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THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF EQUATIONS FOR IDEAL COMPRESSIBLE POLYTROPIC FLUIDS

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Abstract. The local existence of classical solutions for a characteristic initial boundary value problem for the equations of ideal compressible polytropic fluids is proved. The problem is replaced by a system of well–posed problems and then the method of successive approximations is used.

1. Introduction. In this paper we prove the existence and uniqueness of solutions of an initial–boundary value problem for equations of ideal compressible polytropic fluids in a bounded domain with an impermeable boundary.

In the particular case of the barotropic fluid motion the problem was considered in [1], [8].

Since, as in our case, a boundary condition with vanishing normal velocity component is characteristic for the Euler equations we are not able to use the general methods [4], [7], [9] for first order hyperbolic systems.

The proof presented here is close to the method in [8] and is strictly connected with the form of the Euler equations. Only the local existence of classical solutions can be shown.

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Neither classical nor weak global solutions are known. However, the existence of measure–valued solutions of the Euler equations was proved in [5].

2. Statement of the problem. Let us consider the problem [3], [6]

$$
\begin{aligned}
&\rho_t + \operatorname{div}(\rho v) = 0 && \text{in } \Omega^T = \Omega \times (0,T)\,,\\
&\rho v_t + \rho v \cdot \nabla v + \nabla p = \rho f && \text{in } \Omega^T\,,\\
&e_t + \operatorname{div}(v\,(e+p)) = \rho f \cdot v && \text{in } \Omega^T\,,\\
&\rho|_{t=0} = \rho_0\,,\ v|_{t=0} = v_0\,,\ e|_{t=0} = e_0 && \text{in } \Omega\,,\\
&v \cdot \bar{n} = 0 && \text{on } S^T = S \times (0,T)\,,
\end{aligned}
\tag{2.1}
$$

where $\Omega \subset \mathbf{R}^n$, $n = 2,3$, is a bounded domain with boundary S, $\rho = \rho(x,t)$ is the density, $v = v(x,t)$ the velocity vector, $p = p(x,t)$ the pressure, $e = e(x,t)$ the total energy, $\bar{n}$ is the unit outward vector normal to S and $f = f(x,t)$ is the external force field, $\rho_0 \geq \rho^* > 0$, $\rho^* = \text{const}$. The total energy has the form

$$
e = \frac{1}{2}\rho v^2 + \rho\varepsilon\,,
\tag{2.2}
$$

where ε is the specific internal energy.

For the ideal gas we have the following equation of state

$$
p = \rho R \vartheta\,,
\tag{2.3}
$$

where R is the gas constant and $\vartheta = \vartheta(x,t)$ is the absolute temperature.

Our considerations are restricted to polytropic gases which satisfy

$$
\varepsilon = c_v \vartheta\,,
\tag{2.4}
$$

where c_v is the specific heat at constant volume.

Then we have (see [3], Ch. 1)

$$
p = A(S)\rho^\gamma\,,
\tag{2.5}
$$

where

$$
A(S) = (\gamma - 1)e^{\frac{S-S_0}{c_v}} \equiv A_0 e^s\,.
\tag{2.6}
$$

Here S is the density of the entropy, $s = \frac{S}{c_v}$, γ is the adiabatic exponent, $\gamma > 1$, $c_v = \frac{R}{\gamma-1}$ and $A_0 = (\gamma-1)e^{-\frac{S_0}{c_v}}$ is constant.

Therefore, instead of (2.1) we get using the entropy

$$\begin{aligned}
&\rho_t + \operatorname{div}(\rho v) = 0 && \text{in } \Omega^T, \\
&\rho v_t + \rho v \cdot \nabla v + \nabla p = \rho f && \text{in } \Omega^T, \\
&s_t + v \cdot \nabla s = 0 && \text{in } \Omega^T, \\
&\rho|_{t=0} = \rho_0, \ v|_{t=0} = v_0, \ s|_{t=0} = s_0 && \text{in } \Omega, \\
&v \cdot \bar{n} = 0 && \text{on } S^T.
\end{aligned} \tag{2.7}$$

Problem (2.7) has a characteristic boundary as described in [4], [7], [9]. Since we do not know how to solve it directly we replace the problem by a system of uniquely solvable problems.

Let ρ, v, s, p be as smooth as we need. From (2.5) and (2.6) follows

$$p = A_0 e^s \rho^\gamma \tag{2.8}$$

and with $(2.7)_3$ we obtain

$$\partial_t p + v \cdot \nabla p - \frac{\gamma p}{\rho}(\rho_t + v \cdot \nabla \rho) = 0\,. \tag{2.9}$$

Using the continuity equation $(2.7)_1$ we have

$$\partial_t p + v \cdot \nabla p + \gamma p \operatorname{div} v = 0\,. \tag{2.10}$$

Let $\delta = \ln p$. Then

$$\operatorname{div} v = -\frac{1}{\gamma}(\partial_t + v \cdot \nabla)\,\delta\,. \tag{2.11}$$

Applying the operator div to $(2.7)_2$ we get

$$\begin{aligned}
&(\operatorname{div} v)_t + v \cdot \nabla(\operatorname{div} v) + \operatorname{div}\left(\frac{1}{\rho}\nabla p\right) = \\
&= -\partial_{x_j} v_i\, \partial_{x_i} v_j + \operatorname{div}\frac{f}{\rho}\,.
\end{aligned} \tag{2.12}$$

Using (2.11) in (2.12) yields

$$Q^2\delta - \operatorname{div}\left(\gamma\frac{p}{\rho}\nabla\delta\right) = \gamma\left(\partial_{x_j} v_i\, \partial_{x_i} v_j - \operatorname{div} f\right), \tag{2.13}$$

where

$$Q = \partial_t + v \cdot \nabla\,. \tag{2.14}$$

Multiplying $(2.7)_2$ by $\bar{n}$ and projecting the result on S we get with $(2.7)_5$ the boundary condition

$$\left.\frac{\partial\delta}{\partial n}\right|_{S^T} = \left.\frac{\rho(v_i v_j\, \partial_{x_i} n_j + f \cdot \bar{n})}{e^\delta}\right|_{S^T}. \tag{2.15}$$

Furthermore the initial conditions are

$$\begin{aligned} &\delta|_{t=0} = \ln\left(A_0 e^{s_0}\rho_0^{\gamma}\right), \\ &\delta_t|_{t=0} = -v_0\cdot\nabla s_0 - \frac{\gamma}{\rho_0}\left(v_0\cdot\nabla\rho_0 + \rho_0 \operatorname{div} v_0\right), \end{aligned} \tag{2.16}$$

where we used $(2.7)_1$ and $(2.7)_3$.

Hence from (2.13), (2.15), (2.16) for given v and ρ we obtain the following problem for δ

$$\begin{aligned} &Q^2\delta - \operatorname{div}\left(\frac{\gamma e^{\delta}}{\rho}\nabla\delta\right) = \gamma\left(\partial_{x_i}v_j\,\partial_{x_j}v_i - \operatorname{div} f\right), \\ &\delta|_{t=0} = \ln\left(A_0 e^{s_0}\rho_0^{\gamma}\right), \\ &\delta_t|_{t=0} = -v_0\cdot\nabla s_0 - \frac{\gamma}{\rho_0}\left(v_0\cdot\nabla\rho_0 + \rho_0 \operatorname{div} v_0\right), \\ &\frac{\partial\delta}{\partial n}\bigg|_{S^T} = \frac{\rho(v_i v_j\, n_{i,x_j} + f\cdot\bar{n})}{e^{\delta}}\bigg|_{S^T}. \end{aligned} \tag{a}$$

Applying the operator rot to $(2.7)_2$ we obtain the problem for the vorticity vector $\omega = \operatorname{rot} v$

$$\begin{aligned} &\omega_t + v\cdot\nabla\omega - \omega\cdot\nabla v + \operatorname{div} v\,\omega + \nabla\frac{1}{\rho}\times\nabla e^{\delta} = \operatorname{rot} f, \\ &\omega|_{t=0} = \omega_0 \equiv \operatorname{rot} v_0, \end{aligned} \tag{b}$$

where ρ, δ and v are treated as given functions.

Now for given ω and ρ we have the elliptic problem for v

$$\begin{aligned} &\operatorname{rot} v = \omega, \\ &\operatorname{div} v = -\frac{1}{\rho}Q\rho, \\ &v\cdot\bar{n}|_S = 0. \end{aligned} \tag{c}$$

Finally, s is a solution of the problem

$$\begin{aligned} &s_t + v\cdot\nabla s = 0, \\ &s|_{t=0} = s_0, \end{aligned} \tag{d}$$

where v is given.

Let us emphasize that we calculate ρ from (2.8) for given s and δ by the relation

$$\rho = \left(\frac{1}{A_0}e^{\delta}e^{-s}\right)^{\frac{1}{\gamma}} \equiv z(\delta, s). \tag{e}$$

The next sections are organized in the following way. After providing the notation used in the paper we present in Section 4 the method of successive approximations in order to solve the problems (a)–(d). In Section 5 we prove the boundedness of the sequence constructed in Section 4, and in Section 6 we show its convergence. The existence and uniqueness of the solution of problem (2.7) are shown in Section 7.

