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ON TRANSLATIONS OF SETS AND FUNCTIONS

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Abstract. We consider some properties of sets and functions in connection with their translations. We give an application of those properties to the theory of invariant measures (in particular, to the theory of invariant extensions of the classical Lebesgue measure).

Let E be a basic set (we assume, as a rule, that E is infinite) and let G be a group of transformations of E. In such a case the pair (E, G) is usually called a space equipped with a transformation group. If G acts transitively in E, then E is called a homogeneous space (with respect to G).

Let X be an arbitrary infinite subset of a homogeneous space (E, G). Then it is reasonable to study a behaviour of X under the action of transformations from the group G. Namely, the following two general problems arise in a natural way:

1. Find the cardinality of the family of sets
$$\{g(X) \ : \ g \in G\}.$$
2. Find the supremum of the family of cardinal numbers
$$\{card(g(X)\triangle X) \ : \ g \in G\},$$

where the symbol $\triangle$ denotes the operation of symmetric difference of sets.

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Of course, the solution of each of these two problems essentially depends on the algebraic (or, if one prefers, on the geometric) properties of the original objects E, G and X. Notice that in the classical situation, when E coincides with the real line $\mathbf{R}$ and G coincides with the group of all translations of $\mathbf{R}$, these problems, in fact, were completely investigated by Sierpiński (see, e.g., [5] and [4]).

The following two examples can be applied, in particular, to the classical situation mentioned above.

Example 1. Let $(G, +)$ be an arbitrary uncountable commutative group. As usual, we identify G with the group of all translations of G. Let α be an arbitrary infinite cardinal number less or equal to the cardinality of G. Then it can be shown that there exists a subset X of G satisfying the equality

$$card(\{g + X \ : \ g \in G\}) = \alpha.$$

Moreover, X can be a subgroup of the group G (see, for instance, [6] where a much stronger assertion is established).

In connection with the presented result, notice that it is essentially based on the Axiom of Choice. Indeed, let us consider a particular case when $G = \mathbf{R}$ and $\alpha = \omega$. Suppose that Y is a subset of $\mathbf{R}$ satisfying the relation

$$card(\{g + Y \ : \ g \in \mathbf{R}\}) = \omega.$$

Let us put

$$\Gamma = \{g \in \mathbf{R} \ : \ g + Y = Y\}.$$

Then it is easy to see that Γ is a subgroup of $\mathbf{R}$ and the equality

$$card(\mathbf{R}/\Gamma) = \omega$$

holds. Now, using the standard argument (see, e.g., [3]), we can deduce that Γ is a Lebesgue nonmeasurable subset of the real line (and, in addition, Γ does not have the Baire property with respect to the usual Euclidean topology on the real line). Thus we can conclude that any proof of the existence of the set Y mentioned above needs an uncountable form of the Axiom of Choice. On the other hand, applying a Hamel basis of $\mathbf{R}$, it is not difficult to show the existence of the set Y. Namely, let us consider $\mathbf{R}$ as a vector space over the field $\mathbf{Q}$ of all rational numbers and let us take a vector hyperplane Y in this space. Then it is clear that the family of all translates of Y is infinite and countable.

Suppose now that Γ is an arbitrary proper subgroup of $\mathbf{R}$ measurable in the Lebesgue sense (hence, Γ is a Lebesgue measure zero subgroup of $\mathbf{R}$). Then the equality

$$card(\mathbf{R}/\Gamma) = \mathbf{c}$$

is fulfilled. The same equality also holds if Γ is any proper subgroup of $\mathbf{R}$ having the Baire property (i.e. if Γ is a first category subgroup of $\mathbf{R}$). From these facts it can be deduced that if X is a nonempty proper subset of $\mathbf{R}$ measurable in the Lebesgue sense (respectively, having the Baire property), then

$$card(\{g + X \ : \ g \in \mathbf{R}\}) = \mathbf{c}.$$

Notice also that if X is an arbitrary nonempty proper subset of $\mathbf{R}$, then the inequality

$$card(\{g + X \ : \ g \in \mathbf{R}\}) \geq \omega$$

holds (this result is due to Sierpiński, too). Actually, it can be proved that, if G is an arbitrary nontrivial (i.e. nonzero) commutative divisible group and X is an arbitrary nonempty proper subset of G, then the family $\{g + X \ : \ g \in G\}$ is infinite. In particular, from this fact we immediately obtain that any nontrivial commutative divisible group is infinite.

