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PROJECTIONS, EXTENDABILITY OF OPERATORS AND THE GATEAUX DERIVATIVE OF THE NORM

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Dedicated to the memory of Prof. T. Świątkowski

Abstract. The Hahn–Banach extension theorem is generalized to the case of continuous linear operators mapping a subspace Y of a normed space X into a normed space V. In contrast with known results of this kind, we do not equip V with a partial ordering neither impose any restrictions on V. The extension property is fully characterized by the sign of the one sided Gateaux derivative of the norm $\|\cdot\|_X$. Other characterizations, involving e.g. Birkhoff's orthogonality, are also provided.

1. Introduction. Let $(X, \|\cdot\|_X)$ be a normed space. Let X^* denote the space of continuous linear functionals on X. For $f \in X^*$,

$$\|f\|_{X^*} = \sup\{|f(x)| : \ x \in X, \qquad \|x\| \leq 1\}$$

defines a norm on X^*.

It follows from the fundamental Hahn–Banach extension theorem (see [2]) that for any linear subspace Y of X and any continuous linear functional

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on Y, there is an extension $F \in X^*$ of f such that $\|F\|_{X^*} = \|f\|_{Y^*}$ (the subscript on $\|\cdot\|$ indicates where this norm is taken). This result plays a crucial role in geometric functional analysis (see, e.g., [4], p. 95 or [7], Section 2.4).

However, it is known that a similar result need not hold for the class $L(X,V)$ of continuous linear operators from X to a normed space $(V, \|\cdot\|_V)$ even when V is two–dimensional only. Therefore, the following question arises: under what properties of a given normed space V, any operator $f \in L(Y,V)$ can be extended to $F \in L(X,V)$ with preserving its norm, i.e. in such a way that

$$\|F\|_{L(X,V)} = \|f\|_{L(Y,V)},$$

where X is an arbitrary normed space and Y is its subspace. This question was investigated by Nachbin [10], who showed that if every family $\mathcal{B}$ of balls in V, for which any two balls in $\mathcal{B}$ have a common point, has a nonempty intersection, then every linear operator $f \in L(Y,V)$ can be extended to some $F \in L(X,V)$ with preserving its norm, no matter what are the properties of the normed spaces X and $Y \subset X$. Another characterization of such spaces V was given by Kelley [8] (see also [7]). The above property is very restrictive however, and it excludes the case V is the Euclidean space $\mathbb{R}^2$, for instance. This limits very much a potential use of such a generalization of the Hahn–Banach theorem.

In this paper we study somewhat different problem: when, for a given normed space X and its subspace Y, any operator $f \in L(Y,V)$ can be extended to $F \in L(X,V)$ with (see Theorem 2.1) or without (see Corollary 2.1)) preserving its norm, where V is assumed to be an arbitrary normed space. A solution to this problem is fully characterized by the sign of the one sided Gateaux derivative of the norm $\|\cdot\|_X$. Theorem 2.1 gives also two other conditions equivalent to the extension property, the first involving the so-called Birkhoff's orthogonality, and the second making use of some special continuous linear functionals on X. In contrast with the results of Nachbin [10] and Kelly [8], the properties of the space V play no role in characterizing the extension property of $f \in L(Y,V)$. Thus Theorem 2.1 and the results of Nachbin and Kelley are somehow complementary. If one does not want to impose any restrictions on X and Y, then the results of [10] and [8] are applicable; if no restrictions on V are demanded, then Theorem 2.1 and Corollary 3.1 may be useful. In Section 3 we consider three examples concerning the problem of existence of some planes (i.e., two–dimensional subspaces) in $C([0,1])$, the space of all real and continuous functions on the unit interval. In particular, our Example 3.1 shows that there is a plane $Y_0 \subset C([0,1])$ such that for none of the subspaces $X$of $C([0,1])$, $X \supseteq Y_0$, there is a projection of norm one from X onto Y_0 unless $X = Y_0$. We emphasize here that, by the Kakutani theorem [6], one could conclude only that each subspace $X \subseteq C([0,1])$ of dimension ≥ 3 contains a plane Y (depending on X), for which there is no projection of norm one from X onto Y, whereas our plane Y_0 is a universal one having such a property.

Finally, it should be stressed up that we do not assume any order structure in V, in contrast with the approach presented e.g. in [1].

2. The Hahn–Banach type theorem. From now on we shall omit the subscript on the norm whenever it is clear from the context which norm is considered. Let $\tau_+(y;x)$ denote one sided Gateaux derivative of the norm at $y \in X$ in the direction $x \in X$, i.e.,

$$\tau_+(y;x) = \lim_{s \searrow 0} \frac{\|y + sx\| - \|y\|}{s}.$$

The above limit is known to exist (see Mazur [9]).

