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THE USE OF YOUNG MEASURES FOR CONSTRUCTING MINIMIZING SEQUENCES IN THE CALCULUS OF VARIATIONS

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Abstract. The goal of this note is to construct — via the notion of Young measure–minimizing sequences for problems of calculus of variations that do not admit minimizers.

1. Introduction. We denote by Ω a bounded domain of $\mathbf{R}^2$ with boundary Γ. If $W^{1,\infty}(\Omega)$ is the set of Lipschitz continuous functions with values in $\mathbf{R}$, and $W_0^{1,\infty}(\Omega)$ the set of functions in $W^{1,\infty}(\Omega)$ vanishing on the boundary of Ω (see [15] for information on these spaces), we would like to

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consider the following model problems:

$$\inf_{W_0^{1,\infty}(\Omega)} \int_\Omega v_x^2 + (v_y^2 - 1)^2 \; dxdy, \tag{1.1}$$

$$\inf_{W^{1,\infty}(\Omega)} \int_\Omega v_x^2 + (v_y^2 - 1)^2 + v^2 \; dxdy, \tag{1.2}$$

$$\inf_{W_0^{1,\infty}(\Omega)} \int_\Omega v_x^2 + (v_y^2 - x^2)^2 \; dxdy, \tag{1.3}$$

$$\inf_{W^{1,\infty}(\Omega)} \int_\Omega v_x^2 + (v_y^2 - x^2)^2 + v^2 \; dxdy, \tag{1.4}$$

where $v_x,\ v_y$ denote respectively $\partial v/\partial x,\ \partial v/\partial y$. (See [17] for the link with austenite-martensite transformation). The usual issue in the Calculus of Variations is to find a minimizer i.e. a function u for which one of the above infima is achieved. In fact here, all these problems are sharing the same property to have their infima equal to 0 but no minimizer to achieve it. So, there exist minimizing sequences with energy converging toward 0. As we will see, for a given problem all the minimizing sequences have the same pattern. Information about these minimizing sequences can be discovered through the Young measures that they are generating. This is one of the issues that we would like to address here: we will establish that each of the above problems has minimizing sequences defining a unique Young measure. From this we will show how to built corresponding minimizing sequences. Problems of the above types arise in material science. We refer for instance to [9] for a more complex situation. However, the main features of these problems can be carried out in the context of these four simple problems. For more general consideration we refer to [2], [3], [7].

2. Minimizing sequences.

Let us first show:

THEOREM 2.1. *Each of the above infima is equal to 0. None of the above problems admits a minimizer.*

Proof. Let us show that the infimum (1.1) is equal to 0. The fact that the other ones are also equal to 0 will be a consequence of subsequent analysis. Set

$$\varphi(\xi_1, \xi_2) = \xi_1^2 + (\xi_2^2 - 1)^2. \tag{2.1}$$

If φ^{**} denotes the convex envelope of φ i.e. the largest convex function below φ and if $|\Omega|$ denotes the measure of Ω it is well known (see [13]) that

$$\inf_{W_0^{1,\infty}(\Omega)} \frac{1}{|\Omega|} \int_\Omega \varphi(v_x, v_y)\, dxdy = \varphi^{**}(0,0). \tag{2.2}$$

Since $0 \leq \varphi$ and since the function identical equal to 0 is convex one has clearly

$$0 \leq \varphi^{**}.$$

Moreover, since

$$\varphi(0, \pm 1) = 0$$

one has

$$0 \leq \varphi^{**}(0, \pm 1) \leq \varphi(0, \pm 1) = 0.$$

From the convexity of φ^{**} it then follows that

$$\varphi^{**}(0,0) = 0$$

and by (2.2) the infimum (1.1) is equal to 0. This completes in this case the proof of the first part of Theorem 2.1.

Let us assume next that the infimum (1.1) is achieved for some $u \in W_0^{1,\infty}(\Omega)$ i.e. assume

$$\int_\Omega u_x^2 + (u_y^2 - 1)^2\, dxdy = 0.$$

Necessarily $u_x = 0$ so that u is independent of x. But since u vanishes on the boundary of Ω u is identical to 0. Clearly, if u is a minimizer of (1.2) one must have also $u = 0$ and similarly for u solution to (1.3), (1.4). So, the only possible minimizer for the above problems is the function identical equal to 0. Now, for $v = 0$ all the integrals in (1.1)–(1.4) are equal to $|\Omega| \neq 0$. Hence a contradiction. □