Example 2. Let $(G, +)$ be again an arbitrary uncountable commutative group. Then, using an argument essentially due to Sierpiński (see [5] or [4]), it can be shown that there exists a partition $\{X, Y\}$ of G such that

1) $card(X) = card(Y) = card(G)$;

2) for each translation $g \in G$ we have the inequalities

$$card((g + X)\triangle X) < card(G), \ card((g + Y)\triangle Y) < card(G).$$

Indeed, let α be the first ordinal number of the cardinality of G. Obviously, there exists a family

$$\{G_\beta \ : \ \beta < \alpha\}$$

consisting of subsets of G and satisfying the following conditions:

a) this family is increasing with respect to inclusion;

b) the union of this family coincides with G;

c) $card(G_\beta) \leq card(\beta) + \omega$, for all $\beta < \alpha$;

d) G_β is a subgroup of G, for all $\beta < \alpha$.

Applying the method of transfinite recursion, we can construct two injective families

$$\{x_\beta \ : \ \beta < \alpha\}, \ \{y_\beta \ : \ \beta < \alpha\}$$

of elements of G such that

$$(G_\beta + x_\beta) \cap (G_\theta + y_\theta) = \emptyset,$$

for all ordinals $\beta < \alpha$ and $\theta < \alpha$.

Now, if we put

$$X = \cup\{G_\beta + x_\beta \ : \ \beta < \alpha\},$$
$$Y = G \setminus X,$$

then it is easy to check that the partition $\{X, Y\}$ of G satisfies relations 1) and 2).

The sets X and Y are usually called almost G–invariant subsets of the given group G.

For some applications of almost invariant subsets of the real line $\mathbf{R}$ to the theory of invariant measures (in particular, to the theory of invariant extensions of the classical Lebesgue measure), see monograph [3].

Let $\mathbf{c}$ denote, as usual, the cardinality of the continuum. From the above result it follows, in particular, that if the Continuum Hypothesis holds and $G = \mathbf{R}$, then there exists a partition $\{A, B\}$ of $\mathbf{R}$ such that

$$card(A) = card(B) = \mathbf{c}$$

and, for any translation $g \in \mathbf{R}$, we have the inequalities

$$card((g + A)\triangle A) \leq \omega, \ card((g + B)\triangle B) \leq \omega.$$

Conversely, it is not difficult to show that if there exists a partition $\{A, B\}$ of $\mathbf{R}$ satisfying these inequalities, for all translations $g \in \mathbf{R}$, and satisfying the relation $card(A) = card(B) = \mathbf{c}$, then the Continuum Hypothesis is true.

Also it is not difficult to prove that, for an arbitrary partition $\{Z_1, Z_2\}$ of the group G into two infinite subsets, there exists a translation $g \in G$ such that

$$card((g + Z_1) \cap Z_2) \geq \omega.$$

For $G = \mathbf{R}$ this fact was established many years ago by Trzeciakiewicz (see [8]), with the aid of the well known Vitali partition of the real line $\mathbf{R}$. The proof of this result applying the method of Trzeciakiewicz is presented in monograph [4]. The same result can be established in another way. Indeed, let $\{Z_1, Z_2\}$ be a partition of $\mathbf{R}$ into two infinite subsets. Evidently,

$$card(Z_1) > \omega \text{ or } card(Z_2) > \omega.$$

Without loss of generality we may assume that

$$card(Z_2) > \omega.$$

Suppose now that

$$card((g + Z_1) \cap Z_2) < \omega,$$

for each translation $g \in \mathbf{R}$. Then we have

$$card((g + Z_1)\triangle Z_1) < \omega,$$

for each $g \in \mathbf{R}$. Let us fix an infinite countable set

$$Z \subseteq Z_1.$$

It is easy to check the inclusion

$$Z_2 \subseteq \cup\{(Z_1 - z) \setminus Z_1 \ : \ z \in Z\}.$$

From this inclusion we obtain the inequality

$$card(Z_2) \leq \omega,$$

which gives us a contradiction.