Further, an element $x \in X$ is said to be orthogonal to an element y ($x \perp y$) iff $\|x + ty\| \geq \|x\|$, for all t from the set $\mathbb{R}$ of all real numbers. This definition was introduced by G. Birkhoff [3]. If A and B are subsets of X, we say that A is orthogonal to B ($A \perp B$) iff $a \perp b$, for all $a \in A$ and $b \in B$. When one of the sets is a singleton, say $B = \{b\}$, we use the notation $A \perp b$ instead of $A \perp \{b\}$.

Lemma 2.1. *Let $b \in X$ and A be a symmetric subset of X, i.e., $a \in A$ implies that $-a \in A$. Then*

$$A \perp b \quad iff \quad \tau_+(a,b) \geq 0, \quad for\ all\ a \in A.$$

Proof. ($\Rightarrow$) Fix an $a \in A$. By hypothesis, $\|a + tb\| \geq \|a\|$, for all $t \in \mathbb{R}$. Hence, $\frac{\|a+tb\|-\|a\|}{t} \geq 0$, for $t > 0$, which implies that $\tau_+(a,b) \geq 0$.
($\Leftarrow$) For $a \in A$, define the function φ_a by

$$\varphi_a(t) := \frac{\|a + tb\| - \|a\|}{t}, \quad \text{for } t > 0.$$

Since φ_a is non-decreasing (see, e.g., [11], Lemma 1.2), we get that $\varphi_a(t) \geq \lim_{s \searrow 0} \varphi_a(s) = \tau_+(a,b) \geq 0$, which gives $\|a+tb\| \geq \|a\|$, for $a \in A$ and $t > 0$. Hence, replacing an element a by $-a$, we may infer that $\|a + tb\| \geq \|a\|$, for $a \in A$ and $t < 0$. So finally, the above inequality holds for all $t \in \mathbb{R}$, i.e., $a \perp b$ for $a \in A$. $\square$

Lemma 2.2. *Assume that* $\dim X \geq 2$ *and $x, y \in X$. Then $x \perp y$ iff there exists an $f \in X^* \setminus \{0\}$ such that $|f(x)| = \|f\|\,\|x\|$ and $f(y) = 0$.*

Proof. ($\Rightarrow$) If $x = 0$ or $y = 0$, the existence of f follows from the Hahn–Banach theorem and the fact that $\dim X \geq 2$. So assume that $x \neq 0 \neq y$. Then, by the orthogonality, x and y are linearly independent. Define the functional f_0 by

$$f_0(tx + sy) := t, \quad \text{for } t, s \in \mathbb{R}.$$

Then f_0 is continuous and linear on the plane $\operatorname{span}\{x, y\}$. Since $f_0\left(\frac{x}{\|x\|}\right) = \frac{1}{\|x\|}$, we infer that $\|f_0\| \geq \frac{1}{\|x\|}$. Simultaneously, for $t \neq 0$, we get, keeping in mind $x \perp y$,

$$\|tx + sy\| = |t| \, \|x + \frac{s}{t}y\| \geq |t| \, \|x\| = |f_0(tx + sy)| \, \|x\|,$$

which gives $\|f_0\| \leq \frac{1}{\|x\|}$. So $\|f_0\| = \frac{1}{\|x\|}$. By the Hahn–Banach theorem, there is an extension $f \in X^*$ of f_0 with the same norm as f_0. Then f is a desirable functional.

($\Leftarrow$) For all $t \in \mathbb{R}$, we have

$$\|f\| \cdot \|x\| = |f(x)| = |f(x + ty)| \leq \|f\| \, \|x + ty\|,$$

which implies $\|x\| \leq \|x + ty\|$, i.e., $x \perp y$. □

THEOREM 2.1. *Let X be a normed space with $\dim X \geq 2$ and Y be a linear subspace of X. The following statements are equivalent.*

(i) *For any normed space V and $f \in L(Y, V)$, there exists an extension $F \in L(X, V)$ of f such that $\|F\| = \|f\|$.*

(ii) *There exists a continuous linear projection P of X onto Y such that $\|P\| = 1$.*

(iii) *There exists a linear subspace Z of X such that*

$$X = Y + Z \quad \textit{and} \quad Y \perp Z.$$

(iv) *There exists a linear subspace Z of X such that*

$$X = Y + Z \qquad \textit{and} \quad \tau_+(y, z) \geq 0, \quad \textit{for all } y \in Y \textit{ and } z \in Z.$$