3. Notation. In this paper we consider a simply–connected domain $\Omega \subset \mathbf{R}^n$, $n = 2, 3$, with the boundary S of class C^5. We assume that in a neighbourhood of the boundary there exists a vector field $\bar{n}(x) \in C^4$ such that $\bar{n}(x)|_{x\in S}$ is the unit outward vector normal to S.

We denote the norms of the spaces $L_p(\Omega)$, $p \in [1, \infty]$ and the Sobolev spaces $H^l(\Omega)$, $l \in \mathbf{N} \cup \{0\}$, by $| \cdot |_{p,\Omega}$ and $\| \cdot \|_{l,\Omega}$, respectively.

Let B be a Banach space with the norm $\| \cdot \|_B$, $k \in \mathbf{N} \cup \{0\}$ and T some positive constant. Then $L^k_r(0, T; B)$ denotes a Banach space of functions $f(t)$ on $[0, T]$ with values in B for every fixed $t \in [0, T]$, whose k–th derivative with recpect to t has a bounded norm $\left(\int_0^T \| \cdot \|^r_B \, dt\right)^{\frac{1}{r}}$, where $r = \infty$ or $r = 2$.

Moreover, let us introduce the spaces $\Pi^l_{k,r}(\Omega^T) = \bigcap_{i=k}^l L^{l-i}_r(0, T; H^i(\Omega))$ with the norm

$$|u|_{l,k,r,\Omega^T} = \sum_{i=k}^{l} \left(\int_0^T \| \partial_t^{l-i} u \|^r_{i,\Omega} \, dt \right)^{\frac{1}{r}}$$

and $\Gamma^l_k(\Omega)$ with the norm

$$|u|_{l,k,\Omega} = \sum_{i=k}^{l} \| \partial_t^{l-i} u \|_{i,\Omega} \ .$$

4. The method of successive approximations. We prove the existence of solutions of the problem (a, b, c, d, e) by the following method of succesive approximation. Let $v_{m-1}, \delta_{m-1}, s_{m-1}$ be given. Then δ_m is a solution of the problem

$$\begin{aligned}
&Q^2_{m-1}\delta_m - \operatorname{div}\left(\frac{\gamma e^{\delta_{m-1}}}{z(\delta_{m-1}, s_{m-1})} \nabla \delta_m \right) = \\
&= \gamma \left(\partial_{x_i} v_{m-1,j} \, \partial_{x_j} v_{m-1,i} - \operatorname{div} f \right) , \qquad (a_m)\\
&\delta_m|_{t=0} = \ln \left(A_0 e^{s_0} \rho_0^{\gamma} \right) , \\
&\delta_{m,t}|_{t=0} = - v_0 \cdot \nabla s_0 - \frac{\gamma}{\rho_0} \left(v_0 \cdot \nabla \rho_0 + \rho_0 \operatorname{div} v_0 \right) , \\
&\left.\frac{\partial \delta_m}{\partial n}\right|_{S^T} = \left.\frac{z(\delta_{m-1}, s_{m-1})(v_{m-1,i} v_{m-1,j} \, n_{i,x_j} + f \cdot \bar{n})}{e^{\delta_{m-1}}}\right|_{S^T} ,
\end{aligned}$$

where $Q_{m-1} = \partial_t + v_{m-1} \cdot \nabla$.

For given $v_{m-1}, \delta_{m-1}, s_{m-1}$ the function ω_m is a solution of the problem

$$\begin{aligned} &\omega_{m,t} + v_{m-1} \cdot \nabla\omega_m - \omega_m \cdot \nabla v_{m-1} + \operatorname{div} v_{m-1}\, \omega_m \; + \\ &+ \nabla \frac{1}{z(\delta_{m-1}, s_{m-1})} \times \nabla e^{\delta_{m-1}} = \operatorname{rot} f\,, \\ &\omega_m|_{t=0} = \omega_0\,. \end{aligned} \tag{b_m}$$

For v_m we obtain the elliptic problem

$$\begin{aligned} &\operatorname{rot} v_m = \omega_m\,, \\ &\operatorname{div} v_m = -\,Q_{m-1} \ln z(\delta_m, s_m) + \frac{1}{|\Omega|}\int_\Omega Q_{m-1} \ln z(\delta_m, s_m)\, dx\,, \\ &v_m \cdot \bar{n}|_S = 0\,, \end{aligned} \tag{c_m}$$

where ω_m, s_m, δ_m are given.

Finally, s_m is determined for a given v_{m-1} from

$$\begin{aligned} &s_{m,t} + v_{m-1} \cdot \nabla s_m = 0\,, \\ &s_m|_{t=0} = s_0\,. \end{aligned} \tag{d_m}$$

and ρ_m from

$$\rho_m = z(\delta_m, s_m)\,. \tag{e_m}$$

The additional term on the right–hand side of $(c_m)_2$ is necessary in order to satisfy the compatibility condition for problem (c).

Let

$$\begin{aligned} &F_1 = \| \rho_0 \|_{2,\Omega} + \| s_0 \|_{2,\Omega} + \| v_0 \|_{2,\Omega}\,, \\ &F_2 = \| \rho_0 \|_{3,\Omega} + \| s_0 \|_{3,\Omega} + \| v_0 \|_{3,\Omega}\,, \\ &F_3 = \| f(0) \|_{1,\Omega} + \| \operatorname{rot} f(0) \|_{1,\Omega}\,, \\ &F_4 = \| \operatorname{div} f(0) \|_{1,\Omega}\,, \\ &F_5 = |\operatorname{rot} f|_{2,0,1,\Omega^T}\,, \\ &F_6 = |\operatorname{div} f|_{2,1,\infty,\Omega^T}\,, \\ &F_7 = |f \cdot \bar{n}|_{3,1,\infty,\Omega^T}\,. \end{aligned} \tag{4.1}$$

In the following lemmas g_i, h_i, f_i, G_i will be different positive, increasing functions of its arguments. Different constants are denoted by C.

Lemma 4.1. *Let $s_0, \rho_0, v_0 \in H^3(\Omega)$, $\mathit{rot}\, f \in \Pi^2_{0,1}(\Omega^T)$, $\mathit{rot}\, f(0) \in H^1(\Omega)$, $f(0) \in H^1(\Omega)$ and $v_{m-1}, \delta_{m-1}, s_{m-1} \in \Pi^3_{1,\infty}(\Omega^T)$. Then there exists a* *unique solution of the problem* (b_m) *such that* $\omega_m \in \Pi^2_{0,\infty}(\Omega^T)$ *and the estimate*

$$
\begin{aligned}
|\omega_m|_{2,0,\Omega} \le e^{cT|v_{m-1}|_{3,1,\infty,\Omega^T}} \cdot \Big[h_1(T(|\delta_{m-1}|_{3,2,\infty,\Omega^T} + \\
+\ |s_{m-1}|_{3,2,\infty,\Omega^T}) + F_1)(1 + |\delta_{m-1}|^4_{3,1,\infty,\Omega^T} + \\
+\ |s_{m-1}|^4_{3,1,\infty,\Omega^T})T + F_5 + h_2(F_2)(F_2 + F_3) \Big]
\end{aligned}
\tag{4.2}
$$

holds.

Moreover, $\omega_m \in C(0,T;\Gamma^2_0(\Omega))$.

Proof. The existence of the solution ω_m follows from the method of characteristics.

Let us show estimate (4.2). From

$$
\begin{aligned}
&\sum_{\mu\le 2} \int_\Omega D^\mu_{t,x} \Bigg(\omega_{m,t} + v_{m-1} \cdot \nabla\omega_m - \omega_m \cdot \nabla v_{m-1} + \\
&+\ \operatorname{div} v_{m-1} \cdot \omega_m + \nabla \frac{1}{z(\delta_{m-1}, s_{m-1})} \times \nabla e^{\delta_{m-1}} \Bigg) D^\mu_{t,x}\, \omega_m dx = \\
&= \sum_{\mu\le 2} \int_\Omega D^\mu_{t,x} (\operatorname{rot} f) D^\mu_{t,x}\, \omega_m dx\,,
\end{aligned}
$$

where

$$
D^\mu_{t,x} = \sum_{i_0+i_1+\ldots+i_3\le\mu} \frac{\partial^{i_0}}{\partial t^{i_0}} \frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_3}}{\partial x_3^{i_3}}\,,
$$

we get using formula (e_m)

$$
\begin{aligned}
&\frac{d}{dt}|\omega_m|_{2,0,\Omega} \le |v_{m-1}|_{3,1,\Omega}\, |\omega_m|_{2,0,\Omega} + \\
&+\ g_1\,(|s_{m-1}|_{2,2,\Omega}, |\delta_{m-1}|_{2,2,\Omega}) \cdot \\
&\cdot |\delta_{m-1}|_{3,1,\Omega}\, \left(|s_{m-1}|_{3,1,\Omega} + |\delta_{m-1}|_{3,1,\Omega}\right) \cdot \\
&\cdot \left(1 + |s_{m-1}|^2_{3,1,\Omega} + |\delta_{m-1}|^2_{3,1,\Omega}\right) + |\operatorname{rot} f|_{2,0,\Omega}\,.
\end{aligned}
\tag{4.3}
$$

