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COINITIAL FAMILIES OF PERFECT SETS

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Abstract. Let Perf(X) denote the family of all perfect subsets of a perfect Polish space X. We show several properties and characterizations of coinitial subfamilies of Perf(X). Connections with the Vietoris topology and Morgan's category bases are observed. We characterize the least cardinality of a family of coinitial ideals with noncoinitial intersection. Some examples of special noncoinitial families in Perf(X) are presented. Much place is devoted to perfect isomorphisms between ideals of subsets of X.

1. Introduction. Hereditary and finitely additive families of small sets (i.e. ideals) appear frequently in real analysis, measure theory, topology and combinatorics (cf. [31], [12], [21], [26], [28], [32], [14]). It often happens that those families considered in metric spaces contain many perfect sets. (We mean a perfect set as nonempty, closed and dense–in–itself.) Below we describe that situation more explicitly.

Through the paper, X is a fixed perfect Polish space. For any set E, let Pow(E) denote the power set of E. If $E \subseteq X$ then Perf(E) stands for the family of all perfect subsets of E. Consider the partial ordering on Perf(X) given by $P \leq Q$ iff $P \subseteq Q$. In the case when X is the set of reals, that

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ordering is known to produce the Sacks forcing (cf. [18, p.284]). We are interested in subfamilies $\mathcal{F}$ of $\mathrm{Perf}(X)$ which are *coinitial* with respect to $\leq$, i.e.

$$(\forall P \in \mathrm{Perf}(X))(\exists Q \in \mathcal{F})(Q \leq P).$$

In particular, we consider $\mathcal{F}$ to be hereditary or to form an ideal or a σ–ideal in $\mathrm{Perf}(X)$ (the respective definitions are given in Section 2). Note that coinitial hereditary families of perfect sets were considered by Marczewski in [38]. He showed some applications to sets with the Baire property in the restricted sense and observed connections with (s_0)–sets and (s)–sets. Independently of Marczewski's investigations, systematic studies of a related property $(\mathbf{P})$ of ideals $\mathcal{I}$ in $\mathrm{Pow}(X)$ are initiated in [1] and [2]. Namely, $(\mathbf{P})$ states that for each perfect set $P \subseteq X$ there exists a perfect set $Q \subseteq P$ such that $Q \in \mathcal{I}$. Obviously $(\mathbf{P})$ holds true iff $\mathcal{I} \cap \mathrm{Perf}(X)$ is coinitial in $\mathrm{Perf}(X)$. Also in [1] and [2], some applications of property $(\mathbf{P})$, based on the Sierpiński-Erdös theorem [31, Th.19.5] and dealing with isomorphisms between ideals, are shown. In [3], property $(\mathbf{P})$ of an ideal $\mathcal{I}$ turned out useful to show that the σ–ideal of Marczewski (s_0)–sets is not contained in $\mathcal{I}$. Further applications to isomorphism problems are found in [4], and connections with residual families in the hyperspace of perfect sets are observed in [6]. Here we develop some of those ideas and give a survey of known facts. The old results of Marczewski from [38] turn out very useful. We recall some of them with proofs to sake the completeness (note that [38] is written in French). To get a better description, we restrict our considerations to perfect sets. Thus, we would rather investigate ideals in $\mathrm{Perf}(X)$ than in $\mathrm{Pow}(X)$.

In Section 3 we recall the result of Marczewski stating that the intersection of a sequence of coinitial weak ideals is again a coinitial weak ideal. Then we discuss problems concerning intersections of a greater number of coinitial ideals. Here we use some special cardinals. Next, we obtain topological characterizations of certain coinitial families (Section 4). One of them is applied to produce residual sets in the space $\mathrm{Perf}(X)$ endowed with the relative Vietoris topology when X is metric and compact (Section 5). We find connections between coinitial families of perfect sets and Morgan's perfect category bases (Section 6). The paper contains several examples of various coinitial families of perfect sets. We also consider some special invariant noncoinitial families in $\mathrm{Perf}(X)$ provided that X forms a metric group (Section 7). In the last section we study the notion of a perfect isomorphism between σ–ideals in $\mathrm{Pow}(X)$. We compare it with other kinds of isomorphisms. Finally, some consistency problems are posed.

2. Ideals of perfect sets. First, let us state some elementary properties of coinitial families in $\mathrm{Perf}(X)$. Since there are continuum mamy disjoint perfect sets in X, each coinitial family is of size $\mathfrak{c}$ (the cardinality of continuum). Now, we will show that a coinitial family $\mathcal{F} \subseteq \mathrm{Perf}(X)$ need not satisfy $\bigcup \mathcal{F} = X$. Recall that $A \subseteq X$ is an (s_0)*–set* if, for each $P \in \mathrm{Perf}(X)$, there is a $Q \in \mathrm{Perf}(P)$ such that $Q \cap A = \emptyset$ [38]. There exists an (s_0)–set of size $\mathfrak{c}$ [26] but no (s_0)–set contains a perfect set.

The ball in X with centre x and radius r will be written as $B(x, r)$, and the closure of $E \subseteq X$ – as $\mathrm{cl}(E)$.

Theorem 2.1. *A set $E \subseteq X$ satisfies $X \setminus E = \bigcup \mathcal{F}$ for a coinitial family $\mathcal{F}$ in Perf(X) if and only if E is an (s_0)–set.*

Proof. "$\Rightarrow$" Let $P \in \mathrm{Perf}(X)$. Since $\mathcal{F}$ is coinitial, there exists a $Q \in \mathcal{F} \cap \mathrm{Perf}(P)$. Since $\bigcup \mathcal{F} = X \setminus E$, we have $Q \cap E = \emptyset$. Hence E is an (s_0)–set.

"$\Leftarrow$" Let E be an (s_0)–set. We will show that $\mathcal{F} = \mathrm{Perf}(X \setminus E)$ satisfies the assertion. Obviously $\bigcup \mathcal{F} \subseteq X \setminus E$. To prove the converse, consider any $x \in X \setminus E$. Since X is perfect, we can extract a perfect part P_n of $\mathrm{cl}B(x, 1/(n+1))$ for each $n \in \omega$. Since E is an (s_0)–set, for each $n \in \omega$ there is a $Q_n \in \mathrm{Perf}(P_n)$ such that $Q_n \cap E = \emptyset$. Then $Q = \{x\} \cup \bigcup_{n\in\omega} Q_n$ is perfect and $x \in Q \subseteq X \setminus E$. Thus $X \setminus E \subseteq \bigcup \mathcal{F}$. It remains to show that $\mathcal{F}$ is coinitial. Let $D \in \mathrm{Perf}(X)$. Since E is an (s_0)–set, there is a $P \in \mathrm{Perf}(D)$ such that $P \cap E = \emptyset$. Then $P \in \mathcal{F}$ as desired. $\square$

Sometimes it is natural to assume that a coinitial family $\mathcal{F}$ satisfies $\bigcup \mathcal{F} = X$. For instance, it holds true if X forms a group and $\mathcal{F}$ is invariant.

The notion of an ideal of sets is usually associated with a fixed algebra of sets. Here we consider it in connection with the family $\mathrm{Perf}(X)$ which does not form an algebra (it is not stable under complementation).

We say that $\mathcal{I} \subseteq \mathrm{Perf}(X)$ is:

(a) *hereditary*, if for any $A \in \mathcal{I}$ and $B \in \mathrm{Perf}(A)$ we have $B \in \mathcal{I}$;

(b) (*weakly*) *finitely additive*, if for any (disjoint) sets $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$;

(c) *countably additive*, if for each $\{A_n\}_{n\in\omega} \subseteq \mathcal{I}$, from $\bigcup_{n\in\omega} A_n \in \mathrm{Perf}(X)$ it follows that $\bigcup_{n\in\omega} A_n \in \mathcal{I}$;

(d) a (*weak*) *ideal*, if it is hereditary and (weakly) finitely additive;

(e) a σ*–ideal*, if it is hereditary and countably additive.

An ideal $\mathcal{I}$ in $\mathrm{Perf}(X)$ is called *principal* if it is of the form $\mathrm{Perf}(A)$ for some $A \subseteq X$. By Theorem 2.1, if $\mathcal{I}$ is a principal coinitial ideal in $\mathrm{Perf}(X)$, then $\mathcal{I} = \mathrm{Perf}(X \setminus E)$ where E is an (s_0)–set.

Assume that $\mathcal{F} \subseteq \mathrm{Perf}(X)$. Then the family $\mathcal{I}$ of all sets $D \in \mathrm{Perf}(X)$ included in finite (resp. countable) unions of sets from $\mathcal{F}$ forms an ideal (resp. σ–ideal) in $\mathrm{Perf}(X)$ called the *ideal* (resp. *σ–ideal*) *generated* by $\mathcal{F}$. Obviously, if $\mathcal{F}$ is coinitial, so is $\mathcal{I}$.

Let us note that hereditary families and ideals can be also considered in $\mathrm{CL}(X)$, the family of all closed subsets of X. Several interesting examples of σ–ideals in $\mathrm{CL}(X)$ (when X is metric and compact) are given in [21]. Clearly, if $\mathcal{I}$ is a (weak) ideal in $\mathrm{CL}(X)$ then $\mathcal{I} \cap \mathrm{Perf}(X)$ forms a (weak) ideal in $\mathrm{Perf}(X)$.

Now, we are going to give some examples of coinitial hereditary families of perfect sets. The family of nowhere dense perfect sets in X, or the family of Lebesgue null perfect sets in $X = \mathbb{R}$ are well-known coinitial σ–ideals in $\mathrm{Perf}(X)$. Another example can be derived from [9, Lemma 2] where the σ–ideal of Ramsey null perfect sets in $X = [\omega]^\omega$ (the space of all infinite subsets of $[\omega]^\omega$) turns out coinitial. Some natural examples of coinitial families of perfect sets appear in the theory of real functions. For instance, Mazurkiewicz in [25] proved that, for each sequence f^* of continuous functions $f_n : [0,1] \to \mathbb{R}$, bounded by the same constant, and for each $P \in \mathrm{Perf}([0,1])$, there is a $Q \in \mathrm{Perf}(P)$ such that f^{**} is pointwise convergent on Q for some subsequence f^{**} of f^*. Thus, for any fixed f^* with the above properties, the family of all $P \in \mathrm{Perf}([0,1])$ such that a subsequence of f^* is pointwise convergent on P forms a coinitial family in $\mathrm{Perf}([0,1])$. A weak ideal in $\mathrm{Perf}([0,1])$ can be associated with the result of [11] where it was shown that, for each continuous function $f : [0,1] \to \mathbb{R}$ and for each $P \in \mathrm{Perf}([0,1])$, there is a $Q \in \mathrm{Perf}(P)$ such that $f|Q$ is differentiable in the extended sense (i.e. $f'(x)$ can be $+\infty$ or $-\infty$). A general concept of that kind of theorems is presented in [10, pp. 511–512]. Note that in harmonic analysis there are many natural hereditary families $\mathcal{F} \subseteq \mathrm{Pow}(\mathbb{R})$ of small sets, stable under translations and containing perfect sets [12]. One can build from them coinitial ideals in $\mathrm{Perf}(\mathbb{R})$.