(v) *There exists a linear subspace Z of X such that $X = Y + Z$ and for all $y \in Y \setminus \{0\}$ and $z \in Z$ there is an $f \in X^* \setminus \{0\}$ such that f attains its norm at $\frac{y}{\|y\|}$ and vanishes at z.*

Proof. The equivalence ((i)$\Leftrightarrow$(ii)) is well–known (see, e.g., [10], p. 28). To prove that (ii) implies (iii) put $Z := \ker P$ $(= P^{-1}(0))$. Then, for all $y \in Y$ and $z \in Z$, we get $\|y\| = \|P(y + tz)\| \leq \|y + tz\|$, for $t \in \mathbb{R}$, so $y \perp z$.

To prove that (iii) implies (ii), define $P(x) := y$, $(x \in X)$, where $x = y + z$ (given $x \in X$, such an y is unique since $Y \perp Z$ implies $Y \cap Z = \{0\}$). Then P

is linear and, by orthogonality, $\|P(x)\| = \|y\| \le \|y+z\| = \|x\|$, so $\|P\| \le 1$. Simultaneously, $\|P\| \ge 1$, since P is a projection.

Finally, the equivalences ((iii)⇔(iv)) and ((iii)⇔ (v)) follow immediately from Lemmas 2.1 and 2.2, respectively. □

Corollary 2.1. *Let X be a normed space and Y be a linear subspace of X. The following statements are equivalent.*

(i) *For every normed space V and $f \in L(Y,V)$, there exists an extension $F \in L(X,V)$ of f (the norm of f need not be preserved).*

(ii) *There exists a continuous linear projection of X onto Y.*

(iii) *There are: an equivalent norm on X, say $|||\cdot|||$, and a projection $P \in L(X,Y)$ such that*

$$|||P(x)||| \le |||x||| \quad \text{for all } x \in X.$$

(iv) *There are: an equivalent norm on X, say $|||\cdot|||$, and a linear subspace Z of X such that*

$$X = Y + Z \quad \text{and} \quad \tilde{\tau}_+(y,z) \ge 0 \quad \text{for all } y \in Y,\ z \in Z,$$

where

$$\tilde{\tau}_+(y,z) = \lim_{t \searrow 0} \frac{|||y+tz||| - |||y|||}{t}.$$

Proof. In view of Theorem 2.1 the equivalences (i)⇔(ii) and (iii)⇔(iv) are obvious. In order to get that (ii)⇒(iii), it suffices to put

$$|||x||| = \|P(x)\| + \|x - P(x)\|.$$

Obviously, (iii) implies (ii). □

Sometimes $f \in L(Y,V)$ has some additional property, like e.g. monotonicity. The following two remarks may be helpful if we want to extend f with preserving such a property too.

Remark 2.1. Let X and V be normed spaces, Y be a linear subspace of X. Let ϱ be a relation in X (i.e. $\varrho \subseteq X^2$) and ν be a relation in V. A function $f: X \to V$ is called monotone on a set $A \subseteq X$ if for every $x_1, x_2 \in A$ such that $x_1 \varrho x_2$, it holds $f(x_1) \nu f(x_2)$. Let $f \in L(Y,V)$ be monotone on Y. Assume that there is a decomposition of X, $X = Y + Z$, such that $Y \perp Z$. Then, by Theorem 2.1 (ii) and (iii), f can be extended to some $F \in L(X,V)$ with preserving its norm. If, additionally, the linear projection $P: X \to Y$, defined by the decomposition $X = Y + Z$, is monotone (i.e. $x_1 \varrho x_2$ implies $P(x_1) \varrho P(x_2)$), then f can be extended to some

$F \in L(X, V)$ with preserving not only the norm but also the monotonicity property. To see this it is enough to define F by $F(x) = f \circ P(x)$, $x \in X$.

Remark 2.2. Let X and V be normed spaces, Y be a linear subspace of X. Let $M: X \to 2^V$ be a given set–valued map. Assume that $f \in L(Y, V)$ is such that

$$f(y) \in M(y) \qquad \text{for all } y \in Y.$$

If there is a decomposition $X = Y + Z$ of X such that $Y \perp Z$, then there exists an extension $F \in L(X, V)$ of f such that $\|F\| = \|f\|$ and

$$F(x) \in M(P(x)) \qquad \text{for all } x \in X,$$

where P is a continuous linear projection generated by the decomposition $X = Y + Z$. For example, let C be a convex cone in V, $m: X \to V$, and let M be defined by

$$M(x) = \{v \in V \mid v \in m(x) + C\}.$$

If $m(P(x)) \in m(x) + C$, for all $x \in X$, then we have additionally that

$$F(x) \in M(x) \qquad \text{for all } x \in X.$$

3. Projections of norm one onto some subspaces of C([0,1]). We begin with the following theorem which is due to R. C. James ([5], Theorem 4.5).