From the relation

$$
u(t) = \int_0^t u_\tau(\tau) d\tau + u(0)
$$

we obtain

$$
|u|_{2,s,\infty,\Omega^t} \le t\,|u|_{3,s,\infty,\Omega^t} + C\,|u(0)|_{2,s,\Omega}\,,
\tag{4.4}
$$

for $s \leq 2$. Integrating (4.3) with respect to time and applying (4.4) to the arguments of g_1 we get

$$\begin{aligned}&|\omega_m|_{2,0,\Omega} \leq e^{ct|v_{m-1}|_{3,1,\infty,\Omega^t}} \big[g_1\,(t\,|\delta_{m-1}|_{3,2,\infty,\Omega^t} + F_1\,,\\ &t|s_{m-1}|_{3,2,\infty,\Omega^t} + F_1)\,t\,(1 + |\delta_{m-1}|^4_{3,1,\infty,\Omega^t} + \\ &+ |s_{m-1}|^4_{3,1,\infty,\Omega^t}) + |\mathrm{rot}\, f|_{2,0,1,\Omega^t} + |\omega_m(0)|_{2,0,\Omega}\big]\,.\end{aligned} \tag{4.5}$$

Estimating $\omega_m|_{t=0}$, $\omega_{m,t}|_{t=0}$, $\omega_{m,tt}|_{t=0}$ using (b_m) for $t = 0$ it can be shown that

$$\begin{aligned}&|\omega_m(0)|_{2,0,\Omega} \leq C \parallel v_0 \parallel_{3,\Omega} (1+ \parallel v_0 \parallel_{3,\Omega}) \cdot \\ &\cdot (1 + g_2(\parallel \rho_0 \parallel_{2,\Omega}\,, \parallel s_0 \parallel_{2,\Omega}) \parallel v_0 \parallel_{3,\Omega}) \cdot \\ &\cdot (\parallel s_0 \parallel_{3,\Omega} + \parallel \rho_0 \parallel_{3,\Omega})(1+ \parallel s_0 \parallel^2_{3,\Omega} + \parallel \rho_0 \parallel^2_{3,\Omega}) + \\ &+ C(1+ \parallel v_0 \parallel_{3,\Omega}) \cdot (\parallel \mathrm{rot}\, f(0) \parallel_{1,\Omega} + \parallel f(0) \parallel_{1,\Omega})\,.\end{aligned} \tag{4.6}$$

Substituting (4.6) into (4.5), using the notation (4.1) we obtain (4.2).

The continuity of $|\omega_m|_{2,0,\Omega}$ with respect to time follows from (4.3). This concludes the proof. $\square$

LEMMA 4.2. *Let $\delta_m, \delta_{m-1}, s_{m-1} \in \Pi^3_{1,\infty}(\Omega^T)$, $v_{m-1}, \omega_m \in \Pi^2_{0,\infty}(\Omega^T)$, $v_{m-1,xt} \in L_\infty(0,T;L_4(\Omega))$, $\mathrm{div}\, f_t \in L_\infty(0,T;L_2(\Omega))$, then there exists a unique solution of the problem (c_m) such that $v_m \in \Pi^3_{1,\infty}(\Omega^T)$ and*

$$\begin{aligned}&|v_m|_{3,1,\infty,\Omega^T} \leq C\,|\omega_m|_{2,1,\infty,\Omega^T} + \\ &+ h_3(T|v_{m-1}|_{3,1,\infty,\Omega^T} + h_4(F_1)F_1 + F_3\,, \\ &T|\delta_{m-1}|_{3,1,\infty,\Omega^T} + h_5(F_1)F_1\,, \\ &T|s_{m-1}|_{3,1,\infty,\Omega^T} + h_6(F_1)F_1)\,|\delta_m|_{3,1,\infty,\Omega^T} + \\ &+ C(T|v_{m-1}|_{3,1,\infty,\Omega^T} + h_7(F_1)F_1 + F_3) \sup_{t\leq T} |v_{m-1,tx}|_{4,\Omega} + \\ &+ CF_6\,.\end{aligned} \tag{4.7}$$

Proof. Using (d_m) and (e_m) we obtain

$$Q_{m-1} \ln z(\delta_m, s_m) = \frac{1}{\gamma}\, Q_{m-1}\, \delta_m\,.$$

Then (c_m) simplifies to

$$\begin{aligned}&\mathrm{rot}\, v_m = \omega_m\,, \\ &\mathrm{div}\, v_m = -\frac{1}{\gamma}\, Q_{m-1}\delta_m + \frac{1}{\gamma}\,\frac{1}{|\Omega|} \int_\Omega Q_{m-1}\delta_m\, dx\,, \\ &v_m \cdot \bar{n}|_S = 0\,.\end{aligned} \tag{4.8}$$

The existence of the solution of (4.8) follows from [2] and we have the estimate

$$
\begin{aligned}
|v_m|_{3,2,\infty,\Omega^t} \le C\,\{\,&|\omega_m|_{2,1,\infty,\Omega^t} + \\
&+ (|v_{m-1}|_{2,1,\infty,\Omega^t}+1)\,|\delta_m|_{3,1,\infty,\Omega^t}\}\,.
\end{aligned}
\tag{4.9}
$$

In order to estimate $|v_{m,tt}|_{1,1,\infty,\Omega^t}$ we differentiate $(4.8)_2$ twice with respect to time and replace the third time derivative of δ_m by $(a_m)_1$. We obtain

$$
\begin{aligned}
\operatorname{div} v_{m,tt} &= -\frac{1}{\gamma}\,\partial_t\,(\partial_t\,Q_{m-1}\delta_m) + \frac{1}{\gamma}\,\frac{1}{|\Omega|}\int_\Omega \partial_t\,(\partial_t\,Q_{m-1}\delta_m)dx = \\
&= -\frac{1}{\gamma}\,\partial_t\,(Q^2_{m-1}\delta_m - v_{m-1}\cdot\nabla Q_{m-1}\delta_m) + \\
&\quad + \frac{1}{\gamma}\,\frac{1}{|\Omega|}\int_\Omega \partial_t\,(Q^2_{m-1}\,\delta_m - v_{m-1}\cdot\nabla Q_{m-1}\delta_m)\,dx = \\
&= -\frac{1}{\gamma}\,\partial_t\,\Big(\operatorname{div}\Big(\frac{\gamma e^{\delta_{m-1}}}{z(\delta_{m-1},s_{m-1})}\nabla\delta_m\Big) + \\
&\quad + \gamma\partial_{x_i}v_{m-1,j}\,\partial_{x_j}v_{m-1,i} - \gamma\operatorname{div} f - v_{m-1}\cdot\nabla Q_{m-1}\delta_m\Big) + \\
&\quad + \frac{1}{\gamma}\,\frac{1}{|\Omega|}\int_\Omega \partial_t\,\Big(\operatorname{div}\Big(\frac{\gamma e^{\delta_{m-1}}}{z(\delta_{m-1},s_{m-1})}\nabla\delta_m\Big) + \\
&\quad + \gamma\partial_{x_i}v_{m-1,j}\,\partial_{x_j}v_{m-1,i} - \gamma\operatorname{div} f - v_{m-1}\cdot\nabla Q_{m-1}\delta_m\Big)\,dx \equiv \\
&\equiv H\,.
\end{aligned}
$$

Then the problem for $v_{m,tt}$ has the form

$$
\begin{aligned}
&\operatorname{rot} v_{m,tt} = \omega_{m,tt}\,,\\
&\operatorname{div} v_{m,tt} = H\,,\\
&v_{m,tt}\cdot\bar{n}|_S = 0
\end{aligned}
\tag{4.10}
$$

and we get the estimate

$$
\begin{aligned}
&|v_{m,tt}|_{1,1,\infty,\Omega^t} \le g_1\,(|\delta_{m-1}|_{2,2,\infty,\Omega^t}, |s_{m-1}|_{2,2,\infty,\Omega^t})\,\cdot\\
&\cdot|\delta_m|_{3,2,\infty,\Omega^t}\,(1 + |\delta_{m-1}|_{2,1,\infty,\Omega^t} + |s_{m-1}|_{2,1,\infty,\Omega^t}) + \\
&+ |v_{m-1}|_{2,2,\infty,\Omega^t}\sup_{\tau\le t}|v_{m-1,xt}(\tau)|_{4,\Omega} + \\
&+ \big(|v_{m-1}|_{2,1,\infty,\Omega^t}+1\big)\,|v_{m-1}|_{2,1,\infty,\Omega^t}|\delta_m|_{3,1,\infty,\Omega^t} + \\
&+ |\operatorname{div} f_t|_{0,0,\infty,\Omega^t}\,.
\end{aligned}
\tag{4.11}
$$