A similar argument can be applied to an arbitrary uncountable commutative group G. More precisely, it can be proved that if G is an uncountable commutative group, $\{Z_1, Z_2\}$ is a partition of G into two subsets such that

$$card(Z_1) = card(Z_2) = card(G)$$

and β is a cardinal number strictly less than $card(G)$, then there exists an element $g \in G$ satisfying the inequality

$$card((g + Z_1) \cap Z_2) \geq \beta.$$

Example 3. Let $\mathbf{Q}$ be the additive group of all rational numbers and let $\{X, Y\}$ be a partition of $\mathbf{Q}$ into two infinite subsets. It is mentioned in [7] that there always exists an element $q \in \mathbf{Q}$ such that

$$card((q + X) \cap Y) = \omega.$$

This fact is proved in detail in paper [1]. Moreover, it is shown in [1] that an analogous fact is true for any countable, torsion free, not finitely generated commutative group. On the other hand, if $(G, +)$ is an arbitrary infinite countable periodic commutative group, then it is not difficult to prove that there exists a partition $\{A, B\}$ of G into two infinite subsets satisfying the relations

$$card((g + A)\triangle A) < \omega, \ card((g + B)\triangle B) < \omega,$$

for each element $g \in G$.

Example 4. Let $\{X, Y\}$ be a partition of the real line $\mathbf{R}$ into two uncountable Borel subsets. It is known that in this case there exists a translation $g \in \mathbf{R}$ satisfying the equality

$$card((g + X) \cap Y) = \mathbf{c}.$$

Notice that a nice proof of this fact is given in [2], using Shoenfield's absoluteness theorem from mathematical logic. Obviously, an analogous result is valid for an arbitrary uncountable commutative Polish topological group. More generally, let $(G, +)$ be a commutative Polish topological group and let $\{X_n \ : \ n < \omega\}$ be a countable partition of G into uncountable Borel subsets. Then there exist two distinct natural numbers n, m and an element $g \in G$ such that

$$card((g + X_n) \cap X_m) = \mathbf{c}.$$

In connection with this result, the following question arises in a natural way: is an analogous fact true for any ω_1–partition $\{X_\xi \ : \ \xi < \omega_1\}$ of G into uncountable Borel subsets? It turns out that we cannot answer this question within theory **ZFC**. Indeed, it is not difficult to show that if the negation of the Continuum Hypothesis holds, then the answer is positive, i.e. there exist two distinct ordinals $\xi < \omega_1$, $\zeta < \omega_1$ and an element $g \in G$ such that

$$card((g + X_\xi) \cap X_\zeta) = \mathbf{c}.$$

On the other hand, suppose that the Continuum Hypothesis holds. Then it can be proved (cf. [4]) that there exists a partition $\{A_\xi \ : \ \xi < \omega_1\}$ of the real line $\mathbf{R}$, satisfying the following conditions:

1) for each ordinal $\xi < \omega_1$, the set A_ξ is an uncountable G_δ–subset of $\mathbf{R}$;

2) for any translation g of $\mathbf{R}$ and for any two distinct ordinals $\xi < \omega_1$ and $\zeta < \omega_1$, we have

$$card((g + A_\xi) \cap A_\zeta) \leq 1.$$

Hence, we see that in this situation the answer to the question formulated above is negative. Notice that a much simpler partition can be defined for the three–dimensional Euclidean space $\mathbf{R}^3$. Namely, applying some elementary geometrical properties of hyperboloids in the space $\mathbf{R}^3$, we can effectively construct a partition of $\mathbf{R}^3$ consisting of straight lines such that any two distinct lines of this partition do not lie in one plane (consequently, they are not parallel). Obviously, if the Continuum Hypothesis holds, then the corresponding analogs of conditions 1) and 2) are fulfilled for this partition.