The next type of examples, which will be useful in the sequel, was considered in [3]. We say that $\mathcal{F} \subseteq \mathrm{Perf}(X)$ is an *almost disjoint family* (in short, adf), if $|P \cap Q| \leq \omega$ for any distinct $P, Q \in \mathcal{F}$. By Zorn's lemma, each adf can be extended to a maximal adf. Observe that, if $\mathcal{F}$ is an adf of size $< \mathfrak{c}$ and $X \setminus \bigcup \mathcal{F}$ has no perfect subset, then $\mathcal{F}$ is already maximal. Indeed, let $P \in \mathrm{Perf}(X) \setminus \mathcal{F}$ and suppose that $|P \cap Q| \leq \omega$ for each $Q \in \mathcal{F}$. Thus $|P \cap \bigcup \mathcal{F}| < \mathfrak{c}$ and consequently $P \cap \bigcup \mathcal{F}$ is an (s_0)–set [40, Th.2.1]. Hence there is a $Q \in \mathrm{Perf}(P)$ such that $Q \subseteq P \setminus \bigcup \mathcal{F}$, a contradiction. Note that, by [27], in a model of ZFC in which CH fails, there exisis a disjoint family of size ω_1 consisting of closed sets whose union is X. Taking perfect parts of those closed sets we get a disjoint family $\mathcal{F} \subseteq \mathrm{Perf}(X)$ of size ω_1

and fulfilling $|\mathcal{F}| = \omega_1 < \mathfrak{c}$ and $|X \setminus \bigcup \mathcal{F}| \leq \omega_1$. Thus we have a situation considered above.

3. Intersections of coinitial families. Evidently, the intersection of a finite number of coinitial families in $\mathrm{Perf}(X)$ is again a coinitial hereditary family. An analogous statement can be false for a sequence of coinitial hereditary families. Indeed, for each $n \in \omega$, let $\mathcal{I}_n$ denote the collection of all $P \in \mathrm{Perf}(X)$ with diameters $\leq 1/n$. Then $\mathcal{I}_n$ are coinitial hereditary families in $\mathrm{Perf}(X)$ but $\bigcap \mathcal{I}_n = \emptyset$. For weak coinitial ideals we have the following theorem of Marczewski:

Theorem 3.1. *([38, 2.1], cf. also [2, p.25].) If $\{I_n\}_{n\in\omega}$ is a sequence of weak coinitial ideals in Perf(X) then $\mathcal{I} = \bigcap_{n\in\omega} \mathcal{I}_n$ is a weak ideal (coinitial) in Perf(X).*

Proof. First, we show that $\mathcal{I}$ is coinitial. Let $P \in \mathrm{Perf}(X)$. Denote by Seq the set of all finite sequences of zeros and ones. The empty sequence is written as $\langle\ \rangle$. The set of all sequences from Seq with length n is denoted by Seq_n. We define by induction a family $\{P_s : s \in \mathrm{Seq}\} \subseteq \mathrm{Perf}(X)$ as follows. Pick $Q_{\langle\ \rangle} \in \mathcal{I}_0 \cap \mathrm{Perf}(P)$ with diameter ≤ 1. If $n \in \omega$ and sets $Q_s \in \mathcal{I}_n$ ($s \in \mathrm{Seq}_n$) with diameters $\leq 1/(n+1)$ are defined, pick two disjoint perfect sets $P_{s0}, P_{s1} \subseteq Q_s$ with diameters $\leq 1/(n+2)$ (where si, $i \in \{0,1\}$, is the respective extension of the sequence s). Then choose perfect sets $Q_{s0}, Q_{s1} \in \mathcal{I}_{n+1}$ contained respectively in P_{s0}, P_{s1}. Finally, put $Q = \bigcap_{n\in\omega} \bigcup_{s\in\mathrm{Seq}_n} Q_s$. By the fusion lemma [18, p.285], the set Q is perfect. Evidently, $Q \subseteq P$. From the pairwise disjointness of the sets $Q_s \in \mathcal{I}_n$ ($s \in \mathrm{Seq}_n$) and from the weak additivity of $\mathcal{I}_n$ it follows that $\bigcup_{s\in\mathrm{Seq}_n} Q_s \in \mathcal{I}_n$ for each $n \in \omega$. Since every $\mathcal{I}_n$ is hereditary in $\mathrm{Perf}(X)$, we have $Q \in \mathcal{I}$. It easily follows from the definition that $\mathcal{I}$ is a weak ideal in $\mathrm{Perf}(X)$. □

Assume that $\mathcal{G}$ is a family of coinitial weak ideals in $\mathrm{Perf}(X)$. Observe that, if $\bigcap \mathcal{G} \neq \emptyset$, then $\bigcap \mathcal{G}$ is a weak ideal in $\mathrm{Perf}(X)$. It would be interesting to describe those uncountable families $\mathcal{G}$ for which $\bigcap \mathcal{G}$ is coinitial. Let us give some examples.

Example 1. Assume CH. List all elements of X as $\{x_\alpha : \alpha < \omega_1\}$ and let $\mathcal{I}_\gamma = \mathrm{Perf}(X \setminus \{x_\alpha : \alpha < \gamma\})$ for $\gamma < \omega_1$. Then each $\mathcal{I}_\gamma$ ($\gamma < \omega_1$) is a coinitial σ–ideal in $\mathrm{Perf}(X)$ but $\bigcap_{\gamma<\omega_1} \mathcal{I}_\gamma = \emptyset$. Note that I_γ are principal ideals and they do not satisfy $\bigcup \mathcal{I}_\gamma = X$.

Example 2. We will modify Example 1 to work with nonprincipal ideals. Pick $P, Q \in \mathrm{Perf}(X)$ such that $P \cup Q = X$ and $|P \setminus Q| = |Q \setminus P| = \mathfrak{c}$.

Assuming CH let $P = \{x_\alpha : \alpha < \omega_1\}$. For each $\gamma < \omega_1$, let $\mathcal{I}_\gamma$ consist of all $D \in \mathrm{Perf}(X)$ such that $D \cap Q$ is nowhere dense in Q and $D \cap \{x_\alpha : \alpha < \gamma\} = \emptyset$. Then each $\mathcal{I}_\gamma$ ($\gamma < \omega_1$) is a nonprincipal coinitial σ–ideal in $\mathrm{Perf}(X)$. Note that P has no perfect subset in $\mathcal{I} = \bigcap_{\gamma<\omega_1} \mathcal{I}_\gamma$, so $\mathcal{I}$ is not coinitial in $\mathrm{Perf}(X)$.

Example 3. Let $\mathcal{G}$ be the family of all coinitial nonprincipal ideals $\mathcal{I}$ in $\mathrm{Perf}(X)$ (we may add that $\bigcup \mathcal{I} = X$). Then $\bigcap \mathcal{G} = \emptyset$. Indeed, suppose that $P \in \bigcap \mathcal{G}$. Let $\mathcal{I}$ consist of all perfect sets $Q \subseteq X$ such that $Q \cap P$ is nowhere dense in P. Then $\mathcal{I} \in \mathcal{G}$ but $P \notin \mathcal{I}$, a contradiction.

We may improve the result of Theorem 3.1 if we restrict ourselves to coinitial ideals in $\mathrm{Perf}(X)$ (so omitting "weak"). We want to give a more precise estimation of the cardinal describing how large families of coinitial ideals in $\mathrm{Perf}(X)$ provide a coinitial intersection. For this we will need the following definitions.

(a) $add_{Fin}(s_0)$ is the minimal size of a collection $\mathcal{A}$ of maximal almost disjoint families in $\mathrm{Perf}(2^\omega)$ such that there is no perfect set $P \in \mathrm{Perf}(2^\omega)$ satisfying

$$(\forall \mathcal{F} \in \mathcal{A})(\exists \mathcal{F}^* \in [\mathcal{F}]^{<\omega})(P \subseteq \bigcup \mathcal{F}^*).$$

(b) The additivity $add(\mathcal{I})$ of an ideal $\mathcal{I}$ in $\mathrm{Pow}(X)$ is

$$add(\mathcal{I}) = \min\{|\mathcal{F}| : (\mathcal{F} \subseteq \mathcal{I}) \ \&\ (\bigcup \mathcal{F} \notin \mathcal{I})\}.$$

(c) For a partial order $\mathbb{P}$ and a cardinal κ, $MA_\kappa(\mathbb{P})$ is the following sentence

for each $p \in \mathbb{P}$ and a family $\mathcal{A}$ of dense subsets of $\mathbb{P}$,
if $|\mathcal{A}| < \kappa$ then there is a filter $G \subseteq \mathbb{P}$ such that $p \in G$ and

$$(\forall \mathcal{F} \in \mathcal{A})(\mathcal{F} \cap G \neq \emptyset).$$

(d) Sacks Amoeba forcing $\mathfrak{A}$ is the partial order consisting of all pairs (n, T) such that $T \subseteq 2^{<\omega}$ is a perfect tree and $n \in \omega$. The ordering is such that $(n, T) \leq (m, S)$ ((n, T) is stronger than (m, S)) if and only if $n \geq m$, $T \subseteq S$ and $S \cap 2^m = T \cap 2^m$.

The following proposition shows the inequalities between the cardinals appearing above and it explains our notation.

Proposition 3.2.

$$\omega_1 \leq \min\{\kappa : MA_\kappa(\mathfrak{A}) \text{ fails}\} \leq add_{Fin}(s_0) \leq add(s_0) \leq \mathfrak{c}.$$

Proof. The first and the last inequalities should be obvious by the definitions.

For the second inequality suppose that $MA_\kappa(\mathfrak{A})$ holds true and let $\mathcal{A}$ be a set of maximal almost disjoint families in Perf(2^ω), $|\mathcal{A}| < \kappa$. For $\mathcal{F} \in \mathcal{A}$ let

$$\mathcal{F}^+ = \{(n,T) \in \mathfrak{A} : \ (\forall s \in T \cap 2^n)(\exists P \in \mathcal{F})([T_s] \subseteq P)\}$$

(here, $[T]$ stands for the set of all ω—branches through the tree T, and T_s denotes the part of the tree T above the node s, that is $T_s = \{t \in T : t \subseteq s \text{ or } s \subseteq t\}$). Clearly each $\mathcal{F}^+$ is dense in $\mathfrak{A}$ (remember all $\mathcal{F} \in \mathcal{A}$ are maximal adfs). Further, for each $m \in \omega$ let $\mathcal{R}_m \subseteq \mathfrak{A}$ consist of all conditions $(n,T) \in \mathfrak{A}$ such that if $s \in T \cap 2^n$ then $|\{k < n : s|k \text{ is a ramification point of } T\}| \geq m$. The sets $\mathcal{R}_m$ are dense in $\mathfrak{A}$ too. Consequently, by $MA_\kappa(\mathfrak{A})$, we find a filter $G \subseteq \mathfrak{A}$ such that

$$(\forall \mathcal{F} \in \mathcal{A})(\mathcal{F}^+ \cap G \neq \emptyset) \quad \text{and} \quad (\forall m \in \omega)(\mathcal{R}_m \cap G \neq \emptyset).$$

Now take $P^* = \bigcap\{[T] : (\exists n \in \omega)((n,T) \in G)\}$. It is a routine to check that $P^* \in \text{Perf}(2^\omega)$. Moreover, if $(n,T) \in G \cap \mathcal{F}^+$ then $P^* \subseteq \bigcup\limits_{s \in T \cap 2^n} [T_s]$ and $(\forall s \in T \cap 2^n)(\exists P \in \mathcal{F})([T_s] \subseteq P)$. Consequently

$$(\forall \mathcal{F} \in \mathcal{A})(\exists \mathcal{F}^* \in [\mathcal{F}]^{<\omega})(P^* \subseteq \bigcup \mathcal{F}^*).$$

This shows that $\kappa \leq add_{Fin}(s_0)$.