Theorem 3.1. *If f $(\neq 0)$ and g are elements of the space $C([0,1])$ with the sup–norm then*

$$\tau_+(f; g) = \max\{g(t) \operatorname{sign} f(t) : t \in M_f\},$$

where

$$M_f = \{t \in [0,1] : |f(t)| = \|f\|\}.$$

Corollary 3.1. *Let $g \in C([0,1])$ and X be a subspace of $C([0,1])$. Define the set X_s by putting*

$$X_s := \{f \in X : M_f \text{ is a singleton}\}.$$

(i) *If $X \perp g$, then g vanishes on the set $\operatorname{cl}\left(\bigcup_{f \in X_s} M_f\right)$, where $\operatorname{cl}(\cdot)$ denotes the closure operation.*
(ii) *If g vanishes on the set $\bigcup_{f \in X \setminus \{0\}} M_f$, then $X \perp g$.*

Proof. (i) By Lemma 2.1, $X \perp g$ implies that $\tau_+(f,g) \geq 0$, for all $f \in X$. In particular, if $f \in X_s$ then $M_f = \{t_0\}$ for some $t_0 \in [0,1]$ so, by Theorem 3.1, $g(t_0)\operatorname{sign} f(t_0) \geq 0$. Since $-f \in X$ and $M_f = M_{-f}$, we may conclude in the same way, that $g(t_0)\operatorname{sign}(-f(t_0)) \geq 0$. Hence, $g(t_0) = 0$, because f does not vanish at t_0. By continuity, $g(t) = 0$ for all $t \in \operatorname{cl}\left(\bigcup_{f\in X_s} M_f\right)$.

(ii) By hypothesis, for any $f \in X \setminus \{0\}$, g vanishes on the set M_f so, by Theorem 3.1, $\tau_+(f,g) = 0$. Hence and by Lemma 2.1, we get that $X \perp g$. □

Example 3.1. Let Y_0 be the plane in $C([0,1])$ defined by $Y_0 := \operatorname{span}\{f_1, f_2\}$, where

$$f_1(t) := 1 - t^2 \text{ and } f_2(t) := 1 + t, \qquad \text{for } t \in [0,1].$$

Let X be any subspace of $C([0,1])$ such that $Y_0 \subset X \neq Y_0$. We show that there does not exist a projection of norm one from X onto Y_0. Suppose, on the contrary, there is such a projection. Then, by Theorem 2.1, there exists a subspace $Z \subset X$ such that $X = Y_0 + Z$ and $Y_0 \perp Z$. Let $g \in C([0,1])$ and $Y_0 \perp g$. Observe that, for any $t_0 \in [0,1]$, there is an $f \in Y_0$ such that $M_f = \{t_0\}$ (it suffices to put $f := f_1 + 2t_0 f_2$). Therefore, by Corollary 3.1, we may conclude that $g = 0$. Hence, $Z = \{0\}$ so $X = Y_0$, a contradiction.

The following problem arises naturally in the context of Example 3.1: Give a general characterization of these non–Euclidean normed spaces X of dimension ≥ 3 which contain a two–dimensional subspace Y with the property that the identity map on Y is the only projection of norm one from a subspace of X onto Y. Observe that e.g. Theorem 2.1 (iii) demands to verify the existence of subspaces Y and Z of X such that $Y \perp Z$ and $X = Y + Z$. However, as is shown in Example 3.1, even if X is given, it may be troublesome to check the above condition.

Example 3.2. Let Y_1 be the plane in $C([0,1])$ defined by

$$Y_1 := \operatorname{span}\{e_1, e_2\} \quad \text{where } e_1(t) := t \text{ and } e_2(t) := t^2, \quad \text{for } t \in [0,1].$$

Elementary computations show that, for $f \in Y_1 \setminus \{0\}$, the set M_f is either a singleton $\{t_0\}$ with $t_0 \in (-1+\sqrt{2}, 1]$, or $M_f = \{-1+\sqrt{2}, 1\}$. Thus, we may apply Corollary 3.1 to conclude that $Y_1 \perp g$ iff $g\big|_{[-1+\sqrt{2},1]} = 0$. Let Z_1 be the set of all such functions g. Then, by Theorem 2.1, we may infer that the space $X_1 := Y_1 + Z_1$ is the largest (with respect to the inclusion $\subseteq$) subspace of $C([0,1])$, for which there exists a projection of norm one from X_1 onto Y_1. On the other hand, there exists a subspace X dense in $C([0,1])$ such that, for any subspace $X' \neq Y_1$ satysfying $Y_1 \subset X' \subseteq X$, there is no projection of norm one from X' onto Y_1 (it suffices to consider the space X

of all real–valued polynomials restricted to $[0,1]$; in this case, $g \in X$ and $Y_1 \perp g$ imply $g = 0$).