In the absence of minimizer we turn to the study of the minimizing sequences. Due to the fact that the only "possible" minimizer has to be 0 they have a behaviour connected to this point. We have:

Theorem 2.2. *Every uniformly bounded minimizing sequence of the problems (1.1)–(1.4) converges uniformly towards 0.*

Proof. By a uniformly bounded sequence we mean a sequence v_ε such that

$$|v_\varepsilon|_\infty,\ ||\nabla v_\varepsilon||_\infty \leq C \tag{2.3}$$

where C is a constant independent on ε and $|\ |_\infty$ the usual $L^\infty(\Omega)$-norm. Now, for such a sequence, one can extract a subsequence that we will still label by v_ε such that

$$v_\varepsilon \to v \text{ uniformly in } \Omega\ ,\ \nabla v_\varepsilon \rightharpoonup \nabla v \text{ in } L^\infty(\Omega)\ *\text{weak.} \tag{2.4}$$

Since v_ε is a minimizing sequence one has

$$v_{\varepsilon x} \to 0 \text{ in } L^2(\Omega)$$

and in cases (1.2), (1.4)

$$v_\varepsilon \to 0 \text{ in } L^2(\Omega).$$

By uniqueness of the limit in $L^2(\Omega)$ we then deduce that $v = 0$ (in cases (1.1), (1.3) $v_x = 0$ implies $v = 0$ since v vanishes on the boundary of Ω). Thus, the only possible limit for v_ε is 0 and the result follows. □

REMARK 2.1 . From (2.4) one deduces that

$$\nabla v_\varepsilon \rightharpoonup 0 \text{ in } L^\infty(\Omega)\ *\text{weak.}$$

3. Young measures. A uniformly bounded "sequence" $w_\varepsilon \in (L^\infty(\Omega))^m$ i.e. such that

$$||w_\varepsilon||_\infty \le C \tag{3.1}$$

where C is a constant independent on ε and $|\ |$ the usual euclidean norm in $\mathbf{R}^m$ defines a Young measure in the sense that there exists a subsequence of w_ε still labelled by ε and a probability measure ν_x on $\mathbf{R}^m$ parametrized by x such that

$$\varphi(w_\varepsilon) \rightharpoonup \int_{\mathbf{R}^m} \varphi(\lambda) d\nu_x(\lambda) \text{ in } L^\infty(\Omega)\ *\text{weak} \tag{3.2}$$

for any continuous function φ (see [1], [18], [14], [16], [13]). For instance, a bounded minimizing sequence $w_\varepsilon = v_\varepsilon$ of (1.1) or (1.2)–(1.4) defines a unique Young measure given for any x by

$$\nu_x = \delta_0$$

where δ_0 denotes the Dirac mass at 0. Indeed, since

$$v_\varepsilon \to 0 \text{ uniformly in } \Omega$$

one has for any continuous function φ

$$\varphi(v_\varepsilon) \to \varphi(0) \text{ uniformly in } \Omega$$

thus also in $L^\infty(\Omega)$ $*$weak so that for any continuous φ one has

$$\int_{\mathbf{R}} \varphi(\lambda) d\nu_x(\lambda) = \varphi(0)$$

which concludes.

For a uniformly bounded minimizing sequence of (1.1)–(1.4), i.e. satisfying (2.3), $w_\varepsilon = \nabla v_\varepsilon$ defines also a Young measure. This Young measure is unique and we have:

THEOREM 3.1. *Let v_ε a uniformly bounded minimizing sequence of (1.1)–(1.4), then ∇v_ε defines a unique Young measure on $\mathbf{R}^2$ given by*

$$\nu_x = \frac{1}{2}\delta_{W_1} + \frac{1}{2}\delta_{W_2}\ ,\ W_1 = (0,1)\ ,\ W_2 = (0,-1) \tag{3.3}$$

in cases (1.1), (1.2) and

$$\nu_x = \frac{1}{2}\delta_{W_1(x)} + \frac{1}{2}\delta_{W_2(x)}\ ,\ W_1(x) = (0,x)\ ,\ W_2 = (0,-x) \tag{3.4}$$

in cases (1.3), (1.4). As before δ_A stands for the Dirac mass at A.