Combining (4.9), (4.11) and using (4.4) we obtain

$$
\begin{aligned}
&|v_m|_{3,1,\infty,\Omega^t} \le C|\omega_m|_{2,1,\infty,\Omega^t} + \\
&+ g_6(t|v_{m-1}|_{3,1,\infty,\Omega^t} + g_2(F_1)F_1 + F_3)\cdot \\
&\cdot|\delta_m|_{3,1,\infty,\Omega^t} + g_3(t|\delta_{m-1}|_{3,2,\infty,\Omega^t} + F_1, t|s_{m-1}|_{3,1,\infty,\Omega^t} + \\
&+ F_1)\cdot(1+t|\delta_{m-1}|_{3,1,\infty,\Omega^t} + t|s_{m-1}|_{3,1,\infty,\Omega^t} + \\
&+ F_1\cdot g_4(F_1))|\delta_m|_{3,2,\infty,\Omega^t} + \\
&+ (t|v_{m-1}|_{3,1,\infty,\Omega^t} + g_5(F_1)F_1 + F_3)\cdot \\
&\cdot \sup_{\tau\le t}|v_{m-1,xt}(\tau)|_{4,\Omega} + |\operatorname{div} f_t|_{0,0,\infty,\Omega^t}\,.
\end{aligned}
\tag{4.12}
$$

Simplifying (4.12) we get (4.7). This concludes the proof. □

LEMMA 4.3. *Assume that* $S\in C^5$, $v_0\cdot\bar n|_S=0$, $s_0,\rho_0,v_0\in H^3(\Omega)$, $\operatorname{div} f\in \Pi^2_{1,2}(\Omega^T)$, $s_{m-1},\delta_{m-1},v_{m-1}\in\Pi^2_{1,\infty}(\Omega^T)$, $f\cdot\bar n\in\Pi^3_{1,\infty}(\Omega^T)$. *Then there exists a unique solution of the problem* (a_m) *such that* $\delta_m\in\Pi^3_{1,\infty}(\Omega^T)$ *and it holds*

$$
\begin{aligned}
&|\delta_m|_{3,1,\infty,\Omega^T} \le h_8(T^a(|\delta_{m-1}|_{3,1,\infty,\Omega^T} + |s_{m-1}|_{3,1,\infty,\Omega^T} + \\
&+ |v_{m-1}|_{3,1,\infty,\Omega^T}) + h_9(F_1)F_1 + F_3, F_2, T)\cdot \\
&\cdot\Big(F_2+F_3+F_4+F_6+F_7+T^a|v_{m-1}|_{3,1,\infty,\Omega^T}\Big)\,,
\end{aligned}
\tag{4.13}
$$

where $a>0$.

Proof. In order to apply Proposition 8.2 from [1], we have to show that

$$
\gamma A_0^{\frac{1}{\gamma}} e^{\frac{\gamma-1}{\gamma}\delta_{m-1}} e^{\frac{1}{\gamma}s_{m-1}} \ge C_m > 0\,,
$$

where $C_m=\text{const}$. This follows from the assumption that $s_{m-1},\delta_{m-1}\in \Pi^3_{1,\infty}(\Omega^T)$.

Then we have (see formula (8.4) in [1])

$$
\begin{aligned}
&|\delta_m|_{3,1,\infty,\Omega^t} \le P_0\left(\|\,\delta(0)\,\|_{3,\Omega} + \|\,\delta_t(0)\,\|_{2,\Omega} + \right.\\
&\left.+ \|\,v^2_{0,x}\,\|_{1,\Omega} + \|\operatorname{div} f(0)\,\|_{1,\Omega}\right) e^{t\,P_2} + \\
&+ P_1\,e^{t\,P_2}\,t^{\frac12}\Big[|v_{m-1}|^2_{3,2,\infty,\Omega^t} + |\operatorname{div} f|_{2,1,\infty,\Omega^t} + \\
&+ g_1\left(|\delta_{m-1}|_{2,2,\infty,\Omega^t}, |s_{m-1}|_{2,2,\infty,\Omega^t}\right)\cdot \\
&\cdot\Big(1+|\delta_{m-1}|^3_{3,1,\infty,\Omega^t} + |s_{m-1}|^3_{3,1,\infty,\Omega^t}\Big)\Big(|v_{m-1}|^2_{3,1,\infty,\Omega^t} + \\
&+ |f\cdot\bar n|_{3,1,\infty,\Omega^t})\Big]\,,
\end{aligned}
\tag{4.14}
$$

where

$$\begin{aligned}P_0 =&P_0\left(|v_{m-1}|_{2,1,\infty,\Omega^t}, |\delta_{m-1}|_{2,1,\infty,\Omega^t}, |s_{m-1}|_{2,1,\infty,\Omega^t},\right.\\ &\left. e^{C|\delta_{m-1}|_{2,2,\infty,\Omega^t}}, e^{C|s_{m-1}|_{2,2,\infty,\Omega^t}}\right),\\ P_1 =&P_1\left(|v_{m-1}|_{3,2,\infty,\Omega^t}, |\delta_{m-1}|_{3,2,\infty,\Omega^t}, |s_{m-1}|_{3,2,\infty,\Omega^t},\right.\\ &\left. e^{C|\delta_{m-1}|_{2,2,\infty,\Omega^t}}, e^{C|s_{m-1}|_{2,2,\infty,\Omega^t}}\right),\\ P_2 =&P_2\left(|v_{m-1}|_{3,1,\infty,\Omega^t}, |s_{m-1}|_{3,1,\infty,\Omega^t}, |\delta_{m-1}|_{3,1,\infty,\Omega^t},\right.\\ &\left. e^{C|\delta_{m-1}|_{2,2,\infty,\Omega^t}}, e^{C|s_{m-1}|_{2,2,\infty,\Omega^t}}\right)\end{aligned}$$

are polynomials. From the relation (4.4) follows for $s = 1$

$$\begin{aligned}|v_{m-1}|_{2,1,\infty,\Omega^t} &\le t|v_{m-1}|_{3,1,\infty,\Omega^t} + g_2(F_1)F_1 + C\cdot F_3\,,\\ |\delta_{m-1}|_{2,1,\infty,\Omega^t} &\le t|\delta_{m-1}|_{3,1,\infty,\Omega^t} + g_3(F_1)F_1\,,\\ |s_{m-1}|_{2,1,\infty,\Omega^t} &\le t|s_{m-1}|_{3,1,\infty,\Omega^t} + g_4(F_1)F_1\,.\end{aligned} \tag{4.15}$$

Using (4.15) in (4.14) we see that for a certain $a > 0$ the expression t^a is a coefficient of $|v_{m-1}|_{3,1,\infty,\Omega^t}$, $|s_{m-1}|_{3,1,\infty,\Omega^t}$, $|\delta_{m-1}|_{3,1,\infty,\Omega^t}$.
Moreover,

$$\begin{aligned}\| \delta(0) \|_{3,\Omega} &\le g_5(F_1)F_2\,,\\ \| \delta_t(0) \|_{2,\Omega} &\le g_6(F_1)F_2\,.\end{aligned}$$

From the above considerations it follows that

$$\begin{aligned}|\delta_m|_{3,1,\infty,\Omega^t} \le g_7\,&(t|\delta_{m-1}|_{3,1,\infty,\Omega^t} + g_3(F_1)F_1\;,\\ &t|s_{m-1}|_{3,1,\infty,\Omega^t} + g_4(F_1)F_1\,,\, t|v_{m-1}|_{3,1,\infty,\Omega^t} +\\ &+\, g_8(F_1)F_1 + F_3\,,\, t^a|\delta_{m-1}|_{3,1,\infty,\Omega^t},\\ &t^a|s_{m-1}|_{3,1,\infty,\Omega^t},\, t^a|v_{m-1}|_{3,1,\infty,\Omega^t}, F_2, t)\,\cdot\\ &\cdot\left(F_2 + F_3 + F_4 + F_6 + F_7 + t^a|v_{m-1}|_{3,1,\infty,\Omega^t}\right)\,.\end{aligned} \tag{4.16}$$

Simplifying (4.16) we get (4.13). This concludes the proof. □

LEMMA 4.4 *Let $v_{m-1} \in \Pi^3_{1,\infty}(\Omega^T)$, $s_0 \in H^3(\Omega)$, $\rho_0 \in H^2(\Omega)$, $f(0) \in H^1(\Omega)$. Then there exists a unique solution of (d_m) such that $s_m \in \Pi^3_{1,\infty}(\Omega^T)$ and*

$$|s_m|_{3,1,\infty,\Omega^T} \le e^{CT|v_{m-1}|_{3,1,\infty,\Omega^T}}\cdot\left(h_{10}(F_1) + F_3\right)F_2\,. \tag{4.17}$$