Now suppose that $\mathcal{A}^*$ is a family of (s_0)–sets in 2^ω, $|\mathcal{A}^*| < add_{Fin}(s_0)$. For each $X \in \mathcal{A}^*$ choose a maximal adf $\mathcal{F}_X$ in Perf(2^ω) such that $(\forall Q \in \mathcal{F}_X)(Q \cap X = \emptyset)$. By the definition of $add_{Fin}(s_0)$, we find a set $P \in \text{Perf}(2^\omega)$ such that

$$(\forall X \in \mathcal{A}^*)(\exists \mathcal{F}^* \in [\mathcal{F}_X]^{<\omega})(P \subseteq \bigcup \mathcal{F}^*).$$

Plainly, this implies that $P \cap \bigcup \mathcal{A}^* = \emptyset$. Thus we have shown that for each family of (s_0)–sets of size less than $add_{Fin}(s_0)$ there is a perfect set disjoint from the union of the family. This is enough to claim that $add_{Fin}(s_0) \leq add(s_0)$, as we may apply this statement "below each perfect". □

Problem 3.3. *Are the three middle cardinals of 3.2 equal (in ZFC)?*

For more information on Sacks Amoeba forcing and Martin's Axiom we refer the reader for instance to [13]. Let us note here only that $\mathfrak{A}$ is a proper forcing notion, so PFA implies $MA_{\omega_1}(\mathfrak{A})$. Of course, the simplest model of $MA_{\omega_1}(\mathfrak{A})$ is obtained by countable support iteration of length ω_2 of Sacks Amoeba forcing notions over a model of CH. On the other hand, it is known that consistently $add(s_0) < \mathfrak{c}$ even if we additionally require MA (see [19], an upper bound for $add(s_0)$ is given in [34]).

Theorem 3.4. *$add_{Fin}(s_0)$ is equal to the least cardinality of a family $\mathcal{G}$ of coinitial ideals in Perf(2^ω) such that $\bigcap \mathcal{G}$ is not coinitial in Perf(2^ω).*
[We may replace "not coinitial" by "empty".]

Proof. Suppose that $\mathcal{G}$ is a family of coinitial ideals in $\mathrm{Perf}(2^\omega)$ of size less than $add_{Fin}(s_0)$ and let $P \in \mathrm{Perf}(2^\omega)$. For each ideal $\mathcal{I} \in \mathcal{G}$ choose a maximal (in $\mathrm{Perf}(2^\omega)$) adf $\mathcal{F}_\mathcal{I} \subseteq \mathcal{I}$ (possible as each $\mathcal{I} \in \mathcal{G}$ is coinitial in $\mathrm{Perf}(2^\omega)$). By the definition of $add_{Fin}(s_0)$ we find a perfect set $Q \subseteq P$ such that

$$(\forall \mathcal{I} \in \mathcal{G})(\exists \mathcal{F}^* \in [\mathcal{F}_\mathcal{I}]^{<\omega})(Q \subseteq \bigcup \mathcal{F}^*)$$

(remember that P is homeomorphic with 2^ω so we can apply the definition of $add_{Fin}(s_0)$ "below P"). But, as each $\mathcal{I} \in \mathcal{G}$ is an ideal, we conclude that $Q \in \bigcap \mathcal{G}$.

Assume now that $\mathcal{A}$ is a set of maximal almost disjoint families in $\mathrm{Perf}(2^\omega)$ such that there is no perfect set $P \in \mathrm{Perf}(2^\omega)$ satisfying

$$(\forall \mathcal{F} \in \mathcal{A})(\exists \mathcal{F}^* \in [\mathcal{F}]^{<\omega})(P \subseteq \bigcup \mathcal{F}^*).$$

Clearly we may assume that each $\mathcal{F} \in \mathcal{A}$ is infinite. For $\mathcal{F} \in \mathcal{A}$ let $\mathcal{I}_\mathcal{F}$ be the ideal in $\mathrm{Perf}(2^\omega)$ generated by $\mathcal{F}$ (so $\mathcal{I}_\mathcal{F}$ consists of all perfect sets which can be covered by finitely many members of $\mathcal{F}$). The ideals $\mathcal{I}_\mathcal{F}$ are proper and coinitial in $\mathrm{Perf}(2^\omega)$, but, by the choice of $\mathcal{A}$, we have $\bigcap\{\mathcal{I}_\mathcal{F} : \mathcal{F} \in \mathcal{A}\} = \emptyset$. □

The next phenomena concerning intersections of coinitial families of perfect sets deal with game ideals considered in [30], [32], [33], [1], [5], [4]. For each $A \subseteq 2^\omega$ and $K \in [\omega]^\omega$, we denote by $\Gamma(A,K)$ the infinite game with perfect information defined as follows (cf. [30]). Two players choose consecutive terms of $x = (x_1, x_2, ...) \in 2^\omega$: player I chooses x_i for $i \in \omega \setminus K$, and player II – for $i \in K$. Player I wins if $x \in A$, and player II – when $x \notin A$. The family of all sets $A \subseteq 2^\omega$ for which player II has a winning strategy in $\Gamma(A,K)$ is denoted by $V(K)$. For a precise definition of a strategy, see for instance [33]. Observe that player II has in $\Gamma(A,K)$ a winning strategy which does not depend on the moves of player II iff $A|K \neq 2^K$ where $A|K = \{f|K : f \in A\}$. Put $V^*(K) = \{A \subseteq 2^\omega : A|K \neq 2^K\}$. Note that $V^*(K) \subseteq V(K)$ and the inclusion can be proper. Clearly, $V(K)$ and $V^*(K)$ are hereditary families in $\mathrm{Pow}(X)$. Using the representation of perfect sets in 2^ω by perfect trees [18, p.37], it is not hard to check that $\mathrm{Perf}(2^\omega) \cap V^*(K)$ is a hereditary coinitial family in $\mathrm{Perf}(2^\omega)$ for each $K \in [\omega]^\omega$. The same holds true in the case of $\mathrm{Perf}(2^\omega) \cap V(K)$ (cf. [1]). For $\mathcal{K} \subseteq [\omega]^\omega$ we denote

$$\mathbb{M}(\mathcal{K}) = \bigcap_{K \in \mathcal{K}} V(K) \quad \text{and} \quad \mathbb{M}^*(\mathcal{K}) = \bigcap_{K \in \mathcal{K}} V^*(K).$$

A family $\mathcal{K} \subseteq [\omega]^\omega$ is called a *normal system* [33] if each $K \in \mathcal{K}$ contains disjoint sets $K_1, K_2 \in \mathcal{K}$. One can show that, for a normal system $\mathcal{K}$, the families $\mathbb{M}(\mathcal{K})$ and $\mathbb{M}^*(\mathcal{K})$ form σ–ideals. If additionally $\mathcal{K}$ is countable,

$\mathbb{M}(\mathcal{K})$ is known as a *Mycielski ideal* [30]. For the σ–ideal $\mathcal{I} = \mathbb{M}([\omega]^\omega)$, the intersection $\mathrm{Perf}(X) \cap \mathcal{I}$ is coinitial in $\mathrm{Perf}(X)$ [32],[2]. Hence $\mathrm{Perf}(X) \cap \mathbb{M}(\mathcal{K})$ is coinitial for each $\mathcal{K} \subseteq [\omega]^\omega$. Families $\mathbb{M}^*(\mathcal{K})$ behave differently since $\mathrm{Perf}(X) \cap \mathbb{M}^*([\omega]^\omega)$ is noncoinitial [15] although it contains perfect sets [32]. It was proved in [4] that $\mathrm{Perf}(X) \cap \mathbb{M}^*(\mathcal{K})$ is coinitial for each $\mathcal{K} \subseteq [\omega]^\omega$ of size less than the dominating number $\mathfrak{d}$. Recall that $\omega_1 \leq \mathfrak{d} \leq \mathfrak{c}$ and it is consistent that $\mathfrak{d} < \mathfrak{c}$ [39]. It is still not known whether $\mathrm{Perf}(X) \cap \mathbb{M}^*(\mathcal{K})$ is coinitial for each $\mathcal{K} \subseteq [\omega]^\omega$ of size less than $\mathfrak{c}$ (that problem was posed in [4]).

4. Some characterizations. The hyperspace $\mathrm{CL}(X)$ is usually endowed with the *Vietoris topology* generated by the base consisting of sets of the form

$$V(G_0; G_1, ..., G_n) = \{F \in \mathrm{CL}(X) : (F \subseteq G_0) \ \& \ (\forall i \in \{1, ..., n\}) \ (F \cap G_i \neq \emptyset)\}$$

where $G_0, G_1, ..., G_n$ are open in X (cf. [22, §42]). The empty set is treated as an isolated point of $\mathrm{CL}(X)$. If $P \in \mathrm{Perf}(X)$, we equip $\mathrm{Perf}(P)$ with the Vietoris topology inherited from $\mathrm{CL}(P)$. Sets dense (residual) in $\mathrm{Perf}(P)$ with respect to that topology will be called *Vietoris dense* (*residual*).

Theorem 4.1. *Let $\mathcal{I} \subseteq$ Perf(X) be weakly finitely additive. Then $\mathcal{I}$ is coinitial in Perf(X) if and only if $\mathcal{I} \cap$ Perf(P) is Vietoris dense in Perf(P) for each $P \in$ Perf(X).*

Proof. "$\Rightarrow$" Let $P \in \mathrm{Perf}(X)$ and consider a nonempty set

$$W = V(G_0; G_1, ..., G_n) \cap \mathrm{Perf}(P)$$

from the base of the Vietoris topology in $\mathrm{Perf}(P)$, where $G_0, G_1, ..., G_n$ are open sets in P. We will show that $W \cap \mathcal{I} \neq \emptyset$. We may assume that $\bigcup_{i=1}^n G_i \subseteq G_0$ and (after shrinking $G_1, ..., G_n$, if necessary) that $G_1, ..., G_n$ are pairwise disjoint. By the Alexandrov-Hausdorff theorem [22, §37I], we can choose perfect sets $P_i \subseteq G_i$, and next (by the assumption) – perfect subsets $Q_i \in \mathcal{I}$ of P_i for $i = 1, ..., n$. Put $Q = \bigcup_{i=1}^n Q_i$. Clearly, $Q \in W$. Since $\mathcal{I}$ is weakly finitely additive, we have $Q \in \mathcal{I}$.

"$\Leftarrow$" Obvious. □

Recall that a family $\mathcal{F}$ of sets in a topological space is said to be a *network* if, for each nonempty open set U and for each $x \in U$, there is an $E \in \mathcal{F}$ such that $x \in E \subseteq U$.