Proof. Let us first consider the case of (1.1), (1.2). Since v_ε is a minimizing sequence and if φ denotes the function defined by (2.1) one has

$$\int_\Omega \varphi(v_{\varepsilon x}, v_{\varepsilon y})\ dxdy \to 0$$

hence from (3.2) it follows that

$$\int_{\mathbf{R}^2} \varphi(\lambda) d\nu_x(\lambda) = 0.$$

This implies that the support of ν_x is included in $W_1 \cup W_2$ where the W_i are defined in (3.3). Since ν_x is a probability measure one has

$$\nu_x = \alpha_1(x)\delta_{W_1} + (1 - \alpha_1(x))\delta_{W_2}.$$

Note that -above and latter- when no confusion is possible x denotes in fact for simplicity the point (x, y). Now, from remark 2.1 and (3.2) for any function $f \in L^1(\Omega)$ one deduces

$$0 = \lim_\varepsilon \int_\Omega v_{\varepsilon x} f\ dxdy = \int_\Omega f \int_{\mathbf{R}^2} \lambda_1 d\nu_x(\lambda)\ dxdy,$$
$$0 = \lim_\varepsilon \int_\Omega v_{\varepsilon y} f\ dxdy = \int_\Omega f \int_{\mathbf{R}^2} \lambda_2 d\nu_x(\lambda)\ dxdy.$$

Since this is true for any f it follows that

$$0 = \int_{\mathbf{R}^2} \lambda d\nu_x(\lambda) = \alpha_1(x)W_1 + (1 - \alpha_1(x))W_2 = (2\alpha_1(x) - 1)W_1.$$

From this it results that $\alpha_1 \equiv \frac{1}{2}$ which completes the proof in case of problems (1.1), (1.2).

If now v_ε is a minimizing sequence for (1.3) or (1.4) then

$$\lim_{\varepsilon} \int_\Omega v_{\varepsilon x}^2 + (v_{\varepsilon y}^2 - x^2)^2 \; dxdy = 0$$

from which by (3.2) and expending $(v_{\varepsilon y}^2 - x^2)^2$ we deduce

$$\int_\Omega \int_{\mathbf{R}^2} \lambda_1^2 + (\lambda_2^2 - x^2)^2 d\nu_x(\lambda) = 0.$$

Hence for almost every (x, y)

$$\int_{\mathbf{R}^2} \lambda_1^2 + (\lambda_2^2 - x^2)^2 d\nu_x(\lambda) = 0.$$

Thus

$$\nu_x = \alpha_1(x,y)\delta_{W_1(x)} + (1 - \alpha_1(x,y))\delta_{W_2(x)}$$

where the $W_i(x)$ are defined in (3.4). Using again the remark 2.1 we obtain as above that $\alpha_1 \equiv \frac{1}{2}$. □

Remark 3.1. If ν_x is a Young measure defined through a bounded sequence w_ε then $\nu_x(\lambda)$ is the probability that -at the limit- the function w_ε takes the value λ. So, in case of (1.1) and (1.2) and if v_ε is a minimizing sequence at the limit ∇v_ε takes around any points the values W_1, W_2 with the same probability $\frac{1}{2}$. In the case (1.3), (1.4) if v_ε is a minimizing sequence at the limit ∇v_ε takes around any points the values $W_1(x)$, $W_2(x)$ with the same probability. Of course here $W_1(x)$, $W_2(x)$ are changing with x, the associated Young measure is not homogeneous.

4. Construction of minimizing sequences. The analysis conducted in the previous paragraph gives us some information on the minimizing sequences associated to the problems (1.1)–(1.4). First, the uniqueness of the associated Young measure is the indication that they all have the same pattern. This pattern can be predicted as soon as we know what are the gradients used at the limit. Let us first consider the case of problem (1.2). For $\varepsilon > 0$ set

$$v_\varepsilon(x,y) = \begin{cases} y & \text{if } \; y \in (0, \varepsilon), \\ -y + 2\varepsilon & \text{if } \; y \in (\varepsilon, 2\varepsilon). \end{cases} \tag{4.1}$$

Suppose now that v_ε is extended by periodicity of period 2ε in the y-direction on the whole $\mathbf{R}^2$. Let us denote again by v_ε this new function. Clearly

$$0 \le v_\varepsilon \le \varepsilon, \tag{4.2}$$

so that

$$\int_\Omega v_{\varepsilon x}^2 + (v_{\varepsilon y}^2 - 1)^2 + v_\varepsilon^2 \; dxdy \le \varepsilon^2 |\Omega|$$

and the above integral converges towards 0 with ε. This provides us with a minimizing sequence (it is clear that the infimum (1.2) is nonnegative) and shows at the same time that the infimum (1.2) is indeed 0. We also can see that the "wells" $W_1 = (0, 1)$, $W_2 = (0, -1)$ are used -at the limit- with the same probability around each points.

