3) *has no positive integer solution* (m, n, x, y) *with* $n \ge 3$ *and* $x < y \le 2x$.

Proof. We exploit the conclusion of Lemma 8. We use the fact that

$$\lambda = \min\{(m-n-1)x, n\} \ge \min\{y/2, m/2 - 2\}. \tag{26}$$

If the minimum on the right above is $y/2$, then inequality (25) gives

$$y < 10^{11} \log y \qquad \text{giving} \qquad y < 2 \times 10^{11} \times \log(10^{11}) < 6 \times 10^{12},$$

so

$$B < 1.5 \times 10^{13}.$$

If on the other hand the minimum in (26) is $m/2-2$, we then get, using also Lemma 2, that

$$\begin{aligned} m/2 - 2 &< 5 \times 10^{10} \log(2 \times 10^{13} m^2 \log m) \\ &< 5 \times 10^{10} (\log(2 \times 10^{13}) + 3\log m) \\ &< 5 \times 10^{10} \times (23 \log m), \end{aligned}$$

where we used the fact that $\log(2\times 10^{13})+3\log m < 30+3\log m < 23\log m$ for $m\ge 5$ (in fact, $m\ge 1000$, so a slightly better inequality holds at this step). Hence,

$$m/2-2 < 105\times 10^{10}\log m \qquad \text{giving} \qquad m < 3\times 10^{12}\log m.$$

This last inequality leads to

$$m < 2\times 3\times 10^{12}\log(3\times 10^{12}) < 2\times 10^{14},$$

so that

$$B = 2y+3 \le 3+2\times 2\times 10^{13}m^2\log m < 10^{44}.$$

Suppose now that $\lambda > 10$. Then $13/\alpha^\lambda < 1/2$, and so inequality (18) implies by a standard argument

$$|(ay-mx)\log\alpha-(y-x)\log\sqrt{5}| < \frac{26}{\alpha^\lambda},$$

or

$$\left|\frac{ay-mx}{y-x}-\frac{\log\sqrt{5}}{\log\alpha}\right| < \frac{26}{(\log\alpha)(y-x)\alpha^\lambda} < \frac{55}{(y-x)\alpha^\lambda}. \qquad (27)$$

Let $[a_0,a_1,\ldots,a_{99}] = p_{99}/q_{99}$ be the 99th convergent of $\eta = (\log\sqrt{5})/\log\alpha$. The maximal a_i for $i=0,\ldots,99$ is $a_{20}=29$. Furthermore, we also have $q_{99} > 10^{48} > B$. Hence,

$$\left|\frac{ay-mx}{y-x}-\frac{\log\sqrt{5}}{\log\alpha}\right| > \frac{1}{(29+2)(y-x)^2} = \frac{1}{31(y-x)^2}. \qquad (28)$$

Thus, we get, from inequalities (27) and (28),

$$\frac{1}{31(y-x)^2} < \frac{54}{(y-x)\alpha^\lambda}$$

giving

$$\alpha^\lambda \le 54\times 31\times (y-x) < 2000\times B < 2\times 10^{47},$$

leading to

$$\lambda \le \frac{\log(2\times 10^{47})}{\log\alpha} < 227.$$

If $\lambda = n$, we then get that $m/2-2\le n < 227$, so $m < 458$, contradicting the fact that $m\ge 1000$. If $\lambda = (m-n-1)x$, then $(m-n-1)x < 227$. In particular, $x < 227$ and $(m-n)x \le 2(m-n-1)x < 454$. Further, inequality (27) shows that

$$\frac{ay-mx}{y-x} < \frac{\log\sqrt{5}}{\log\alpha}+\frac{55}{(y-x)\alpha} < 2+\frac{34}{y-x},$$

so

$$ay-mx < 2(y-x)+34.$$

If $a = n$, then

$$ay - mx = ny - mx = n(y-x) - (m-n)x < 2(y-x) + 34,$$

so

$$n \leq \frac{(m-n)x}{y-x} + 2 + \frac{34}{y-x} < 454 + 2 + 34 = 490,$$

therefore $m \leq (m-n)+n < 454+490 = 944$, contradicting the fact that $m \geq 1000$. If $a = n+1$, then

$$ay - mx = (n+1)y - mx = (n+1)(y-x) - (m-n-1)x < 2(y-x) + 34,$$

so

$$n+1 < \frac{(m-n-1)x}{y-x} + 2 + \frac{34}{y-x} < 227 + 2 + 34 = 263,$$

so $m \leq (m-n) + n < 454 + 262 = 716$, contradicting the fact that $m \geq 1000$. □