Theorem 4.2. *Let $\mathcal{I} \subseteq$ Perf(X) be a σ–ideal fulfilling $\bigcup \mathcal{I} = X$. The following conditions are equivalent:*
(a) $\mathcal{I}$ is coinitial in Perf(X);
(b) for each $P \in$ Perf(X), the family $\mathcal{I} \cap$ Perf(P) forms a network of the

topology in P;
(c) $\bigcup \mathcal{I} \cap Perf(P) = P$ for each $P \in Perf(X)$.

Proof. $(a) \Rightarrow (b)$ Let U be open in $P \in \mathrm{Perf}(X)$ and $x \in U$. By the Alexandrov-Hausdorff theorem, for each $n \in \omega$, we can find a perfect set $P_n \subseteq U \cap B(x, 1/(n+1))$. By (a) we pick $Q_n \in \mathcal{I} \cap \mathrm{Perf}(P_n)$ for each $n \in \omega$. Since $\bigcup \mathcal{I} = X$, there is a $D \in \mathcal{I} \cap \mathrm{Perf}(X)$ such that $x \in D$. The set $Q = D \cup \bigcup_{n\in\omega} Q_n$ is perfect and, as $\mathcal{I}$ is countably additive in $\mathrm{Perf}(X)$, we have $Q \in \mathcal{I}$. Since $\mathcal{I}$ is hereditary, the perfect set $Q^* = \{x\} \cup \bigcup_{n\in\omega} Q_n$ is also in $\mathcal{I}$ and, clearly $x \in Q^* \subseteq U$ as desired.

$(b) \Rightarrow (c) \Rightarrow (a)$ Obvious. □

5. Applications to Vietoris residual sets.

Corollary 5.1. *If X is a compact metric space and $\mathcal{J} \subseteq CL(X)$ is a G_δ set in $CL(X)$ such that $\mathcal{J} \cap Perf(X)$ is weakly finitely additive and coinitial in $Perf(X)$ then $\mathcal{J} \cap Perf(P)$ is Vietoris residual in $Perf(P)$ for each $P \in Perf(X)$.*

Proof. Since X is metric and compect, so is $\mathrm{CL}(X)$ with the Vietoris topology [22, §42I,II]. Hence $\mathrm{CL}(X)$ is Polish. Fix any $P \in \mathrm{Perf}(X)$. The set $\mathrm{CL}(P)$ is closed in $\mathrm{CL}(X)$ [22, §42III], so it forms a Polish space. The space $\mathrm{Perf}(P)$, being a G_δ subset of a Polish space $\mathrm{CL}(P)$ [22, §42III] is Polish [22, §33VI]. Since $\mathcal{J}$ is a G_δ set in $\mathrm{CL}(X)$ and $\mathrm{Perf}(P) \subseteq \mathrm{CL}(X)$, therefore $\mathcal{J} \cap \mathrm{Perf}(P)$ is a G_δ set in $\mathrm{Perf}(P)$. From Theorem 4.1 it follows that $\mathcal{J} \cap \mathrm{Perf}(P)$ is Vietoris dense in $\mathrm{Perf}(P)$. Hence, being of type G_δ, it is residual in $\mathrm{Perf}(P)$, by the Baire category theorem. □

For compact metric spaces X, there are several known G_δ ideals $\mathcal{J}$ in $\mathrm{CL}(X)$ with $\mathcal{J} \cap \mathrm{Perf}(X)$ coinitial in $\mathrm{Perf}(X)$. Then Corollary 5.1 works. Let us give some examples of such families $\mathcal{J}$.

(I) $\mathcal{J} =$ the family of all closed Lebesgue null sets, or the family of closed null sets with respect to the Hausdorff measure μ^h (for the adequate function h) on $X = [0, 1]$ (cf. [21, p.417]).

(II) $\mathcal{J} =$ the family of closed nowhere dense sets in X (cf. [21, p.417]).

(III) $\mathcal{J} =$ the family of closed sets from a fixed Mycielski ideal on the Cantor space 2^ω [30] (see Section 3). It is shown in [5] that $\mathcal{J}$ is a G_δ residual set in $\mathrm{CL}(2^\omega)$, and it is proved in [1] that $\mathcal{J} \cap \mathrm{Perf}(2^\omega)$ is coinitial in $\mathrm{Perf}(2^\omega)$.

(IV) Let $X = [-1, 1]$. A set $E \subseteq X$ is called *strongly porous* [41] if $p(E, x) = 1$ for each $x \in E$, where

$$p(E, x) = \limsup_{r\to 0+} (\gamma(E, x, r)/r)$$

and $\gamma(E, x, r)$ is the length of a longest interval $(a, b) \subseteq (x - r, x + r) \setminus E$. Let $\mathcal{J}$ denote the family of all closed strongly porous sets in X. Then $\mathcal{J}$ is a G_δ residual set in $\mathrm{CL}(X)$ [23]. In [2] it is proved that $\mathcal{J} \cap \mathrm{Perf}(X)$ is coinitial in $\mathrm{Perf}(X)$. (Note that in [6] the analogous results are shown for a more restrictive kind of porosity.) It is easy to check that $\mathcal{J} \cap \mathrm{Perf}(X)$ forms a weak ideal in $\mathrm{Perf}(X)$. So, the scheme of Corollary 5.1 is applicable. Observe that $\mathcal{J} \cap \mathrm{Perf}(X)$ is not an ideal in $\mathrm{Perf}(X)$. Indeed, for each positive integer n, consider a perfect strongly porous set $P_n \subseteq [1/(2n+1), 1/(2n)]$ with $\min P_n = 1/(2n+1)$ and $\max P_n = 1/(2n)$ (cf. [41]). Then $P = \{0\} \cup \bigcup_{n=1}^{\infty} P_n$ is perfect strongly porous and so is $-P = \{-x : x \in P\}$. However $Q = P \cup -P$ is not strongly porous since

$$p(Q, 0) = \lim_{n \to \infty} \frac{1/(2n-1) - 1/(2n)}{1/(2n-1)} = 0.$$

6. Connections with perfect category bases. Category bases were defined and studied by J. C. Morgan [28]. A *category base* on a set Y [28, p.8] is a pair $(Y, \mathcal{C})$ such that $Y \neq \emptyset$ and $\mathcal{C}$ is a family of nonempty subsets of Y, called *regions*, satisfying the conditions:
(1) $\bigcup \mathcal{C} = Y$;
(2) Let A be a region and let $\mathcal{D}$ be a nonempty family of pairwise disjoint regions which has cardinality less than $|\mathcal{C}|$. Then
(i) if $A \cap (\bigcup \mathcal{D})$ contains a region, then there is a region $B \in \mathcal{D}$ such that $A \cap B$ contains a region,
(ii) if $A \cap (\bigcup \mathcal{D})$ contains no region, then there is a region $B \subseteq A$ disjoint from $\bigcup \mathcal{D}$.

A category base consisting of perfect sets is called a *perfect base* if, for each region A, each positive integer n, and each pair x_1, x_2 of different points in A, there exist disjoint regions A_1, A_2 such that $x_i \in A_i$, $A_i \subseteq A$ and the diameter of A_i is $\leq 1/n$ for $i = 1, 2$ (cf. [28, pp.144–145]). A set $E \subseteq Y$ is called *$\mathcal{C}$–singular* if, for each region A, there exists a region $B \subseteq A$ such that $B \cap E = \emptyset$. A set $M \subseteq Y$ is called *$\mathcal{C}$–meager* if M is a countable union of $\mathcal{C}$–singular sets. The family of $\mathcal{C}$–meager sets is written as $\mathcal{M}(\mathcal{C})$. A set $G \subseteq Y$ is called *$\mathcal{C}$–Baire* if, for each region A there exists a region $B \subseteq A$ such that $B \cap G \in \mathcal{M}(\mathcal{C})$ or $B \setminus G \in \mathcal{M}(\mathcal{C})$. The family of all $\mathcal{C}$–Baire sets is denoted by $\mathcal{B}(\mathcal{C})$. Two category bases $(Y, \mathcal{C}_1)$ and $(Y, \mathcal{C}_2)$ are called *equivalent* if $\mathcal{B}(\mathcal{C}_1) = \mathcal{B}(\mathcal{C}_2)$ and $\mathcal{M}(\mathcal{C}_1) = \mathcal{M}(\mathcal{C}_2)$ (cf. [28, pp.22–24]).

It is known that, for a perfect Polish space X, the pair $(X, \mathrm{Perf}(X))$ forms a perfect category base [28, Th.33, p.156]. Moreover, for $\mathcal{C} = \mathrm{Perf}(X)$, we have $\mathcal{M}(\mathcal{C}) = (s_0)$-sets, $\mathcal{B}(\mathcal{C}) = (s)$-sets [28, p. 157]. Recall that $A \subseteq X$ is said to be an (s)–*set* if, for each $P \in \mathrm{Perf}(X)$, there exists a $Q \in \mathrm{Perf}(P)$ such that either $Q \subseteq A$ or $Q \cap A = \emptyset$ [38].

Corollary 6.1. *A σ–ideal $\mathcal{I} \subseteq Perf(X)$ satisfying $\bigcup \mathcal{I} = X$ is coinitial in $Perf(X)$ if and only if for each $P \in Perf(X)$ the pair $(P, \mathcal{I} \cap Perf(P))$ forms a perfect category base equivalent to $(P, Perf(P))$.*

Proof. "$\Rightarrow$" Let $P \in \mathrm{Perf}(X)$. From [28, Th.2, p.11] it follows that $(P, \mathcal{I} \cap \mathrm{Perf}(P))$ forms a category base. Since $\mathcal{I}$ is hereditary in $\mathrm{Perf}(X)$, it can be easily shown that this base is perfect. The bases $(P, \mathcal{I} \cap \mathrm{Perf}(P))$ and $(P, \mathrm{Perf}(P))$ are equivalent, by [28, Th.1, p.23].

"$\Leftarrow$" Since $(P, \mathcal{I} \cap \mathrm{Perf}(P))$ forms a category base for each $P \in \mathrm{Perf}(X)$, therefore, by condition (1) given in the definition of the category base, we have $\bigcup(\mathcal{I} \cap \mathrm{Perf}(P)) = P$ for each $P \in \mathrm{Perf}(X)$. Then it suffices to apply Theorem 4.2. □

7. Some noncoinitial families in Perf(X). Here we assume that X forms an Abelian metric (Polish) additive group with the neutral element e. Invariant noncoinitial σ–ideals $\mathcal{I}$ in $\mathrm{Perf}(X)$ satisfying $\bigcup \mathcal{I} = X$ will be called INC *σ–ideals*. By the definition, $\mathcal{F} \subseteq \mathrm{Perf}(X)$ is noncoinitial iff there is a $Q \in \mathrm{Perf}(X)$ with no perfect subset in $\mathcal{F}$. So, by the Alexandrov-Haudsorff theorem, a hereditary family $\mathcal{F}$ is noncoinitial in $\mathrm{Perf}(X)$ iff there is a $Q \in \mathrm{Perf}(X)$ such that $|P \cap Q| \leq \omega$ for each $P \in \mathcal{F}$. Thus, the simplest way to construct an INC σ–ideal $\mathcal{I}$ in $\mathrm{Perf}(X)$ is to find perfect sets P, Q such that $|(P + x) \cap Q| \leq \omega$ for all $x \in X$ and put $\mathcal{I} = \mathcal{I}(P)$ where $\mathcal{I}(P)$ denotes the σ–ideal in $\mathrm{Perf}(X)$ generated by $\{P + x : x \in X\}$. At the same time, we get another INC σ–ideal $\mathcal{I}(Q)$. Note that $\mathcal{I}(P) \cap \mathcal{I}(Q) = \emptyset$. So, that method produces a pair of disjoint INC σ–ideals.