To construct a minimizing sequence corresponding to problem (1.1) we replace the sequence v_ε in (4.1) by the same function cut-off by the distance to the boundary i.e. by

$$\hat{v}_\varepsilon = v_\varepsilon \wedge dist(., \Gamma) \tag{4.3}$$

where $\wedge$ denotes the minimum of two numbers. Clearly this function is in $W_0^{1,\infty}(\Omega)$. Moreover, since by (4.2) when $\varepsilon \le dist(x, \Gamma)$ one has $\hat{v}_\varepsilon = v_\varepsilon$

$$\int_\Omega \hat{v}_{\varepsilon x}^2 + (\hat{v}_{\varepsilon y}^2 - 1)^2 \; dxdy \le CV_\varepsilon \tag{4.4}$$

where V_ε denotes a volume of size ε around the boundary Γ and C a constant depending on the gradient of $\hat{v}_\varepsilon$ which is uniformly bounded independently of ε. It follows from (4.4) and theorem (1.1) that $\hat{v}_\varepsilon$ is a minimizing sequence for the problem (1.1). Again, at the limit and around each point, $\hat{v}_\varepsilon$ uses with the same probability $W_1 = (0, 1)$ and $W_2 = (0, -1)$. We turn now to the case of problems (1.3) and (1.4).

A minimizing sequence for (1.3) or (1.4) has to have a more complicated pattern than in the previous cases since the wells used have to change with the position in Ω. Let us consider first the case of problem (1.3). Define for $0 < \varepsilon \le 1$

$$v_\varepsilon(x, y) = \begin{cases} x'y & \text{if } \; y \in (0, \varepsilon), \\ -x'y + 2\varepsilon x' & \text{if } \; y \in (\varepsilon, 2\varepsilon). \end{cases} \tag{4.5}$$

Suppose now that v_ε is extended by periodicity of period 2ε on $\mathbf{R}^2$. If Q is a square of side $\delta > \varepsilon$ and center (x', y') define by $\hat{v}_\varepsilon$ the function

$$\hat{v}_\varepsilon = v_\varepsilon \wedge dist(., \partial Q) \tag{4.6}$$

where ∂Q denotes the boundary of Q. δ will be chosen latter on. It is clear from (4.4) that

$$\int_Q \hat{v}_{\varepsilon x}^2 + (\hat{v}_{\varepsilon y}^2 - x'^2)^2 \; dxdy \le C\varepsilon\delta \tag{4.7}$$

where C is a constant independent of ε and δ. Then we cover Ω by squares Q_i of center (x_i, y_i) (see Figure 1).

On each square Q_i that is completely included in Ω define v_ε as the function $\hat{v}_\varepsilon$ given by (4.6) for $(x', y') = (x_i, y_i)$. On the squares cutting the

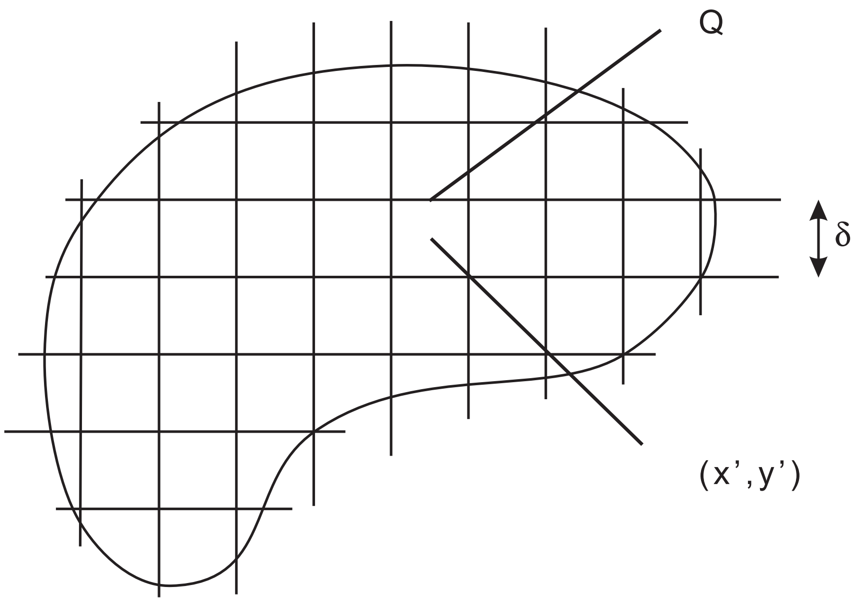

FIGURE 1

boundary Γ define v_ε by 0. If D denotes the part of Ω covered by these squares cutting the boundary one has for some constant C independent of δ

$$\int_D v_{\varepsilon x}^2 + (v_{\varepsilon y}^2 - x^2)^2 \; dxdy \le C\delta. \tag{4.8}$$