Example 5. Let S^2 be the unit sphere in the space $\mathbf{R}^3$. Equip S^2 with the group G of all its rotations. Clearly, G is not a commutative group (but it is a locally commutative group). Let $\{X, Y\}$ be an arbitrary partition of S^2 into two infinite subsets. It can be shown that there exists a rotation $g \in G$ satisfying the inequality

$$card(g(X) \cap Y) \geq \omega.$$

This result is precise in a certain sense. Indeed, assuming the Continuum Hypothesis and applying the method of Sierpiński, we can construct a partition $\{A, B\}$ of S^2 such that

1) $card(A) = card(B) = \mathbf{c}$;

2) for any rotation $g \in G$, we have the inequalities

$$card(g(A)\triangle A) \leq \omega, \; card(g(B)\triangle B) \leq \omega.$$

We see, in particular, that

$$card(g(A) \cap B) \leq \omega,$$

for each rotation $g \in G$.

In our further considerations connected with translations of functions it is convenient to identify any function with its graph.

Let E be a nonempty basic set and let S be a σ–algebra of subsets of E. In other words, the pair (E, S) is a measurable space. We need the following simple lemma.

Lemma 1. *Let f be an arbitrary S–measurable function acting from E into $\mathbf{R}$ and let*

$$\Phi \; : \; E \times \mathbf{R} \rightarrow E \times \mathbf{R}$$

be a mapping defined by the formula

$$\Phi(x, y) = (x, f(x) + y).$$

Then the mapping Φ is a bijection and transforms the family of graphs of all S–measurable functions onto itself. The inverse mapping Φ^{-1} has the same property.

Let E be a nonempty basic set, let S be a σ–algebra of subsets of E and let μ be a σ–finite complete measure defined on S. Equip the real line $\mathbf{R}$ with the standard Lebesgue measure. We need the following simple auxiliary proposition concerning translations of μ–measurable real functions.

Lemma 2. *Let f be an arbitrary μ–measurable function acting from E into $\mathbf{R}$ and let $\{f_n : n < \omega\}$ be an arbitrary sequence of μ–measurable functions also acting from E into $\mathbf{R}$. Then, for almost all $h \in \mathbf{R}$ (in the sense of the Lebesgue measure), the set*

$$\{x \in E \; : \; (x, f(x) + h) \in \cup\{f_n : n < \omega\}\}$$

is a μ–measure zero subset of E.

The proof of Lemma 2 can directly be deduced from Lemma 1, using the classical Fubini theorem. Notice that a proposition analogous to Lemma 2 also is true in the case of real functions having the Baire property. In this case we apply, instead of the Fubini theorem, the classical Kuratowski–Ulam theorem (for the formulation and the proof of the Kuratowski–Ulam theorem and for its various applications, see, e.g., [5]).

Obviously, an analogue of Lemma 2 is true for the unit two–dimensional Euclidean sphere S^2 equipped with the group of all its rotations, and, more generally, for other manifolds M equipped with some transitive groups of diffeomorphisms of M.

The assumption of measurability of functions in Lemma 2 is very essential. This can be shown by the following statement.