Examples. (a) Assume that P, Q are subgroups of X being perfect sets and satisfying $P \cap Q = \{e\}$. Then $|(P + x) \cap Q| \leq 1$ for each $x \in X$. So, the above scheme works. In particular, we can consider $X = \mathbb{R}^2$, $P = \mathbb{R} \times \{0\}$ and $Q = \{0\} \times \mathbb{R}$ which well illustrates our idea.

(b) (Cf. [1, Example 2.4].) A set $E \subseteq X$ is called *algebraically independent* if $\sum_{i=1}^{n} k_i x_i \neq e$ for each finite set $\{x_1, ..., x_n\} \subseteq E$ and for any integer numbers k_i $(i = 1, ..., n)$ [22, §55V] (if the group X contains elements of finite rank then we additionally demand that each $|k_i|$ is smaller than the rank of x_i). There exist perfect algebraically independent sets in several Polish groups [24], [16]. Now, assume that $E \in \mathrm{Perf}(X)$ is algebraically independent. Pick disjoint sets $P, Q \in \mathrm{Perf}(E)$. It is easy to check that $|(P + x) \cap Q| \leq 1$ for each $x \in X$ (cf. [35], [28, p.210]). Thus our scheme is applicable.

(c) Let X be equal to the Cantor group 2^ω with the coordinatewise addition modulo 2. A set $E \subseteq X$ is called *set-theoretically independent* if $\bigcap_{i=1}^n f_i^{-1}[\{k_i\}] \neq \emptyset$ for each finite set $\{f_1, ..., f_n\} \subseteq E$ and any $k_i \in \{0,1\}$ $(i = 1, ..., n)$. It is known that there exist perfect set-theoretically independent sets in X [17]. Let us verify that each set-theoretically independent set is algebraically independent (the converse is not true). Indeed, since $g + g = e$ for each $g \in X$, we have to show that $\sum_{i=1}^n f_i \neq e$ for any fixed set $\{f_1, ..., f_n\} \subseteq E$. Pick $k_i \in \{0,1\}$ $(i = 1, ..., n)$ so that $\sum_{i=1}^n k_i = 1$. Let $m \in \bigcap_{i=1}^n f_i^{-1}[\{k_i\}]$. Then $\sum_{i=1}^n f_i(m) = 1$, as desired. Next, let us show that the family $\mathcal{F}_{al}$ of all perfect algebraically independent sets in X is coinitial in $\mathrm{Perf}(X)$ but the analogous statement for the family $\mathcal{F}_{st}$ of all set perfect set-theoretically independent sets in X is false. Indeed, let $P \in \mathrm{Perf}(X)$. Since the sets $R_n = \{\langle f_1, ..., f_n\rangle \in P^n : \sum_{i=1}^n f_i \neq e\}$, $n \geq 1$, are comeager in P^n, there exists a $Q \in \mathrm{Perf}(P)$ with $\{\langle f_1, ..., f_n\rangle \in Q^n : f_i \neq f_j, \text{ if } i \neq j\} \subseteq R_n$ for each $n \geq 1$, by the Mycielski theorem [29] (see also [20, Th.19.1]). Therefore $Q \in \mathcal{F}_{al}$ and $\mathcal{F}_{al}$ is coinitial. Now, consider the σ–ideal $\mathcal{I} = \mathbb{M}^*([\omega]^\omega)$ in $\mathrm{Pow}(X)$ (see Section 3). It was shown in [32] that $\mathcal{I}$ contains perfect sets. On the other hand, by [15, Th.4], each set-theoretically independent set has no perfect subset in $\mathcal{I}$. Hence $\mathcal{F}_{st}$ is not coinitial. Evidently, it is translation invariant. Also the ideal $\mathcal{I}$ is translation invariant [32] and so is $\mathcal{I}^* = \mathcal{I} \cap \mathrm{Perf}(X)$. Consequently, $\mathcal{I}^*$ and the σ–ideal in $\mathrm{Perf}(X)$ generated by $\mathcal{F}_{st}$ form disjoint INC σ–ideals.

We may elaborate this further. For $Q \in \mathrm{Perf}(X)$ let

$$\mathcal{J}(Q) = \{P \in \mathrm{Perf}(X) : (\forall x \in X)(|(P + x) \cap Q| \leq \omega)\}.$$

The families of the form $\mathcal{J}(Q)$ are canonical examples of INC σ–ideals in $\mathrm{Perf}(2^\omega)$ in the following sense.

Proposition 7.1. *Let $X = 2^\omega$.*

1. *For each $Q \in Perf(X)$, the family $\mathcal{J}(Q)$ is an INC σ–ideal iff $\mathcal{J}(Q) \neq \emptyset$.*
2. *If $P, Q \in Perf(X)$, $P \in \mathcal{J}(Q)$ then $\mathcal{I}(P) \subseteq \mathcal{J}(Q)$ and $\mathcal{J}(Q) \cap \mathcal{I}(Q) = \emptyset$.*
3. *Suppose that $\mathcal{I}$ is an INC ideal in Perf(X). Then there is a perfect algebraically independent set $Q \in Perf(X)$ such that $\mathcal{I} \subseteq \mathcal{J}(Q)$.*

Proof. (1) Clearly $Q \notin \mathcal{J}$ and $\mathcal{J}(Q)$ is hereditary, translation invariant and countably additive. Consequently it forms an INC σ–ideal if and only if it is nonempty.
(2) Clear.
(3) Take a $Q_0 \in \mathrm{Perf}(X)$ such that $\mathrm{Perf}(Q_0) \cap \mathcal{I} = \emptyset$. This means that if $P \in \mathcal{I}$ then $(\forall x \in X)(|(P + x) \cap Q_0| \leq \omega)$. Consequently $\mathcal{I} \subseteq \mathcal{J}(Q_0)$ and

the last is an INC σ–ideal. We may choose a perfect independent subset Q of Q_0 (it follows from Mycielski's theorem [29]). Then $\mathcal{J}(Q_0) \subseteq \mathcal{J}(Q)$, finishing the proof. $\square$

8. Perfect isomorphisms. Recall that a bijection $f : X \to X$ is said to be a *Borel isomorphism* [20, p. 71] if f and f^{-1} are Borel measurable. We say that a bijection $f : X \to X$ is a *perfect isomorphism* if, for each $P \in \mathrm{Perf}(X)$, the sets $f[P]$ and $f^{-1}[P]$ contain perfect sets (cf. [4]). By Souslin's theorem [22, §39V], the image of a perfect set by a one-to-one Borel measurable function is an uncountable Borel set. Thus, by the Alexandrov-Hausdorff theorem, that image contains a perfect set. Consequently, each Borel isomorphism is a perfect isomorphism. It turns out that perfect isomorphisms can be nicely characterized by (s)-sets of Marczewski [38]. Recall that (s)-sets in X form a σ–algebra. A function $f : X \to X$ is called (s)-measurable if the preimage $f^{-1}[G]$ of any open set $G \subseteq X$ is an (s)-set. The following lemma due to Marczewski shows an equivalent definition of (s)-measurable functions given originally by Sierpiński [37].

Lemma 8.1. *(*[38, 4.3]*) A function $f : X \to X$ is (s)-measurable if and only if, for each $P \in Perf(X)$, there exists a $Q \in Perf(P)$ such that $f|Q$ is continuous.*

Proof. "$\Rightarrow$" Fix a base $\{U_n\}_{n\in\omega}$ of open sets in X. For each $n \in \omega$ let $\mathcal{I}_n$ consist of all sets $Q \in \mathrm{Perf}(X)$ such that $Q \cap f^{-1}[U_n]$ is open in Q. Then $\mathcal{I}_n$ forms a weak ideal in $\mathrm{Perf}(X)$. Moreover, $\mathcal{I}_n$ is coinitial since $f^{-1}[U_n]$ is an (s)-set. So, $\mathcal{I} = \bigcap_{n\in\omega} \mathcal{I}_n$ is coinitial by Theorem 3.1. Now, let $P \in \mathrm{Perf}(X)$. Pick $Q \in \mathcal{I} \cap \mathrm{Perf}(P)$. It follows that $f|Q$ is continuous.
"$\Leftarrow$" Let G be an open subset of X and let $P \in \mathrm{Perf}(X)$. Choose a $Q \in \mathrm{Perf}(P)$ such that $f|Q$ is continuous. Thus $(f|Q)^{-1}[G] = Q \cap f^{-1}[G]$ is open in Q. It is easy to find a $D \in \mathrm{Perf}(Q)$ which either is contained in $f^{-1}[G]$ or misses $f^{-1}[G]$. Hence $f^{-1}[G]$ is an (s)-set. $\square$

Theorem 8.2. *Let $f : X \to X$ be a bijection. The following conditions are equivalent:*
(a) f is a perfect isomorphism;
(b) for each $P \in Perf(X)$ there exist (compact) sets $P^*, P^{**} \in Perf(P)$ such that the restrictions $f|P^*, f^{-1}|P^{**}$ are homeomorphisms;
(c) for each $E \subseteq X$, we have E is an (s)-set iff $f[E]$ is an (s)-set;
(d) both functions f and f^{-1} are (s)-measurable.

Proof. $(a) \Rightarrow (b)$ Let $P \in \mathrm{Perf}(X)$. Pick a compact $P_{\langle\,\rangle} \in \mathrm{Perf}(P)$ (homeomorphic with 2^ω) and choose $Q_{\langle\,\rangle} \in \mathrm{Perf}(f[P])$. We may assume that the diameters of $P_{\langle\,\rangle}$ and $Q_{\langle\,\rangle}$ are ≤ 1. Let $n \in \omega$. Having P_t, Q_t, $(t \in$

Seq_n) defined, we pick disjoint sets $P_{t0}, P_{t1} \in \mathrm{Perf}(f^{-1}[Q_t])$ and sets $Q_{t0} \in \mathrm{Perf}(f[P_{t0}])$, $Q_{t1} \in \mathrm{Perf}(f[P_{t1}])$, all of diameters $\leq 1/(n+2)$ (note that Q_{t0}, Q_{t1} are disjoint). Let P^* and Q^* denote the fusions of the families $\{P_t : t \in \mathrm{Seq}\}$ and $\{Q_t : t \in \mathrm{Seq}\}$, respectively. Then for each $t \in \mathrm{Seq}$ we have $f[P_t \cap P^*] = Q_t \cap Q^*$ and thus $f|P^*$ is a homeomorphism from P^* onto Q^*. The respective set P^{**} for f^{-1} can be found analogously.