REMARK 3.1. The results of Example 3.2 can be extended to the planes Y_{kn} defined by

$$Y_{kn} := \operatorname{span}\{e_k, e_n\},$$

where k and n are non–negative integers, $k < n$, and $e_m(t) := t^m$ for $t \in [0,1]$ and $m \in \mathbb{N} \cup \{0\}$. Then, there is a projection of norm one from $C([0,1])$ onto Y_{kn} iff $k = 0$. Further, for $n > k \geq 1$, define a real number $t_{kn} \in (0,1)$ as the solution of the equation

$$(n-k)x^n + nx^{n-k} - k = 0$$

(this equation has a unique solution in $(0,1)$), and put

$$Z_{kn} := \{g \in C([0,1]) : \ g\big|_{[t_{kn},1]} = 0\}.$$

We leave it to the reader to verify that $X_{kn} := Y_{kn} + Z_{kn}$ is the largest subspace of $C([0,1])$, for which there exists a projection of norm one from X_{kn} onto Y_{kn}.

We close our paper with an example of a three–dimensional space X, $X \subset C([0,1])$, and its two–dimensional subspace Y, for which the family of projections of norm one from X onto Y is infinite. We emphasize that such a case could not occur if X was a Hilbert space.

EXAMPLE 3.3. Let X be the subspace of $C([0,1])$ defined by

$$X := \operatorname{span}\{f_1, f_2, f_3\},$$

where $f_1(t) := \left|t - \frac{1}{2}\right|$ and $f_2(t) := 1$ for $t \in [0,1]$, and $f_3(t) := \left|t - \frac{1}{2}\right|$ for $t \in \left[0, \frac{1}{2}\right]$ and $f_3(t) := 0$ for $t \in \left(\frac{1}{2}, 1\right]$.

Clearly, the functions f_1, f_2, f_3 are linearly independent, so $\dim X = 3$. Further, let

$$Y := \operatorname{span}\{f_1, f_2\}.$$

It is easy to verify that for $f \in Y$, either $M_f = \left\{\frac{1}{2}\right\}$, or $M_f = \{0,1\}$, or $M_f = \left\{0, \frac{1}{2}, 1\right\}$, or $M_f = [0,1]$. Applying Lemma 2.1 and Theorem 3.1 it can be obtained that for a function $g \in C([0,1])$,

$$Y \perp g \quad \text{iff} \quad g\left(\frac{1}{2}\right) = 0 \quad \text{and} \quad g(0) \cdot g(1) \leq 0.$$

Hence, we easily get that if $g \in X$ and $g = \alpha f_1 + \beta f_2 + \gamma f_3$, then $Y \perp g$ iff $\beta = 0$ and $\alpha(\alpha+\gamma) \leq 0$. This leads to the following equivalence: for $g \in X$,

$$Y \perp g \Longleftrightarrow \bigvee_{\lambda\in\mathbb{R}} \bigvee_{f\in[f_3-f_1,f_3]} g = \lambda f,$$

where $[f_3 - f_1, f_3]$ denotes the segment with ends at the points $f_3 - f_1$ and f_3. That means that the set of all elements in X which are orthogonal (from the right) to Y is the symmetric cone generated by the segment $[f_3 - f_1, f_3]$ with 0 as its vertical. For $\lambda \in [0, 1]$, let

$$g_\lambda := f_3 - \lambda f_1 \quad \text{and} \quad Z_\lambda := \operatorname{span}\{g_\lambda\}.$$

Then $Y \perp Z_\lambda$ and $X = Y + Z_\lambda$ for $\lambda \in [0, 1]$. Let P_λ denote a continuous linear projection from X onto Y such that $\|P_\lambda\| = 1$, the existence of which is guaranteed by Theorem 2.1 (see the proof of (iii) $\Rightarrow$ (ii)). It is clear that, for λ', $\lambda'' \in [0, 1]$, $P_{\lambda'} \neq P_{\lambda''}$ if $\lambda' \neq \lambda''$. Therefore, the family $\{P_\lambda : \lambda \in [0, 1]\}$ is infinite. Furthermore, it contains all possible projections of norm one from X onto Y.

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