Proof. The existence follows from the method of characteristics.
As in the proof of Lemma 4.1 we obtain

$$\frac{d}{dt}|s_m|_{3,1,\Omega} \le C\,|v_{m-1}|_{3,1,\Omega}\,|s_m|_{3,1,\Omega}\,. \tag{4.18}$$

The integration of (4.18) yields

$$|s_m|_{3,1,\Omega} \le e^{C\int_0^t |v_{m-1}(0)|_{3,1,\Omega}\,ds}\, |s_m(0)|_{3,1,\Omega}\,, \tag{4.19}$$

where

$$\begin{aligned} |s_m(0)|_{3,1,\Omega} \le C \parallel s_0 \parallel_{3,\Omega} \Big(1+ \parallel v_0 \parallel_{2,\Omega}^2 \; + \\ + \; g_1(\parallel s_0 \parallel_{2,\Omega}, \parallel \rho_0 \parallel_{2,\Omega})\,) + \parallel f(0) \parallel_{1,\Omega} \parallel s_0 \parallel_{3,\Omega} \;. \end{aligned} \tag{4.20}$$

Inserting (4.20) into (4.19) using the notation in (4.1) we get (4.17). This concludes the proof. □

5.The boundedness of the sequence. Let us define

$$\alpha_m = |\delta_m|_{3,1,\infty,\Omega^T} + |v_m|_{3,1,\infty,\Omega^T} + |s_m|_{3,1,\infty,\Omega^T}\,, \tag{5.1}$$

$$\beta_m = \sup_{t\le T} |v_{m,xt}|_{4,\Omega} \tag{5.2}$$

and

$$F = F_2 + F_3 + F_4 + F_5 + F_6 + F_7\,. \tag{5.3}$$

First we show

Lemma 5.1. *The sequences $\{\alpha_m\}$, $\{\beta_m\}$ satisfy the following inequalities*

$$\begin{aligned} \alpha_m \;\le\; G(T^a\alpha_{m-1}, F)(T^{a'} + F) \;+ \\ +\; C\,(T\alpha_{m-1} + F)\beta_{m-1} \equiv K_1(T) \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} &\beta_m \le \varepsilon G(T^a\alpha_{m-1}, F)(T^{a'} + F) + \\ &+ C\,\varepsilon\beta_{m-1}(T\alpha_{m-1} + F) \;+\; \frac{C}{\varepsilon}\,[T\,(G\,(T^a\alpha_{m-1}, F)\,\cdot \\ &\cdot (T^{a'} + F) + C\beta_{m-1}(T\alpha_{m-1} + F)\,) \;+\; h_1(F)F] \equiv K_2(T) \end{aligned} \tag{5.5}$$

where $\varepsilon \in (0,1)$, $a, a' > 0$, G is a positive increasing function of its arguments and $T < \infty$.

Proof. From Lemma 4.1 we obtain

$$|\omega_m|_{2,0,\Omega} \le G_1(T^a\alpha_{m-1}, F_2)(T^{a'} + F_5 + F_2 + F_3)\,. \tag{5.6}$$

Lemma 4.2 implies, using (5.6)

$$\begin{aligned} |v_m|_{3,1,\infty,\Omega^T} \le G_1(T^a\alpha_{m-1}, F_2)(T^{a'} + F_2 + F_3 + F_5) \;+ \\ +\; G_2(T\alpha_{m-1} + g_1(F_1)F_1 + F_3)\,|\delta_m|_{3,1,\infty,\Omega^T} \;+ \\ +\; C(T\alpha_{m-1} + g_2(F_1)F_1 + F_3)\beta_{m-1} + CF_6\,. \end{aligned} \tag{5.7}$$

From Lemma 4.3 and Lemma 4.4 follow the relations

$$|\delta_m|_{3,1,\infty,\Omega^T} \leq G_3(T^a\alpha_{m-1} + g_3(F_1)F_1 + F_3, F_2, T) \cdot \\ \cdot (F + T^a\alpha_{m-1}) \tag{5.8}$$

and

$$|s_m|_{3,1,\infty,\Omega^T} \leq G_4\,(T\alpha_{m-1}\,,\, g_4(F_1) + F_3)\,F_2\,, \tag{5.9}$$

where F is as in (5.3) and $a, a' > 0$. Simplifying (5.6)–(5.9) we get

$$|\delta_m|_{3,1,\infty,\Omega^T} \leq G_5(T^a\alpha_{m-1}, F, T)(F + T^a\alpha_{m-1})\,, \tag{5.10}$$

$$|s_m|_{3,1,\infty,\Omega^T} \leq G_6(T\alpha_{m-1}, F)F\,, \tag{5.11}$$

$$|v_m|_{3,1,\infty,\Omega^T} \leq G_7(T^a\alpha_{m-1}, F)(T^{a'} + F)\; + \\ +\; G_8(T^a\alpha_{m-1}, F, T)(F + T^a\alpha_{m-1})\; + \\ +\; C(T\alpha_{m-1} + F)\beta_{m-1}\,. \tag{5.12}$$

The combination of these inqualities leads to (5.4).

In view of the interpolation inequality we get

$$|v_{m,xt}|_{4,\Omega} \leq \varepsilon \parallel v_{m,xxt} \parallel_{0,\Omega} + \frac{C}{\varepsilon} \parallel v_{m,t} \parallel_{0,\Omega} \leq \\ \leq \varepsilon \parallel v_{m,xxt} \parallel_{0,\Omega} + \frac{C}{\varepsilon}(T \sup_{t\leq T} \parallel v_{m,tt} \parallel_{0,\Omega} + \\ + \parallel v_{m,t}(0) \parallel_{0,\Omega}) \tag{5.13}$$

and therefore

$$\beta_m \leq \varepsilon\alpha_m + \frac{C}{\varepsilon}\,(T\alpha_m + g_5(F)F)\;. \tag{5.14}$$

From (5.14) follows (5.5) using (5.4). This concludes the proof. □

To show the boundedness of the sequences $\{\alpha_m\}$ and $\{\beta_m\}$ we need the following

LEMMA 5.2. *There exist $\varepsilon > 0$ and sufficiently large $M = M(\varepsilon)$, $M' = M'(\varepsilon)$ such that if $\alpha_{m-1} \leq M$ and $\beta_{m-1} \leq M'$ it follows that $\alpha_m \leq M$ and $\beta_m \leq M'$ for $T < T_*$, where T_* is sufficiently small.*

Proof. Let $T = 0$ in $K_1(T)$ and $K_2(T)$ defined in (5.4) and (5.5). We are looking for $M, M', \varepsilon > 0$ such that

$$K_1(0) \le G(0,F)\,F + CFM' \le \frac{1}{2}M\,, \tag{5.15}$$
$$K_2(0) \le \varepsilon\, G(0,F)\,F + C\,\varepsilon M'F + \frac{C}{\varepsilon}h_1(F)F \le \le \frac{1}{2}M'\,.$$

Let $\varepsilon\, C\, F = \frac{1}{4}$. Then $(5.15)_2$ implies

$$\frac{1}{4CF}\,G(0,F)\,F + 4CF^2h_1(F) \le \frac{1}{4}M'\,. \tag{5.16}$$

We define M' by (5.16). Inserting it into $(5.15)_1$ we obtain M.

Since K_1 and K_2 in (5.4) and (5.5), respectively, are increasing functions of T, which at $T = 0$ satisfy the relations (5.15), there exists T_* such that for $T \le T_*$

$$K_1(T) \le M\,, \qquad K_2(T) \le M'\,. \tag{5.17}$$

This concludes the proof. □

Finally, we have

CONCLUSION 5.3. *Let $s_0, \rho_0, v_0 \in H^3(\Omega)$, $S \in C^5$, $f \in \Pi^3_{1,\infty}(\Omega^T)$, $v \cdot \bar{n}|_S = 0$. We choose the sequence $\{\delta_m, v_m, s_m\}$ in such a way that $\delta_m|_{m=0} = \delta_0 = \ln(A_0 e^{s_0}\rho_0^\gamma)$, $v_m|_{m=0} = v_0$, $s_m|_{m=0} = s_0$.*

Let

$$\alpha_0 = \|\, v_0\,\|_{3,\Omega} + \|\,\partial_t v(0)\,\|_{2,\Omega} + \|\,\partial_t^2 v(0)\,\|_{1,\Omega} + + \|\, s_0\,\|_{3,\Omega} + \|\,\partial_t s(0)\,\|_{2,\Omega} + \|\,\partial_t^2 s(0)\,\|_{1,\Omega} + + \|\,\delta_0\,\|_{3,\Omega} + \|\,\partial_t \delta(0)\,\|_{2,\Omega} + \|\,\partial_t^2 \delta(0)\,\|_{1,\Omega}$$

and $\beta_0 = |v_{xt}(0)|_{4,\Omega}$, where the time–derivates are calculated from equations (1.1) at $t = 0$. We assume that $\alpha_0 \le M$ and $\beta_0 \le M'$. Then Lemma 5.2 implies that

$$\alpha_m \le M\,, \quad m = 0, 1, \ldots\,. \tag{5.18}$$

6. Convergence of the sequence. In this section we show that the sequence $\{\delta_m, v_m, s_m\}$ converges strongly in $\Pi^2_{1,\infty}(\Omega^T)\times\Pi^2_{1,\infty}(\Omega^T)\times\Pi^2_{1,\infty}(\Omega^T)$ and weakly in $\Pi^3_{1,\infty}(\Omega^T)\times\Pi^3_{1,\infty}(\Omega^T)\times\Pi^3_{1,\infty}(\Omega^T)$.