3.9. A Small Linear Form in Three Logarithms

From now on, we assume that $y > 2x$.

Lemma 10. *Any positive integer solution (m, n, x, y) of equation (3) with $n \geq 3$ and $x \neq y$ satisfies*

$$\left| F_a^y \alpha^{-mx} 5^{x/2} - 1 \right| < \frac{4}{\alpha^{\mu}} \quad \text{for some} \quad a \in \{n, n+1\}, \tag{29}$$

where $\mu = \min\{m, (m-n-2)x\}$.

Proof. Suppose that we work with the left equation (3). Then, by Lemma 5, we have

$$F_n^x + F_{n+1}^y = F_m^x = \frac{\alpha^{mx}}{5^{x/2}} \left(1 + \zeta_{m,x}\right),$$

so

$$\left| F_{n+1}^y \alpha^{-mx} 5^{x/2} - 1 \right| \leq |\zeta_{m,x}| + \frac{F_n^x}{\alpha^{mx}/5^{x/2}}. \tag{30}$$

Estimate (19) gives

$$\frac{F_n^x}{\alpha^{mx}/5^{x/2}} < \frac{3}{2} \left(\frac{F_n^x}{F_m^x} \right) < \frac{2}{\alpha^{(m-n-1)x}},$$

where we used inequalities (5) to say that $F_n < \alpha^{n-1}$ and $F_m > \alpha^{m-2}$. Since $|\zeta_{m,x}| < 2/\alpha^m$ by Lemma 5, we get, from inequality (30), that

$$\left| F_{n+1}^y \alpha^{-mx} 5^{x/2} - 1 \right| < \frac{2}{\alpha^m} + \frac{2}{\alpha^{(m-n-1)x}} \leq \frac{4}{\alpha^{\min\{m, (m-n-1)x\}}}, \tag{31}$$

A similar argument applies to the right equation (3). In that case, we get

$$F_n^y + F_{n+1}^x = F_m^x = \frac{\alpha^{mx}}{5^{x/2}}(1+\zeta_{m,x}),$$

therefore

$$\begin{aligned} \left|F_n^y \alpha^{-mx} 5^{x/2} - 1\right| &< |\zeta_{m,x}| + \frac{F_{n+1}^x}{\alpha^{mx}/5^{x/2}} < \frac{2}{\alpha^m} + \frac{3}{2}\left(\frac{F_{n+1}^x}{F_m^x}\right) \\ &< \frac{2}{\alpha^m} + \frac{2}{\alpha^{(m-n-2)x}} \le \frac{4}{\alpha^{\min\{m,(m-n-2)x\}}}, \end{aligned} \tag{32}$$

which together with inequality (31) completes the proof of this lemma. □

Remark. Lemma 2 (iv) shows that

$$(m-1)x > (n-2)y > 2(n-2)x, \quad \text{so} \quad m-1 > 2n-4 \quad \text{so} \quad m \ge 2n-2.$$

In particular, $m - n - 2 \ge (m-6)/2$. This will be useful later.

Lemma 11. *Any positive solution* (m, n, x, y) *of equation* (3) *with* $n \ge 3$ *and* $x \ne y$ *satisfies:*

(i) $n > 10^{-14} m/\log m$*;*

(ii) $n \ge 1000$.