Equivalences $(b) \iff (c) \iff (d)$ are due to Marczewski [38, 4.4]. We show the remaining arguments for the reader's convenience.

$(b) \Rightarrow (c)$ It is enough to demonstrate that, if E is an (s)-set, then $f[E]$ is an (s)-set. Let $P \in \mathrm{Perf}(X)$. Let $P^{**} \in \mathrm{Perf}(P)$ be a compact set such that $f^{-1}|P^{**}$ is a homeomorphism. Thus $Q = f^{-1}[P^{**}]$ is perfect and compact. Since E is an (s)-set, there exists a $D \in \mathrm{Perf}(Q)$ such that either $D \subseteq E$ or $D \cap E = \emptyset$. Consequently, $f[D]$ is a perfect subset of P which either is contained in $f[E]$ or is disjoint from $f[E]$. Hence $f[E]$ is an (s)-set.

$(c) \Rightarrow (d)$ It suffices to prove that f is (s)-measurable. But this follows from (c) and from the fact that each open set is an (s)-set.

$(d) \Rightarrow (a)$ Let $P \in \mathrm{Perf}(X)$. Since f is (s)-measurable, we infer from Lemma 8.1 that there exists a $Q \in \mathrm{Perf}(P)$ such that $f|Q$ is continuous. Then pick a compact $P^* \in \mathrm{Perf}(Q)$. Thus $f[P^*]$ is a compact perfect subset of $f[P]$. Similarly, $f^{-1}[P]$ contains a perfect set. $\square$

Let $\mathcal{I}$ and $\mathcal{J}$ be subfamilies of $\mathrm{Pow}(X)$. We say that $\mathcal{I}$ and $\mathcal{J}$ are:

(a) *isomorphic*, if there exists a bijection $f : X \to X$ such that $E \in \mathcal{I}$ iff $f[E] \in \mathcal{J}$ for each $E \subseteq X$ (then f is called an *isomorphism* between $\mathcal{I}$ and $\mathcal{J}$);

(b) *perfect isomorphic*, if there exists a perfect isomorphism $f : X \to X$ which is an isomorphism between $\mathcal{I}$ and $\mathcal{J}$;

(c) *Borel isomorphic*, if there exists a Borel isomorphism $f : X \to X$ which is an isomorphism between $\mathcal{I}$ and $\mathcal{J}$.

By $\mathrm{Bor}(X)$ we denote the family of all Borel subsets of X. Observe that the definition (c) still works if $\mathcal{I}, \mathcal{J} \subseteq \mathrm{Bor}(X)$. Note that, if $\mathcal{I}$ and $\mathcal{J}$ are isomorphic (resp. Borel isomorphic) ideals in $\mathrm{Pow}(X)$ (resp. in $\mathrm{Bor}(X)$), then the quotient Boolean algebras $\mathrm{Pow}(X)/\mathcal{I}$ and $\mathrm{Pow}(X)/\mathcal{J}$ (resp. $\mathrm{Bor}(X)/\mathcal{I}$ and $\mathrm{Bor}(X)/\mathcal{J}$) are isomorphic. It was shown in [4] that, if $\mathcal{I}, \mathcal{J} \subseteq \mathrm{Pow}(X)$ are perfect isomorphic and $\mathcal{I} \cap \mathrm{Perf}(X)$ is coinitial in $\mathrm{Perf}(X)$, then so is $\mathcal{J} \cap \mathrm{Perf}(X)$. Thus, if exactly one of the families $\mathcal{I} \cap \mathrm{Perf}(X)$ and $\mathcal{J} \cap \mathrm{Perf}(X)$ is coinitial in $\mathrm{Perf}(X)$, then $\mathcal{I}$ and $\mathcal{J}$ cannot be perfect isomorphic.

For an ideal $\mathcal{I} \subseteq \mathrm{Pow}(X)$, we denote

$$cof(\mathcal{I}) = \min\{|\mathcal{F}| : (\mathcal{F} \subseteq \mathcal{I})\ \&\ (\forall A \in \mathcal{I})(\exists B \in \mathcal{F})(A \subseteq B)\}.$$

We say that $\mathcal{I}$ has a *Borel basis* if each set $A \in \mathcal{I}$ is contained in a set $B \in \mathcal{I} \cap \mathrm{Bor}(X)$.

Theorem 8.3. *There exist isomorphic σ–ideals $\mathcal{I}, \mathcal{J} \subseteq Pow(X)$, fulfilling $\bigcup \mathcal{I} = \bigcup \mathcal{J} = X$ and $add(\mathcal{I}) = add(\mathcal{J}) = \mathfrak{c}$, which are not perfect isomorphic.*

Proof. Consider a partition $\{X_\alpha : \alpha < \mathfrak{c}\}$ of X into pairwise disjoint uncountable Borel sets, and a partition $\{Y_\alpha : \alpha < \mathfrak{c}\}$ of X into pairwise disjoint Bernstein sets [28, Th.27, p.152]. Let

$$\mathcal{I} = \{A \subseteq X : (\exists \alpha < \mathfrak{c})(A \subseteq \bigcup_{\gamma<\alpha} X_\gamma)\}, \qquad \text{and}$$

$$\mathcal{J} = \{A \subseteq X : (\exists \alpha < \mathfrak{c})(A \subseteq \bigcup_{\gamma<\alpha} Y_\gamma)\}.$$

For each $\alpha < \mathfrak{c}$, choose a bijection $f_\alpha : X_\alpha \to Y_\alpha$ and define $f : X \to X$ by $f = f_\alpha$ on X_α ($\alpha < \mathfrak{c}$). Then f is an isomorphism between $\mathcal{I}$ and $\mathcal{J}$. However, no bijection $g : X \to X$ is a perfect isomorphism between $\mathcal{I}$ and $\mathcal{J}$ since $\mathcal{I} \cap \mathrm{Perf}(X) \neq \emptyset$ but $\mathcal{J} \cap \mathrm{Perf}(X) = \emptyset$. $\square$

We denote $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Lemma 8.4. *Let $\{P_\alpha : \alpha < \mathfrak{c}\}$, $\{Q_\alpha : \alpha < \mathfrak{c}\}$ be maximal almost disjoint families of perfect sets in X such that each P_α is homeomorphic with Q_α. Suppose that $|X \setminus \bigcup_{\alpha<\mathfrak{c}} P_\alpha| = |X \setminus \bigcup_{\alpha<\mathfrak{c}} Q_\alpha|$. Then there is a perfect isomorphism $g : X \to X$ such that for each $\alpha < \mathfrak{c}$ we have*

$$(*_\alpha) \qquad g[\bigcup_{\gamma<\alpha} P_\gamma] = \bigcup_{\gamma<\alpha} Q_\gamma \quad \textit{and} \quad |g[P_\alpha] \triangle Q_\alpha| \leq |\alpha| + \omega < \mathfrak{c}.$$

Proof. First we declare that $g|(X \setminus \bigcup_{\alpha<\mathfrak{c}} P_\alpha) : X \setminus \bigcup_{\alpha<\mathfrak{c}} P_\alpha \longrightarrow X \setminus \bigcup_{\alpha<\mathfrak{c}} Q_\alpha$ is any bijection. (Note that both $X \setminus \bigcup_{\alpha<\mathfrak{c}} P_\alpha$ and $X \setminus \bigcup_{\alpha<\mathfrak{c}} Q_\alpha$ are (s_0)–sets.) For each $\alpha < \mathfrak{c}$, fix a homeomorphism $h_\alpha : P_\alpha \to Q_\alpha$. Let $g(x) = h_0(x)$ for each $x \in P_0$. Assume that $\alpha < \mathfrak{c}$ and that g is defined on $\bigcup_{\beta<\alpha} P_\beta$ so that, for each $\beta < \alpha$, condition $(*_\beta)$ holds true, and there are sets $E_\beta \subseteq P_\beta$, $G_\beta \subseteq Q_\beta$ of size $\leq |\beta| + \omega$ such that

$$g|(P_\beta \setminus E_\beta) = h_\beta|(P_\beta \setminus E_\beta), \quad g^{-1}|(Q_\beta \setminus G_\beta) = h_\beta^{-1}|(Q_\beta \setminus G_\beta).$$

We will define g on $P_\alpha \setminus \bigcup_{\beta<\alpha} P_\beta$. We will also show $(*_\alpha)$ and find the respective sets E_α and G_α. Let $A = P_\alpha \cap \bigcup_{\beta<\alpha} P_\beta$ and $B = Q_\alpha \cap \bigcup_{\beta<\alpha} g[P_\beta]$. Then $|A| \leq |\alpha| + \omega$ since $\{P_\alpha : \alpha < \mathfrak{c}\}$ is an adf. Also $|B| \leq |\alpha| + \omega$ since for $D = Q_\alpha \cap \bigcup_{\beta<\alpha} Q_\beta$ we have $|D| \leq |\alpha| + \omega$ and $|B \triangle D| \leq |\alpha| + \omega$ (which follows from $B \triangle D \subseteq \bigcup_{\beta<\alpha} (g[P_\beta] \triangle Q_\beta)$ and the induction hypothesis). Put $A_0 = A \cup h_\alpha^{-1}[B]$. Choose $A_1 \subseteq P_\alpha$ so that $A_1 \cap A_0 = \emptyset$ and $|A_1| = |A_0| + \omega$. Let $E_\alpha = A_0 \cup A_1$. Define $g(x) = h_\alpha(x)$ for each $x \in P_\alpha \setminus E_\alpha$. Let $b : E_\alpha \setminus A \to h_\alpha[A_1]$ be any bijection. Then define $g(x) = b(x)$ for each $x \in E_\alpha \setminus A$. Put $G_\alpha = B \cup h_\alpha[A_1]$. It is not hard to check that $(*_\alpha)$ holds true and that E_α, G_α are as desired. This finishes the inductive definition of g. To show that g is a perfect isomorphism, consider $P \in \mathrm{Perf}(X)$. There is an $\alpha < \mathfrak{c}$ such that $|P \cap P_\alpha| = \mathfrak{c}$. Since $P \cap P_\alpha$ contains $\mathfrak{c}$ pairwise disjoint perfect sets, there is a $P^* \in \mathrm{Perf}(P \cap P_\alpha \setminus E_\alpha)$. Thus $g|P^* = h_\alpha|P^*$ is a homeomorphism. Similarly, there is a $P^{**} \in \mathrm{Perf}(X)$ such that $g^{-1}|P^{**}$ is a homeomorphism. Hence g is a perfect isomorphism, by Theorem 8.3. □

Theorem 8.5. *There are σ–ideals $\mathcal{I}, \mathcal{J} \subseteq$ Pow(X) such that both $\mathcal{I} \cap$ Perf(X) and $\mathcal{J} \cap$ Perf(X) are coinitial and $\mathcal{I}, \mathcal{J}$ are perfect isomorphic but not Borel isomorphic.*

Proof. The main idea is to find two perfect isomorphic σ–ideals such that exactly one of them has a Borel basis. Let $H \subseteq X$ be a Borel set such $|H| = |X \setminus H| = \mathfrak{c}$. Choose a maximal adf $\{P_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathrm{Perf}(X)$ such that each P_α is homeomorphic with 2^ω and

$$(\forall \alpha < \mathfrak{c})(P_\alpha \subseteq H \quad \text{or} \quad P_\alpha \cap H = \emptyset) \qquad \text{and}$$

$$|\{\alpha < \mathfrak{c} \colon P_\alpha \subseteq H\}| = |\{\alpha < \mathfrak{c} \colon P_\alpha \cap H = \emptyset\}| = \mathfrak{c}.$$

Let $\mathcal{I}$ denote the σ–ideal in $\mathrm{Pow}(X)$ generated by the family $\mathcal{G}$ of all $E \in \mathrm{Bor}(X)$ such that

$$|\{\alpha < \mathfrak{c} : |P_\alpha \cap E| = \mathfrak{c} \ \& \ P_\alpha \cap H = \emptyset\}| < \mathfrak{c}.$$

Observe that $H \in \mathcal{I}$ and $P_\alpha \in \mathcal{I}$ for each $\alpha < \mathfrak{c}$ (in fact those sets are in $\mathcal{G}$). However, $X \setminus H \notin \mathcal{I}$ (an easy exercise remembering that $cf(\mathfrak{c}) > \omega$). Since all P_α's are in $\mathcal{I}$, we have that $\mathcal{I} \cap \mathrm{Perf}(X)$ is coinitial.