Let us introduce the following differences $D_m = \delta_m - \delta_{m-1}$, $\Omega_m = \omega_m - \omega_{m-1}$, $V_m = v_m - v_{m-1}$, $R_m = \rho_m - \rho_{m-1}$ and $S_m = s_m - s_{m-1}$, where $m = 1, 2, \ldots$.

From (a_m) we obtain for D_m

$$\begin{aligned}
&Q^2_{m-1} D_m - \operatorname{div}\left(\frac{\gamma e^{\delta_{m-1}}}{z(\delta_{m-1}, s_{m-1})} \nabla D_m\right) = F_{m-1}\,,\\
&D_m|_{t=0} = 0\,,\\
&D_{m,t}|_{t=0} = 0\,,\\
&\frac{\partial D_m}{\partial n}|_{S^T} = G_{m-1}\,,
\end{aligned} \tag{A_m}$$

where

$$\begin{aligned}
F_{m-1} =& - V_{m-1}\nabla(Q_{m-1}\delta_{m-1}) - Q_{m-2}(V_{m-1}\nabla\delta_{m-1}) \;+\\
&+\; A_0^{\frac{1}{\gamma}} \operatorname{div}\Big\{\Big(S_{m-1} e^{\delta_{m-1}(\gamma-1)/\gamma} \int_0^1 e^{\frac{1}{\gamma}(ts_{m-1}+(1-t)s_{m-2})} dt \;+\\
&+\; (\gamma-1) D_{m-1} e^{s_{m-2}/\gamma} \int_0^1 e^{\frac{\gamma-1}{\gamma}(t\delta_{m-1}+(1-t)\delta_{m-2})} dt\Big) \cdot\\
&\cdot \nabla\delta_{m-1}\Big\} + \gamma\left(\partial_{x_i} V_{m-1,j}\partial_{x_j} v_{m-1,i} + \partial_{x_i} v_{m-2,j}\partial_{x_j} V_{m-1,i}\right)\,,
\end{aligned}$$

$$\begin{aligned}
G_{m-1} =& -\; A_0^{-\frac{1}{\gamma}}\gamma^{-1}\Big\{(\gamma-1) D_{m-1} e^{-s_{m-1}/\gamma}\cdot\\
&\cdot \int_0^1 e^{-\frac{\gamma-1}{\gamma}(t\delta_{m-1}+(1-t)\delta_{m-2})}\, dt \;+\\
&+\; S_{m-1} e^{-\delta_{m-2}\frac{\gamma-1}{\gamma}} \int_0^1 e^{-\frac{1}{\gamma}(ts_{m-1}+(1-t)s_{m-2})}\, dt\Big\}\cdot\\
&\cdot (v_{m-1,i}v_{m-1,j}n_{i,x_j} + f\cdot\bar{n})|_S \;+\\
&+\; A_0^{-\frac{1}{\gamma}} e^{-\delta_{m-2}\frac{\gamma-1}{\gamma}} e^{s_{m-2}/\gamma}\left(V_{m-1,i}v_{m-1,j}n_{i,x_j} \;+\right.\\
&\left.+\; v_{m-2,i}V_{m-1,j}n_{i,x_j}\right)|_{S^T}\,.
\end{aligned}$$

Ω_m is a solution of the problem

$$\begin{aligned}
&\Omega_{m,t} + v_{m-1}\cdot\nabla\Omega_m - \Omega_m\cdot\nabla v_{m-1} + \operatorname{div} v_{m-1}\cdot\Omega_m \;= \\
&-\; V_{m-1}\cdot\nabla\omega_{m-1} + \omega_{m-1}\cdot\nabla V_{m-1} - \operatorname{div} V_{m-1}\,\omega_{m-1} \;+ \\
&-\; A_0^{\frac{1}{\gamma}}\gamma^{-1}\nabla\left\{-D_{m-1}\int_0^1 e^{-(t\delta_{m-1}+(1-t)\delta_{m-2})/\gamma}dt\, e^{s_{m-1}/\gamma} \;+\right. \\
&\left.+\; e^{-\delta_{m-2}/\gamma}S_{m-1}\int_0^1 e^{(ts_{m-1}+(1-t)s_{m-2})/\gamma}dt\right\}\times\nabla e^{\delta_{m-2}} \;+ \\
&-\; A_0^{\frac{1}{\gamma}}\nabla\left(e^{(-\delta_{m-1}+s_{m-1})/\gamma}\right)\times \\
&\times\nabla\left(D_{m-1}\int_0^1 e^{t\delta_{m-1}+(1-t)\delta_{m-2}}dt\right), \\
&\Omega_m|_{t=0} = 0\,.
\end{aligned} \tag{B_m}$$

Finally, we get for V_m and S_m the problems

$$\begin{aligned}
&\operatorname{rot} V_m = \Omega_m\,, \\
&\operatorname{div} V_m = -\frac{1}{\gamma}(Q_{m-1}D_m + V_{m-1}\nabla\delta_{m-1}) \;+ \\
&+\; \frac{1}{\gamma}\frac{1}{|\Omega|}\int_\Omega (Q_{m-1}D_m + V_{m-1}\nabla\delta_{m-1})dx\,, \\
&V_m\cdot\bar{n}|_S = 0
\end{aligned} \tag{C_m}$$

and

$$\begin{aligned}
&S_{m,t} + v_{m-1}\cdot\nabla S_m + V_{m-1}\cdot\nabla s_{m-1} = 0\,, \\
&S_m|_{t=0} = 0\,.
\end{aligned} \tag{D_m}$$

Similarly as in Lemma 4.1 and Lemma 4.4 we obtain the estimates

$$\begin{aligned}
&\frac{d}{dt}|\Omega_m|_{1,0,\Omega} \le C|v_{m-1}|_{3,2,\Omega}|\Omega_m|_{1,0,\Omega} \;+ \\
&+\; C\left(|\omega_{m-1}|_{2,1,\Omega}|V_{m-1}|_{2,1,\Omega} + f_1\left(|\delta_{m-1}|_{3,2,\Omega},\right.\right. \\
&\left.|s_{m-1}|_{3,2,\Omega}, |\delta_{m-2}|_{3,2,\Omega}, |s_{m-2}|_{3,2,\Omega}\right)\cdot \\
&\left.\cdot\left(|D_{m-1}|_{2,1,\Omega} + |S_{m-1}|_{2,1,\Omega}\right)\right)
\end{aligned} \tag{6.1}$$

and

$$\begin{aligned}
&\frac{d}{dt}|S_m|_{2,1,\Omega} \le C\left(|v_{m-1}|_{3,2,\Omega}\,|S_m|_{2,1,\Omega} + \right. \\
&+\; \left.|V_{m-1}|_{2,1,\Omega}\,|s_{m-1}|_{3,2,\Omega}\right)\,.
\end{aligned} \tag{6.2}$$

From (C_m) follows the estimate

$$\begin{aligned}&\| V_m \|_{2,\Omega} \leq \| \Omega_m \|_{1,\Omega} + |D_m|_{2,1,\Omega}(1 + \| v_{m-1} \|_{2,\Omega}) + \\ &+ \| V_{m-1} \|_{1,\Omega} \| \delta_{m-1} \|_{3,\Omega} \ .\end{aligned} \tag{6.3}$$

In order to get an estimate for $V_{m,t}$ we differentiate (c_m) with respect to time and calculate $\delta_{m,tt}$ from $(a_m)_1$, i.e.

$$\begin{aligned}
\operatorname{rot} v_{m,t} =& \omega_{m,t}\,, \\
\operatorname{div} v_{m,t} =& -\frac{1}{\gamma}\partial_t(Q_{m-1}\delta_m) + \frac{1}{\gamma}\frac{1}{|\Omega|}\int_\Omega \partial_t(Q_{m-1}\delta_m)dx = \\
&= -\frac{1}{\gamma}\left[Q^2_{m-1}\delta_m - v_{m-1}\nabla(Q_{m-1}\delta_m)\right] + \\
&+ \frac{1}{\gamma}\frac{1}{|\Omega|}\int_\Omega \left(Q^2_{m-1}\delta_m - v_{m-1}\nabla(Q_{m-1}\delta_m)\right)dx \ = \\
&= -\operatorname{div}\left(\frac{e^{\delta_{m-1}}}{z(\delta_{m-1}, s_{m-1})}\nabla\delta_m\right) + \\
&- \partial_{x_i}v_{m-1,j}\partial_{x_j}v_{m-1,i} + \operatorname{div} f + \frac{1}{\gamma}v_{m-1}\nabla(Q_{m-1}\delta_m) + \\
&+ \frac{1}{|\Omega|}\int_\Omega \left[\operatorname{div}\left(\frac{e^{\delta_{m-1}}}{z(\delta_{m-1}, s_{m-1})}\nabla\delta_m\right) + \right. \\
&\left. + \partial_{x_i}v_{m-1,j}\partial_{x_j}v_{m-1,i} - \operatorname{div} f - \frac{1}{\gamma}v_{m-1}\nabla(Q_{m-1}\delta_m)\right]dx\,, \\
& v_{m,t}\cdot\bar{n}|_S = 0\,.
\end{aligned}$$