Next, due to (4.4) on each Q_i one has

$$\int_{Q_i} \hat{v}_{\varepsilon x}^2 + (\hat{v}_{\varepsilon y}^2 - x_i^2)^2 \; dxdy \le C\varepsilon\delta. \tag{4.9}$$

So, we deduce

$$\begin{aligned}\int_\Omega v_{\varepsilon x}^2 + (v_{\varepsilon y}^2 - x^2)^2 \; dxdy &\le C\delta + \sum_i \int_{Q_i} v_{\varepsilon x}^2 + (v_{\varepsilon y}^2 - x^2)^2 \; dxdy \\ &\le C\delta + \sum_i \int_{Q_i} v_{\varepsilon x}^2 + (v_{\varepsilon y}^2 - x_i^2)^2 \; dxdy \qquad (4.10)\\ &+ \sum_i \int_{Q_i} \{v_{\varepsilon x}^2 + (v_{\varepsilon y}^2 - x^2)^2 - v_{\varepsilon x}^2 - (v_{\varepsilon y}^2 - x_i^2)^2\} \; dxdy.\end{aligned}$$

Since for $(x,y) \in Q_i$

$|v_{\varepsilon x}^2 + (v_{\varepsilon y}^2 - x^2)^2 - v_{\varepsilon x}^2 - (v_{\varepsilon y}^2 - x_i^2)^2| = |(x - x_i)(x + x_i)(2v_{\varepsilon y}^2 - x^2 - x_i^2)| \le C\delta$

if N denotes the number of squares Q_i included in Ω one deduces from above

$$\int_\Omega v_{\varepsilon x}^2 + (v_{\varepsilon y}^2 - x^2)^2 \; dxdy \leq C\delta + NC\varepsilon\delta.$$

Now, clearly

$$N\delta^2 \leq |\Omega|$$

so that

$$\int_\Omega v_{\varepsilon x}^2 + (v_{\varepsilon y}^2 - x^2)^2 \; dxdy \leq C\{\delta + \frac{\varepsilon}{\delta}\}.$$

If one selects

$$\delta = \varepsilon^\beta$$

we obtain

$$\int_\Omega v_{\varepsilon x}^2 + (v_{\varepsilon y}^2 - x^2)^2 \; dxdy \leq C\{\varepsilon^\beta + \varepsilon^{1-\beta}\}.$$

Taking $\beta = \frac{1}{2}$ we end up with

$$\int_\Omega v_{\varepsilon x}^2 + (v_{\varepsilon y}^2 - x^2)^2 \; dxdy \leq C\varepsilon^{\frac{1}{2}}.$$

This shows that v_ε is a minimizing sequence and that the infimum (1.3) is 0. Clearly since v_ε is bounded from above by ε one has also for $\varepsilon \leq 1$

$$\int_\Omega v_{\varepsilon x}^2 + (v_{\varepsilon y}^2 - x^2)^2 + v^2 \; dxdy \leq C\varepsilon^{\frac{1}{2}}$$

so that the infimum (1.4) is also 0 and v_ε is a minimizing sequence for (1.4).

Remark 4.1. We considered here uniformly bounded minimizing sequences. Of course it is possible to have unbounded sequences. It is enough to modify slightly one of the above minimizing sequence. For instance, one can modify v_ε defined by (4.1) in one strip of size 2ε in such a way that on this strip S

$$|v_y| = \frac{1}{\varepsilon^\alpha}.$$

Then

$$\int_S v_{\varepsilon x}^2 + (v_{\varepsilon y}^2 - 1)^2 \; dxdy \leq C\varepsilon\frac{1}{\varepsilon^{4\alpha}} \to 0$$

for $\alpha < \frac{1}{4}$. Clearly this modified v_ε is an unbounded minimizing sequence. However, as we can see, its pattern remains the same.

Remark 4.2. It is a challenging problem to succeed to obtain numerically the right pattern for the minimizing sequences associated to problems studied here. For these questions related to computations we refer the interested reader to [4], [5], [6], [8], [9], [10], [11], [12].

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