Proof. We put $\Lambda = F_a^y \alpha^{-mx} 5^{x/2} - 1$ for the form that appears in the left hand side of inequality (29). Since $mx > 0$ and no power of α of positive integer exponent can be a rational number, it follows that $\Lambda \ne 0$. Inequality (29) shows that

$$\log|\Lambda| < \log 4 - \mu \log \alpha. \tag{33}$$

We find a lower bound on $\log|\Lambda|$. We use Theorem 2 with the choices of parameters $t = 3$, $\gamma_1 = F_a$, $\gamma_2 = \alpha$, $\gamma_3 = \sqrt{5}$, $b_1 = y$, $b_2 = -mx$, $b_3 = -x$. We have $\mathbb{K} = \mathbb{Q}(\sqrt{5})$ for which $D = 2$. We take $A_1 = 2n \log \alpha$, $A_2 = \log \alpha$, and $A_3 = \log 5$. We take $B = my$. We then have

$$\log|\Lambda| > -1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1+\log 2)(1+\log(my))(2n\log\alpha)(\log\alpha)(\log 5). \tag{34}$$

Comparing estimates (33) and (34), we get

$$\mu < \frac{4}{\log \alpha} + 1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1+\log 2) \times 2 \times (\log\alpha) \times (\log 5) n (2\log(my)).$$

giving

$$\mu < 4 \times 10^{12} n \log(my).$$

With Lemmas 3 and 10 and the remark following Lemma 10, we get

$$\begin{aligned} m-6 &< 8\times 10^{12} n \log(2\times 10^{13} m^4) = 8\times 10^{12} n(\log(2\times 10^{13}) + 4\log m) \\ &< 8\times 10^{12} n \times (9\log m), \end{aligned}$$

where we used the fact that $\log(2\times 10^{13}) + 4\log m < 31 + 4\log m < 9\log m$ for $m \geq 1000$. Thus, we get

$$m < 6 + 8\times 9\times 10^{12} n\log m < 10^{14} n\log m, \tag{35}$$

which leads to (i). For (ii), assuming that $n < 1000$, we get, by inequality (35), that

$$m < 10^{17}\log m \quad \text{therefore} \quad m < 2\times 10^{17}\log(10^{17}) < 10^{19}.$$

By (ii) of Lemma 3, we have

$$B = my < 10^{13} m^2 \log m < 10^{53}.$$

Clearly, since $\mu \geq m/2 - 3 > 10$, it follows that $4/\alpha^{\mu} < 1/2$. A standard argument implies that inequality (29) leads to

$$|y\log F_a - mx\log\alpha + x\log\sqrt{5}| < \frac{8}{\alpha^{\mu}}, \tag{36}$$

where $a \leq n+1 \leq 1000$ and $\max\{y, mx, x\} \leq B < 10^{53}$. However, the minimum of the expression appearing in the left–hand side of inequality (36) even over all the indices $n < 3000$ and coefficients at most 5×10^{65} in absolute value was bounded from below using LLL in Section 6 of [4]. The lower bound there was $100/1.5^{750}$. Hence, we get that

$$\frac{100}{1.5^{750}} < \frac{8}{\alpha^{\mu}}, \qquad \text{therefore} \qquad \mu < 750\left(\frac{\log 1.5}{\log\alpha}\right) - \frac{\log 12.5}{\log\alpha} < 630.$$

Since in fact $\mu = \min\{m, (m-n-1)x\} \geq \min\{m, (m-6)x/2\}$ and $m \geq 1000$, the only possibility is when $\mu = (m-n-2)x$ and $x = 1$. If $y \geq 3$, then, Lemma 2 (iv) shows that

$$m-1 > (n-2)y \geq 3n-6 \quad \text{so} \quad m \geq 3n-4 \quad \text{so} \quad (m-n-2) \geq 2(m-5)/3,$$

implying that $\mu = (m-n-2)x \geq 2(m-5)/3 > 663$, a contradiction with $\mu < 630$. Hence, $y = 2$. Let us see that this is impossible. Suppose that we work with the left equation (3). Then

$$F_m = F_n + F_{n+1}^2 < F_n^2 + F_{n+1}^2 = F_{2n+1}$$

so $F_m < F_{2n+1}$, therefore $m < 2n$. The case $m = 2n$ is not convenient because F_n and F_{n+1} are coprime, so $m \leq 2n-1$, which is impossible because then

$$F_m \leq F_{2n-1} = F_{n-1}^2 + F_n^2 < (F_n + F_{n-1})^2 = F_{n+1}^2 < F_{n+1}^2 + F_n = F_m.$$