Then we obtain for $V_{m,t}$

$$\begin{aligned}
&\operatorname{rot} V_{m,t} = \Omega_{m,t}\,, \\
&\operatorname{div} V_{m,t} = -H_m + \frac{1}{|\Omega|}\int_\Omega H_m dx\,, \\
&V_{m,t}\cdot\bar{n}|_S = 0\,,
\end{aligned} \tag{6.4}$$

where

$$
\begin{aligned}
H_m =&\operatorname{div}\left(\frac{e^{\delta_{m-1}}}{z(\delta_{m-1},s_{m-1})}\nabla D_m\right)+\\
&+\ A_0^{\frac{1}{\gamma}}\frac{1}{\gamma}\operatorname{div}\left[\left(e^{\delta_{m-1}\frac{(\gamma-1)}{\gamma}}S_{m-1}\int_0^1 e^{(ts_{m-1}+(1-t)s_{m-2})/\gamma}\,dt\ +\right.\right.\\
&+\ \left.\left. e^{s_{m-2}/\gamma}(\gamma-1)D_{m-1}\int_0^1 e^{\frac{(\gamma-1)}{\gamma}(t\delta_{m-1}+(1-t)\delta_{m-2})}\,dt\right)\nabla\delta_{m-1}\right]+\\
&+\ \partial_{x_i}V_{m-1,j}\partial_{x_j}v_{m-1,i}+\partial_{x_i}v_{m-2,j}\partial_{x_j}V_{m-1,i}-\frac{1}{\gamma}V_{m-1}\nabla(Q_{m-1}\delta_m)\\
&-\ \frac{1}{\gamma}v_{m-2}\nabla(Q_{m-1}D_m+V_{m-1}\nabla\delta_{m-1})\,.
\end{aligned}
$$

The estimate for $V_{m,t}$ has the form

$$
\begin{aligned}
&\| V_{m,t}\|_{1,\Omega}\le \|\Omega_{m,t}\|_{0,\Omega}+f_2\left(\|\delta_{m-1}\|_{2,\Omega},\|s_{m-1}\|_{2,\Omega}\right)\cdot \qquad (6.5)\\
&\| D_m\|_{2,\Omega}+f_3\left(\|\delta_{m-1}\|_{3,\Omega},\|s_{m-1}\|_{3,\Omega},\|\delta_{m-2}\|_{3,\Omega},\|s_{m-2}\|_{3,\Omega}\right)\cdot\\
&\cdot\left(\|S_{m-1}\|_{1,\Omega}+\|D_{m-1}\|_{1,\Omega}\right)+\\
&+\ \|V_{m-1}\|_{1,\Omega}\left(\|v_{m-1}\|_{3,\Omega}+\|v_{m-2}\|_{3,\Omega}+(1+\|v_{m-1}\|_{3,\Omega}+\right.\\
&+\ \left.\|v_{m-2}\|_{2,\Omega})\cdot\|\delta_{m-1}\|_{3,\Omega}\right)+\|v_{m-2}\|_{2,\Omega}\cdot\\
&\cdot(1+\|v_{m-1}\|_{2,\Omega})\,\|D_m\|_{2,\Omega}\ .
\end{aligned}
$$

Finally, we obtain for D_m

$$
\begin{aligned}
&|D_m|^2_{2,1,\infty,\Omega^T}\le \qquad (6.6)\\
&\le P_1e^{P_2T}\int_0^T\left(\|F_{m-1}(s)\|^2_{1,\Omega}+|G_{m-1}(s)|^2_{2,1,\Omega}\right)ds\,,
\end{aligned}
$$

where

$$
\begin{aligned}
&\|F_{m-1}\|_{1,\Omega}\le C|V_{m-1}|_{2,1,\Omega}\left(1+\|v_{m-1}\|_{3,\Omega}+\|v_{m-2}\|_{3,\Omega}\right)\cdot\\
&\cdot(1+|\delta_{m-1}|_{3,2,\Omega})+f_4\left(\|\delta_{m-1}\|_{3,\Omega},\|s_{m-1}\|_{3,\Omega},\|s_{m-2}\|_{3,\Omega},\right.\\
&\left.\|\delta_{m-2}\|_{3,\Omega}\right)\left(\|S_{m-1}\|_{2,\Omega}+\|D_{m-1}\|_{2,\Omega}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&|G_{m-1}|_{2,1,2,\Omega^T}\ \le\ f_5\Big(|s_{m-1}|_{3,2,\infty,\Omega^T},\,|s_{m-2}|_{3,2,\infty,\Omega^T},\\
&|\delta_{m-1}|_{3,2,\infty,\Omega^T},\,|\delta_{m-2}|_{3,2,\infty,\Omega^T}\Big)\Big(|D_{m-1}|_{2,1,2,\Omega^T}+|S_{m-1}|_{2,1,2,\Omega^T}\Big)\cdot\\
&\cdot\Big(|v_{m-1}|^2_{2,1,\infty,\Omega^T}+|f|_{2,1,\infty,\Omega^T}\Big)+\Big(|v_{m-1}|_{2,1,\infty,\Omega^T}+|v_{m-2}|_{2,1,\infty,\Omega^T}\Big)\cdot\\
&\cdot f_6\Big(|\delta_{m-2}|_{2,1,\infty,\Omega^T},\,|s_{m-2}|_{2,1,\infty,\Omega^T}\Big)|V_{m-1}|_{2,1,2,\Omega^T}\,.
\end{aligned}
$$

Now we can prove

Lemma 6.1. *Let the assumptions of Conclusion 5.3 hold. Let*

$$y_m = |D_m|_{2,1,\infty,\Omega^T} + |V_m|_{2,1,\infty,\Omega^T} + |S_m|_{2,1,\infty,\Omega^T} \tag{6.7}$$

for $m \geq 1$. Then

$$y_m \leq h_{11}(T,M)\; T^{\frac{1}{2}} y_{m-1} \tag{6.8}$$

and $\{y_m\}$ converges to zero, for sufficiently small T.

Proof. In view of the result of Conclusion 5.3 we get from (6.1), (6.2), (6.3), (6.5) and (6.6)

$$|\Omega_m|_{1,0,\infty,\Omega^T} \leq g_1(T,M)\; T \Big(|V_{m-1}|_{2,1,\infty,\Omega^T} + \tag{6.9}$$

$$+ |D_{m-1}|_{2,1,\infty,\Omega^T} \; + \; |S_{m-1}|_{2,1,\infty,\Omega^T}\Big) \;,$$

$$|S_m|_{2,1,\infty,\Omega^T} \leq g_2(T,M)\; T\; |V_{m-1}|_{2,1,\infty,\Omega^T} \;, \tag{6.10}$$

$$\begin{aligned} |V_m|_{2,1,\infty,\Omega^T} \leq g_3(M)\; & \Big(|\Omega_m|_{1,0,\infty,\Omega^T} + |D_m|_{2,1,\infty,\Omega^T} + \\ & + \; |V_{m-1}|_{1,1,\infty,\Omega^T} + |S_{m-1}|_{1,1,\infty,\Omega^T} + |D_{m-1}|_{1,1,\infty,\Omega^T}\Big) \;, \end{aligned} \tag{6.11}$$

$$|D_m|_{2,1,\infty,\Omega^T} \leq g_4(T,M)\; T^{\frac{1}{2}} \Big(|V_{m-1}|_{2,1,\infty,\Omega^T} + \tag{6.12}$$

$$+ |D_{m-1}|_{2,1,\infty,\Omega^T} \; + \; |S_{m-1}|_{2,1,\infty,\Omega^T}\Big) \;.$$

With

$$U_{m-1} = \int_0^T \partial_t U_{m-1}\, dt$$

we have

$$|U|_{1,1,\infty,\Omega^T} \leq T |U|_{2,1,\infty,\Omega^T} \tag{6.13}$$

and using (6.13) for $V_{m-1}, S_{m-1}, D_{m-1}$ at the right–hand side of (6.11) yields

$$\begin{aligned} |V_m|_{2,1,\infty,\Omega^T} \leq g_3(M)\; & (\; |\Omega_m|_{1,0,\infty,\Omega^T} + |D_m|_{2,1,\infty,\Omega^T} \; + \\ & + \; T\Big(|V_{m-1}|_{2,1,\infty,\Omega^T} + |S_{m-1}|_{2,1,\infty,\Omega^T} + |D_{m-1}|_{2,1,\infty,\Omega^T}\Big)\Big) \;. \end{aligned} \tag{6.14}$$