Now, consider those sets $B \in \mathrm{Bor}(X)$ for which $|\{\alpha < \mathfrak{c} : |P_\alpha \setminus B| = \mathfrak{c}\}| = \mathfrak{c}$ and list them as B_α, $\alpha < \mathfrak{c}$ (note that all P_α's are among them). Fix enumerations $\{\alpha_\gamma : \gamma < \mathfrak{c}\}$ of $\{\alpha < \mathfrak{c} : P_\alpha \subseteq H\}$ and $\{\alpha^*_\gamma : \gamma < \mathfrak{c}\}$ of $\{\alpha < \mathfrak{c} : P_\alpha \cap H = \emptyset\}$. Define a function $f : \mathfrak{c} \to \mathfrak{c}$ as follows. First, let $f(\alpha_0)$ be the first ordinal $\beta < \mathfrak{c}$ such that $|P_\beta \setminus B_0| = \mathfrak{c}$ and let $f(\alpha^*_0)$ be the first element of $\mathfrak{c} \setminus \{f(\alpha_0)\}$. Let $\eta < \mathfrak{c}$ and assume that we have defined $f(\alpha_\gamma)$ and $f(\alpha^*_\gamma)$ for $\gamma < \eta$. Pick $f(\alpha_\eta)$ as the first ordinal $\beta \in \mathfrak{c} \setminus \{f(\alpha_\gamma), f(\alpha^*_\gamma) : \gamma < \eta\}$ such that $|P_\beta \setminus B_\eta| = \mathfrak{c}$ and let $f(\alpha^*_\eta)$ be the first element of $\mathfrak{c} \setminus \{f(\alpha_\gamma), f(\alpha^*_\xi) : \gamma \leq \eta, \ \xi < \eta\}$. The function f defined in this manner is a bijection from $\mathfrak{c}$ onto $\mathfrak{c}$ and has the property that, for each

$B \in \mathrm{Bor}(X)$, either $|\{\alpha < \mathfrak{c} : |P_\alpha \setminus B| = \mathfrak{c}\}| < \mathfrak{c}$ or there exists an $\alpha < \mathfrak{c}$ such that $P_\alpha \subseteq H$ and $|P_{f(\alpha)} \setminus B| = \mathfrak{c}$.

Now, consider the perfect isomorphism $g : X \to X$ obtained in Lemma 8.4 for $\{P_\alpha : \alpha < \mathfrak{c}\}$ and $\{P_{f(\alpha)} : \alpha < \mathfrak{c}\}$ (so this is the same adf, we just permute its elements). Let $\mathcal{J} = \{g[E] : E \in \mathcal{I}\}$. Obviously, $\mathcal{J}$ forms a σ–ideal in $\mathrm{Pow}(X)$ perfect isomorphic with $\mathcal{I}$, and $\mathcal{J} \cap \mathrm{Perf}(X)$ is coinitial. We will show that $\mathcal{I}$ and $\mathcal{J}$ are not Borel isomorphic. It is enough to check that $\mathcal{J}$ does not have a Borel basis. To this end, observe that $g[H] \in \mathcal{J}$ is not contained in a Borel set from $\mathcal{J}$. Suppose that $g[H] \subseteq B \in \mathcal{J} \cap \mathrm{Bor}(X)$. First note that, by the choice of the function g (and Lemma 8.4), for each $\alpha < \mathfrak{c}$ we have

$$|P_\alpha \setminus B| = \mathfrak{c} \quad \text{if and only if} \quad |P_{f^{-1}(\alpha)} \setminus g^{-1}[B]| = \mathfrak{c}.$$

Since f is a bijection, we get

$$|\{\alpha < \mathfrak{c} : |P_\alpha \setminus B| = \mathfrak{c}\}| = |\{\alpha < \mathfrak{c} : |P_\alpha \setminus g^{-1}[B]| = \mathfrak{c}\}|.$$

Now, as $g^{-1}[B] \in \mathcal{I}$, we have

$$|\{\alpha < \mathfrak{c} : |P_\alpha \cap g^{-1}[B]| = \mathfrak{c} \ \&\ P_\alpha \cap H = \emptyset\}| < \mathfrak{c}.$$

As there are continuum many α's such that $P_\alpha \cap H = \emptyset$, we conclude that $|\{\alpha < \mathfrak{c} : |P_\alpha \setminus g^{-1}[B]| = \mathfrak{c}\}| = \mathfrak{c}$ and hence $|\{\alpha < \mathfrak{c} : |P_\alpha \setminus B| = \mathfrak{c}\}| = \mathfrak{c}$. Thus, by the choice of f, there is an $\alpha < \mathfrak{c}$ such that $P_\alpha \subseteq H$ and $|P_{f(\alpha)} \setminus B| = \mathfrak{c}$. Then $g[P_\alpha] \subseteq g[H] \subseteq B$. Since, by Lemma 8.4 we have $|g[P_\alpha] \triangle P_{f(\alpha)}| < \mathfrak{c}$, we obtain a contradiction with $|P_{f(\alpha)} \setminus B| = \mathfrak{c}$. $\square$

Some months ago, the first author was informed by Sz. Plewik that the proof of the classical Sierpiński duality theorem [36] (cf. also [31, Th.19.2]) works (under CH) if one wants to show that there exists a perfect isomorphism between $\mathbb{K}$, the σ–ideal of meager sets in $\mathbb{R}$, and $\mathbb{L}$, the σ–ideal of Lebesgue null sets in $\mathbb{R}$. We will go further and describe more general situations where two σ–ideals in $\mathrm{Pow}(X)$ are perfect isomorphic. Our method mixes modifications of the Sierpiński duality theorem (cf. [14, pp.176–178]) with some tricks using almost disjoint families in $\mathrm{Perf}(X)$.

Let $\kappa < \mathfrak{c}$ be an uncountable cardinal and let $\mathcal{I} \subseteq \mathrm{Pow}(X)$ be a σ–ideal fulfilling $\bigcup \mathcal{I} = X$. We say that a Borel basis $\{B_\alpha : \alpha < \kappa\}$ of $\mathcal{I}$ forms a *perfect Borel tower* (of size κ) if $B_\alpha \subseteq B_\beta$ for all $\alpha, \beta < \kappa$ with $\alpha < \beta$, and $\mathrm{Perf}(B_\alpha \setminus \bigcup_{\gamma<\alpha} B_\gamma) \neq \emptyset$ for each $\alpha < \kappa$.

Theorem 8.6. *For an uncountable cardinal $\kappa \leq \mathfrak{c}$ let Φ_κ denote the family of all σ–ideals $\mathcal{I} \subseteq Pow(X)$ fulfilling $\bigcup \mathcal{I} = X$ and possessing a perfect Borel tower of size κ. Then any two σ–ideals $\mathcal{I}$ and $\mathcal{J}$ from Φ_κ with coinitial $\mathcal{I} \cap Perf(X)$ and $\mathcal{J} \cap Perf(X)$ are perfect isomorphic.*

Proof. Suppose $\mathcal{I} \in \Phi_\kappa$ and $\mathcal{I} \cap \mathrm{Perf}(X)$ is coinitial in $\mathrm{Perf}(X)$. Let $\{B^{\mathcal{I}}_\alpha : \alpha < \kappa\}$ be a perfect Borel tower of $\mathcal{I}$. Choose a maximal adf $\mathcal{F}_{\mathcal{I}} \subseteq \mathrm{Perf}(X)$ consisting of sets homeomorphic with 2^ω and such that

1. $\mathcal{F}_{\mathcal{I}} \subseteq \mathcal{I}$,
2. $\mathcal{F}_{\mathcal{I}} \subseteq \bigcup\limits_{\alpha<\kappa} \mathrm{Perf}(B^{\mathcal{I}}_\alpha \setminus \bigcup\limits_{\gamma<\alpha} B^{\mathcal{I}}_\alpha)$,
3. for each $\alpha < \kappa$
$$|\{P \in \mathcal{F}_{\mathcal{I}} : P \subseteq B^{\mathcal{I}}_\alpha \setminus \bigcup_{\beta<\alpha} B^{\mathcal{I}}_\beta\}| = \mathfrak{c} \quad \text{and} \quad |(B^{\mathcal{I}}_\alpha \setminus \bigcup_{\beta<\alpha} B^{\mathcal{I}}_\beta) \setminus \bigcup \mathcal{F}_{\mathcal{I}}| = \mathfrak{c},$$
4. $\mathcal{F}_{\mathcal{I}} \cap \mathrm{Perf}(B^{\mathcal{I}}_\alpha \setminus \bigcup\limits_{\beta<\alpha} B^{\mathcal{I}}_\beta))$ is a maximal adf in $\mathrm{Perf}(B^{\mathcal{I}}_\alpha \setminus \bigcup\limits_{\beta<\alpha} B^{\mathcal{I}}_\beta))$ (for all $\alpha < \kappa$).

The construction of $\mathcal{F}_{\mathcal{I}}$ can be carried out as follows. For each $\alpha < \kappa$ we choose an almost disjoint family $\mathcal{F}^*_\alpha$ of perfect sets homeomorphic with 2^ω and contained in $B^{\mathcal{I}}_\alpha \setminus \bigcup\limits_{\beta<\alpha} B^{\mathcal{I}}_\beta$, such that:

$(5)_\alpha$ $\mathcal{F}^*_\alpha \subseteq \mathcal{I} \cap \mathrm{Perf}(B^{\mathcal{I}}_\alpha \setminus \bigcup\limits_{\beta<\alpha} B^{\mathcal{I}}_\beta)$ and $\mathcal{F}^*_\alpha$ forms a maximal adf in $\mathrm{Perf}(B^{\mathcal{I}}_\alpha \setminus \bigcup\limits_{\beta<\alpha} B^{\mathcal{I}}_\beta)$ (i.e. if $P \subseteq B^{\mathcal{I}}_\alpha \setminus \bigcup\limits_{\beta<\alpha} B^{\mathcal{I}}_\beta$ is perfect then $(\exists Q \in \mathcal{F}^*_\alpha)(|P \cap Q| > \omega)$),

$(6)_\alpha$ $|(B^{\mathcal{I}}_\alpha \setminus \bigcup\limits_{\beta<\alpha} B^{\mathcal{I}}_\beta) \setminus \bigcup \mathcal{F}^*_\alpha| = \mathfrak{c}$ and $|\mathcal{F}^*_\alpha| = \mathfrak{c}$.