Suppose now that we work with the right equation (3). Then

$$F_m = F_n^2 + F_{n+1} < F_n^2 + F_{n-1}^2 = F_{2n-1} \qquad \text{for} \qquad n > 10.$$

The inequality $n > 10$ holds because $m \geq 1000$, and the last inequality above is implied by $F_{n+1} < F_{n-1}^2$, which holds because $F_{n+1} < 2F_n < 4F_{n-1} < F_{n-1}^2$ for $n > 10$. Hence, $m < 2n-1$. The case $m \leq 2n-3$ leads to a contradiction since then

$$F_m \leq F_{2n-3} = F_{n-1}^2 + F_{n-2}^2 < (F_{n-1} + F_{n-2})^2 = F_n^2 < F_n^2 + F_{n+1} = F_m.$$

Finally, the case $m = 2n-2$, gives

$$F_{2n-2} = F_n^2 + F_{n+1} = F_n(F_n + 1) + F_{n-1}.$$

Since $F_{n-1} \mid F_{2n-2}$ and F_{n-1} is coprime to F_n, we get that F_{n-1} is a divisor of $F_n + 1 = (F_{n-2} + 1) + F_{n-1}$, so F_{n-1} divides $F_{n-2} + 1$, which in turn implies that $F_{n-2} + 1 \geq F_{n-1} = F_{n-2} + F_{n-3}$, or $1 \geq F_{n-3}$, which is false for $n > 10$. □

Lemma 12. *Estimates* (17) *hold.*

Proof. As in the proofs of Lemma 5 and 6, it is enough, in light of the Binet formula (4), to show that the inequality

$$y < \alpha^n \tag{37}$$

holds. By Lemma 3 (ii) and Lemma 11 (i), it suffices that the inequality

$$\log(10^{13} m \log m) < 10^{-14} (\log \alpha) m / \log m$$

holds. The above inequality holds for $m > 10^{18}$. On the other hand, if $m \leq 10^{18}$, then again by Lemma 3 (ii) and Lemma 11 (ii), we have

$$y < 10^{13} m \log m < 10^{13}(10^{18}) \log(10^{18}) < 10^{33} < \alpha^{1000} \leq \alpha^n.$$

This finishes the proof of this lemma. □

Lemma 13. *Equation* (3) *has no positive integer solution* (m, n, x, y) *with* $n \geq 3$ *and* $x \neq y$.

Proof. By Lemmas 6 and 12, inequalities (18) hold. Recall $\lambda = \min\{n, (m-n-1)x\}$. Inequality (22) is

$$\lambda < 5 \times 10^{10} \log y.$$

By the remark following Lemma 10, $m-1 > 2n-4$, so $(m-n-1) \geq n-4$. Hence, for us, $\lambda \geq n-4$. By Lemma 11 (i) and 3 (ii), we get

$$10^{-14} m / \log m - 4 < n - 4 \leq \lambda < 5 \times 10^{10} \log y < 5 \times 10^{10} \log(10^{13} m \log m),$$

giving $m < 10^{30}$, so $y < 10^{13} m \log m < 10^{45}$. We thus get inequality (27), which we recall here under the form

$$\left| \frac{ay - mx}{y - x} - \frac{\log \sqrt{5}}{\log \alpha} \right| < \frac{55}{(y-x)\alpha^{996}}.$$

The calculation with the 99th convergent of $\log \sqrt{5}/\log \alpha$ from the proof of Lemma 10 shows that the left hand side of the above inequality is at least $1/(31(y-x)^2)$. So, we get $\alpha^{996} < 55 \times 31(y-x) < 55 \times 31 \times 10^{45} < 10^{50}$, which is absurd. This finishes the proof of the lemma and of the theorem. □

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