From (6.9), (6.10), (6.12) and (6.14) we obtain (6.7). This concludes the proof. □

7. Existence and uniqueness. Finally, we present the main result of this paper.

THEOREM 7.1. *Let* $s_0 \in H^3(\Omega)$, $\rho_0 \in H^3(\Omega)$, $v_0 \in H^3(\Omega)$, $f \in \Pi^3_{1,\infty}(\Omega^T)$, *div* $f \in \Pi^2_{1,\infty}(\Omega^T)$, $f(0) \in H^1(\Omega)$ *and* $v_0 \cdot \bar{n}|_S = 0$ *where* $\partial\Omega = S \in C^5$. *Assume the compatibility conditions*

$$\frac{\partial}{\partial n}\left(\ln(A_0 e^{s_0}\rho_0^\gamma)\right)\Big|_S = \frac{z(\delta_0, s_0)(v_{0,i}v_{0,j}n_{i,x_j} + f(0)\cdot\bar{n})}{e^{\delta_0}}\Big|_S\,,$$

$$\frac{\partial}{\partial n}\left(-v_0\cdot\nabla s_0 - \frac{\gamma}{\rho_0}(v_0\cdot\nabla\rho_0 + \rho_0\operatorname{div} v_0\right)\Big|_S =$$

$$= \frac{\partial}{\partial t}\left(\frac{z(\delta, s)(v_i v_j n_{i,x_j} + f\cdot\bar{n})}{e^{\delta}}\right)\Big|_{S,t=0}\,,$$

where $\delta_t(0)$, $s_t(0)$, $v_t(0)$ *are calculated from (2.7).*

Then there exists a solution of problem (a)–(d) such that $\omega \in \Pi^2_{0,\infty}(\Omega^T)$, $v, s, \rho \in \Pi^3_{1,\infty}(\Omega^T)$ *for sufficiently small* T.

Proof. From Conclusion 5.3 follows that the sequence $\{\delta_m, \omega_m, v_m, s_m\}$ is bounded in $\Pi^3_{1,\infty}(\Omega^T) \times \Pi^2_{0,\infty}(\Omega^T) \times \Pi^3_{1,\infty}(\Omega^T) \times \Pi^3_{1,\infty}(\Omega^T) \equiv \Lambda$, for sufficiently small T. Therefore there exists a subsequence which converges weakly star in Λ to $\{\delta, \omega, v, s\} \in \Lambda$.

Lemma 6.1 implies that for sufficiently small T the sequence $\{\delta_m, \omega_m, v_m, s_m\}$ converges strongly in $\Pi^2_{1,\infty}(\Omega^T) \times \Pi^1_{0,\infty}(\Omega^T) \times \Pi^2_{1,\infty}(\Omega^T) \times \Pi^2_{1,\infty}(\Omega^T)$, i.e. also to $\{\delta, \omega, v, s\} \in \Lambda$. Moreover, the sequence converges almost everywhere.

Let us show that $\{\delta, \omega, v, s\}$ is a solution of (a,b,c,d). We consider the problems (a_m), (b_m) in the following integral form

$$\int_{\Omega^T}\left(Q^2_{m-1}\delta_m - \operatorname{div}\left(\frac{\gamma e^{\delta_{m-1}}}{z(\delta_{m-1}, s_{m-1})}\nabla\delta_m\right) + \right.$$

$$\left. -\ \gamma\,\partial_{x_i}v_{m-1,j}\partial_{x_j}v_{m-1,i} + \gamma\operatorname{div} f\right)\cdot\phi_1\,dx\,dt = 0\,,$$

$$\int_{\Omega^T}\left(\omega_{m,t} + v_{m-1}\cdot\nabla\omega_m - \omega_m\cdot\nabla v_{m-1} + \operatorname{div} v_{m-1}\cdot\omega_m + \right.$$

$$\left. +\ \nabla\frac{1}{z(\delta_{m-1}, s_{m-1})}\times\nabla e^{\delta_{m-1}} - \operatorname{rot} f\right)\phi_2\,dx\,dt = 0\,,$$

where $\phi_1, \phi_2 \in C^\infty(\Omega^T)$. Since we proved the convergence of the sequence $\{\delta_m, \omega_m, v_m, s_m\}$ we can pass to the limit in the above identities. The limit functions $\{\delta, \omega\}$ are weak solutions of (a), (b).

Since $s_m, v_m \in \Pi^3_{1,\infty}(\Omega^T)$ and $s, v \in \Pi^3_{1,\infty}(\Omega^T)$ the embedding theorem implies that s_m, v_m, s and v are continuous functions. On the other hand we have strong convergence $s_{m,t} \to s_t$ in $\Pi^1_{1,\infty}(\Omega^T)$, $s_{m,x} \to s_x$ in $\Pi^1_{0,\infty}(\Omega^T)$,

$v_{m,x} \to v_x$ in $\Pi^1_{0,\infty}(\Omega^T)$, $\delta_{m,x} \to \delta_x$ in $\Pi^1_{0,\infty}(\Omega^T)$ and $\delta_{m,t} \to \delta_t$ in $\Pi^1_{1,\infty}(\Omega^T)$, i.e. all those sequences converge almost everywhere to their limits. From the continuity of the functions it follows that we can pass to the limits in (c_m), (d_m) classically.

It remains to show that

$$\psi(t) = \frac{1}{|\Omega|}\int_\Omega Q\delta\,dx = 0\,.$$

From (2.16) and $v_0 \cdot \bar{n} = 0$ we have $\psi(0) = 0$. Relation $(a)_1$ implies

$$\begin{aligned}\psi_t =&\frac{1}{|\Omega|}\int_\Omega \partial_t(Q\delta)\,dx = \\ =&\frac{1}{|\Omega|}\int_\Omega Q^2\delta\,dx - \frac{1}{|\Omega|}\int_\Omega v\cdot\nabla(Q\delta)\,dx = \\ =&\frac{1}{|\Omega|}\int_\Omega \left\{\operatorname{div}\left(\frac{\gamma e^\delta}{\rho}\nabla\delta\right) + \gamma\left(\partial_{x_i}v_j\,\partial_{x_j}v_i - \operatorname{div} f\right)\right\}dx\; - \\ &- \frac{1}{|\Omega|}\int_\Omega v\cdot\nabla(Q\delta)\,dx\,.\end{aligned}$$

Performing integration by parts in the first term on the right hand side and using $(a)_4$ we have

$$\begin{aligned}\psi_t =&\frac{\gamma}{|\Omega|}\int_S v_iv_jn_{i,x_j}\,ds + \frac{\gamma}{|\Omega|}\int_S n_iv_j\partial_{x_j}v_i\,ds \\ &- \frac{\gamma}{|\Omega|}\int_\Omega v\cdot\nabla(\operatorname{div} v)\,dx - \frac{1}{|\Omega|}\int_\Omega v\cdot\nabla(Q\delta)\,dx\,.\end{aligned}$$

Finally, we obtain after integration by parts in the 2nd term and using $\operatorname{div} v = -\frac{1}{\gamma}Q\delta + \frac{1}{\gamma}\frac{1}{|\Omega|}\int_\Omega Q\delta\,dx$, $\psi_t = 0$, $\psi(0) = 0$, that $\psi = 0$.

Hence $\{\delta, \omega, v, s\}$ is a solution of problem (a,b,c,d). This concludes the proof. □

Let us show the equivalence of the problems (2.7) and (a,b,c,d,e).

LEMMA 7.2. *Let $v, p, s \in \Pi^3_{1,\infty}(\Omega^T)$. Then problem (2.7) is equivalent to problem (a,b,c,d,e).*

Proof. The proof in one direction follows from the construction of problem (a,b,c,d,e).

From problem $(c)_2$ follows $(2.7)_1$. Relation $(c)_3$ implies the boundary condition $(2.7)_5$. From $(d)_1$ we have $(2.7)_3$. Equations $(a)_1$, $(b)_1$, (e) and

$(2.7)_5$ lead to

$$\begin{aligned} &\operatorname{div} \Phi = 0\,, \\ &\operatorname{rot} \Phi = 0\,, \\ &\Phi \cdot \bar{n}|_S = 0\,, \end{aligned}$$

where $\Phi = \partial_t v + v\nabla v + \frac{1}{\rho}\nabla p - f$.

For simply connected domains it follows that $\Phi = 0$ [2]. This concludes the proof. □

Remark 7.3. By the presented method we proved existence of the weakest possible solutions. However, the solution $\{\delta, v, s\}$ is Hölder continuous. Therefore, the system (2.7) is satisfied in a weak sense only. To show higher regularity we have to apply well-known regularization techniques.

Remark 7.4. The uniqueness of the solution $\{\delta, \omega, v, s\}$ follows in a standard way.

Remark 7.5. More general results for general state equations can be shown in a similar way.

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