Why is the choice of $\mathcal{F}^*_\alpha$ possible? As $B^{\mathcal{I}}_\alpha \setminus \bigcup\limits_{\beta<\alpha} B^{\mathcal{I}}_\beta$ contains a perfect set, and by the fact that there are (s_0)–sets of size $\mathfrak{c}$ (and by Zorn's lemma). Finally, we let $\mathcal{F}_{\mathcal{I}} = \bigcup\limits_{\alpha<\kappa} \mathcal{F}^*_\alpha$. Now, why $\mathcal{F}_{\mathcal{I}}$ is as required? Clauses (1) and (2) follow from the conditions $(5)_\alpha$ for $\mathcal{F}^*_\alpha$, clause (3) is a consequence of $(6)_\alpha$ for $\mathcal{F}^*_\alpha$, and (4) follows from $(5)_\alpha$. What we need to show is that $\mathcal{F}_{\mathcal{I}}$ is a maximal adf in $\mathrm{Perf}(X)$. For this we use the assumption that $\mathcal{I} \cap \mathrm{Perf}(X)$ is coinitial: let $P \in \mathrm{Perf}(X)$. Take $Q \subseteq P$ such that $Q \in \mathcal{I} \cap \mathrm{Perf}(X)$. Then $Q \subseteq B^{\mathcal{I}}_\alpha$ for some $\alpha < \kappa$. Let α_0 be the first ordinal below κ such that $Q \cap B^{\mathcal{I}}_{\alpha_0}$ is uncountable (so $\alpha_0 \leq \alpha$). As $|\alpha_0| < \mathfrak{c}$ we necessarily have $|\bigcup\limits_{\beta<\alpha_0} B^{\mathcal{I}}_\beta \cap Q| < \mathfrak{c}$ and hence there is a perfect set $Q_0 \subseteq Q \cap (B^{\mathcal{I}}_{\alpha_0} \setminus \bigcup\limits_{\beta<\alpha_0} B^{\mathcal{I}}_\beta)$. Now, by $(5)_{\alpha_0}$, there is $Q_1 \in \mathcal{F}^*_{\alpha_0} \subseteq \mathcal{F}_{\mathcal{I}}$ such that $Q_1 \cap Q_0$ is uncountable (and so contains a perfect set).

Suppose now that $\mathcal{I}, \mathcal{J} \in \Phi_\kappa$ have coinitial traces in $\mathrm{Perf}(X)$ and consider the maximal almost disjoint families $\mathcal{F}_{\mathcal{I}}$, $\mathcal{F}_{\mathcal{J}}$. Choose enumerations
$$\{P_\beta : \beta < \mathfrak{c}\} = \mathcal{F}_{\mathcal{I}} \quad \text{and} \quad \{Q_\beta : \beta < \mathfrak{c}\} = \mathcal{F}_{\mathcal{J}}$$
such that for each $\beta < \mathfrak{c}$ and $\alpha < \kappa$
$$P_\beta \subseteq B^{\mathcal{I}}_\alpha \quad \text{if and only if} \quad Q_\beta \subseteq B^{\mathcal{J}}_\alpha$$

(possible by the requirement (3)). Modifying slightly the proof of Lemma 8.4 we find a perfect isomorphism $g : X \to X$ such that for each $\alpha < \kappa$ we have

$$g[\bigcup\{P_\beta : \beta < \mathfrak{c} \ \& \ P_\beta \subseteq B^{\mathcal{I}}_\alpha\}] = \bigcup\{Q_\beta : \beta < \mathfrak{c} \ \& \ Q_\beta \subseteq B^{\mathcal{J}}_\alpha\}.$$

(Just by induction on $\alpha < \kappa$ define $g|(\bigcup\{P_\beta : \beta < \mathfrak{c} \ \& \ P_\beta \subseteq B^{\mathcal{I}}_\alpha \setminus \bigcup\limits_{\gamma<\alpha} B^{\mathcal{I}}_\gamma\})$, proceeding exactly as in 8.4; remember our requirements on $\mathcal{F}_{\mathcal{I}}$, $\mathcal{F}_{\mathcal{J}}$ and their enumerations). We may arbitrarily modify g on an (s_0)–set, so we may additionally require that for each $\alpha < \kappa$

$$g[B^{\mathcal{I}}_\alpha] = B^{\mathcal{J}}_\alpha$$

(remember (3) and (2)). But this immediately implies that g is a perfect isomorphism between $\mathcal{I}$ and $\mathcal{J}$. $\square$

Proposition 8.7. *Assume that a σ–ideal $\mathcal{I} \subseteq Pow(X)$ has a Borel basis and satisfies $\bigcup \mathcal{I} = X$. Consider the following condition*

$$(\otimes) \qquad (\forall A \in \mathcal{I})(\exists P \in \mathcal{I} \cap Perf(X))(A \cap P = \emptyset).$$

(a) If $\mathcal{I} \cap Perf(X)$ is coinitial then $(\otimes)$ holds true.
(b) If $add(\mathcal{I}) = cof(\mathcal{I}) = \kappa$ and $(\otimes)$ holds true then $\mathcal{I} \in \Phi_\kappa$.
(c) If $add(\mathcal{I}) = cof(\mathcal{I}) = \kappa < \mathfrak{c}$ then $(\otimes)$ holds true.
(d) If $cof(\mathcal{I}) = \kappa < \mathfrak{c}$ then $\mathcal{I} \cap Perf(X)$ is coinitial. (Consequently, $\mathcal{I} \cap Perf(X)$ is coinitial for each $\mathcal{I} \in \Phi_\kappa$ with $\kappa < \mathfrak{c}$.)

Proof. (a) If $A \in \mathcal{I}$ then $A \subseteq B$ for some B from a fixed Borel basis of $\mathcal{I}$. Since $X \setminus B$ is an uncountable Borel set, it contains a perfect set P. By assumption, we can pick $Q \in \mathcal{J} \cap \mathrm{Perf}(P)$. Plainly Q is good.
(b) Since $add(\mathcal{I}) = cof(\mathcal{I}) = \kappa$ and $\mathcal{I}$ has a Borel basis, there is a Borel basis $\{B_\alpha : \alpha < \kappa\}$ of $\mathcal{I}$ such that

$$(\forall \alpha, \beta < \kappa)(\alpha < \beta \implies B_\alpha \subseteq B_\beta).$$

As $\bigcup \mathcal{I} = X$, we have $\bigcup_{\alpha<\kappa} B_\alpha = X$. Now, by $(\otimes)$, for each $\alpha < \kappa$, there is $\beta > \alpha$ such that $|B_\beta \setminus B_\alpha| = \mathfrak{c}$. Hence, passing to a subsequence, we may additionally have that $B_\alpha \setminus \bigcup_{\beta<\alpha} B_\beta$ contains a perfect set (for each $\alpha < \kappa$).
(c) As above, we get an increasing Borel basis $\{B_\alpha : \alpha < \kappa\}$ of $\mathcal{I}$ with $\bigcup_{\alpha<\kappa} B_\alpha = X$. If $A \in \mathcal{I}$ then $A \subseteq B_\alpha$ for some $\alpha < \kappa$. Since $X \setminus B_\alpha$ is an uncountable Borel set, we have $|X \setminus B_\alpha| = \mathfrak{c}$. As $X \setminus B_\alpha = \bigcup_{\alpha<\beta<\kappa}(B_\beta \setminus B_\alpha)$, there exists $\beta > \alpha$ for which $B_\beta \setminus B_\alpha$ is uncountable. Hence $B_\beta \setminus B_\alpha$ contains a perfect set P. Then $P \in \mathcal{I}$ and $P \cap A = \emptyset$.
(d) Let $\mathcal{F}$ be a Borel basis of $\mathcal{I}$ with $|\mathcal{F}| = \kappa$. Evidently $\bigcup \mathcal{F} = X$. If $P \in \mathrm{Perf}(X)$ then $P \cap B$ is uncountable for some $B \in \mathcal{F}$. A perfect subset of $P \cap B$ is in $\mathcal{I}$. $\square$

Corollary 8.8. *(a) Let $\mathcal{I}$ and $\mathcal{J}$ be σ–ideals in Pow(X) fulfilling $\bigcup\mathcal{I} = \bigcup\mathcal{J} = X$ and having Borel bases. If add$(\mathcal{I})$ = cof$(\mathcal{I})$ = add$(\mathcal{J})$ = cof$(\mathcal{J}) = \kappa$ then $\mathcal{I}$ and $\mathcal{J}$ are perfect isomorphic, provided that either $\kappa < \mathfrak{c}$ or $\kappa = \mathfrak{c}$ and $\mathcal{I} \cap$ Perf(X), $\mathcal{J} \cap$ Perf(X) are coinitial.*
(b) If add$(\mathbb{K})$ = cof$(\mathbb{K})$ = add$(\mathbb{L})$ = cof$(\mathbb{L})$ then $\mathbb{K}$ and $\mathbb{L}$ are perfect isomorphic.

Proof. Statement (a) is immediate by Theorem 8.6 and Proposition 8.7, and statement (b) follows from (a). □

Final Remarks. 1) Note that the ideals $\mathbb{K}$ and $\mathbb{L}$ are not Borel isomorphic (in ZFC) [31, Chapter 21].
2) Both CH and MA imply that $cof(\mathbb{K}) = add(\mathbb{K}) = cof(\mathbb{L}) = add(\mathbb{K})$ but it is consistent that this condition fails (see e.g. [8]). If it holds true, the σ–ideals $\mathbb{K}$ and $\mathbb{L}$ are perfect isomorphic but not Borel isomorphic.
3) In Proposition 8.7(d) we really need the restriction that $\kappa < \mathfrak{c}$. Assume CH. Consider the σ–ideal $\mathcal{I} \subseteq \mathrm{Pow}(\mathbb{R}^2)$ generated by sets of the form $\{x\}\times B$ for $x \in \mathbb{R}$ and $B \in \mathbb{K}$. Then $cof(\mathcal{I}) = \mathfrak{c}$ but $\mathbb{R} \times \{0\}$ contains no perfect subset in $\mathcal{I}$. It is not clear if one can get a similar example with large continuum.

Problem 8.9.
1. *Is it consistent that $\Phi_{\mathfrak{c}} \neq \emptyset$ and for every $\mathcal{I} \in \Phi_{\mathfrak{c}}$ the family $\mathcal{I} \cap$ Perf(X) is coinitial?*
2. *Is it consistent that the ideals $\mathbb{K}$ and $\mathbb{L}$ are isomorphic but not perfect isomorphic?*
3. *Is it consistent that there exist two isomorphic but not perfect isomorphic σ–ideals having Borel bases and coinitial traces in Perf(X)?*

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