Theorem 1. *Suppose that the Continuum Hypothesis holds. Then there exists a subset Z of the plane $\mathbf{R}^2$, satisfying the next three relations:*

1) $pr_1(Z) = \mathbf{R} \times \{0\}$;

2) for every straight line P lying in $\mathbf{R}^2$ and parallel to the line $\{0\} \times \mathbf{R}$, we have the inequality

$$card(P \cap Z) \leq \omega;$$

3) for every translation g of $\mathbf{R}^2$ parallel to the line $\{0\} \times \mathbf{R}$, we have the inequality

$$card(g(Z) \triangle Z) \leq \omega.$$

Proof. Let $\{x_\xi \ : \ \xi < \omega_1\}$ be an injective family of all points of the straight line $\mathbf{R} \times \{0\}$. Let $\{G_\xi \ : \ \xi < \omega_1\}$ be a family of subgroups of the line $\{0\} \times \mathbf{R}$ such that

(a) this family is increasing with respect to inclusion;

(b) the union of this family coincides with $\{0\} \times \mathbf{R}$;

(c) $card(G_\xi) \leq \omega$, for all ordinals $\xi < \omega_1$.

Now, let us define

$$Z = \cup\{G_\xi + x_\xi \ : \ \xi < \omega_1\}.$$

Then it is not difficult to check that the set Z satisfies relations 1), 2) and 3). □

Furthermore, it can be shown, slightly changing the above argument, that the set Z may also have some additional properties. For instance, Z may be a massive (thick) subset of $\mathbf{R}^2$ in the sense of the Lebesgue measure or in the sense of category. But we do not need here these additional properties of Z.

Now, let $\{f_n \ : \ n < \omega\}$ be a countable family of functions from $\mathbf{R}$ into $\mathbf{R}$ such that

$$Z = \cup\{f_n \ : \ n < \omega\}.$$

Evidently, the existence of such a family of functions follows from relations 1) and 2) of Theorem 1. Let us put

$E = \mathbf{R}$;

$\mu =$ the Lebesgue measure on $\mathbf{R}$;

$f = f_0$.

Then it is easy to see that the assertion of Lemma 2 is false for the function f and for the family of functions $\{f_n : n < \omega\}$. Moreover, if g is any translation of $\mathbf{R}^2$ parallel to the line $\{0\} \times \mathbf{R}$, then we have the inequality

$$card(g(f) \setminus Z) \leq \omega.$$

Remark 1. It can be proved that the existence of the set Z satisfying relations 1), 2) and 3) of Theorem 1 implies the Continuum Hypothesis.

Theorem 2. *Let $\{A, B\}$ be a partition of the unit two–dimensional Euclidean sphere S^2 into two Borel subsets A and B such that*

$$card(A) = card(B) = \mathbf{c}.$$

Then there exists a rotation g of $\mathbf{R}^3$ satisfying the equalities

$$g(S^2) = S^2, \ card(g(A) \cap B) = \mathbf{c}.$$

Proof. For any point x belonging to the straight line $\mathbf{R} \times \{(0,0)\}$, we denote by P_x the plane in $\mathbf{R}^3$ containing x and orthogonal to the line $\mathbf{R} \times \{(0,0)\}$. Only two cases are possible.

1. There exists a point $x \in \mathbf{R} \times \{(0,0)\}$ such that

$$card(P_x \cap A) = card(P_x \cap B) = \mathbf{c}.$$

In this case we can consider the commutative group H of all rotations h of $\mathbf{R}^3$ satisfying the equalities

$$h(S^2) = S^2, \ h(P_x \cap S^2) = P_x \cap S^2.$$

Taking into account the remark made in Example 4, we obtain that there exists a rotation $h \in H$ such that

$$card(h(A) \cap B) = \mathbf{c}.$$

2. For each point $x \in \mathbf{R} \times \{(0,0)\}$, we have

$$card(P_x \cap A) \leq \omega \text{ or } card(P_x \cap B) \leq \omega.$$

In this case there exists a nonempty perfect subset X of $[-1,1] \times \{(0,0)\}$ such that at least one of the following two relations holds:

1) $0 < card(P_x \cap A) \leq \omega$, for all points $x \in X$;

2) $0 < card(P_x \cap B) \leq \omega$, for all points $x \in X$.

Without loss of generality we may assume that relation 1) is fulfilled. Let us consider the Borel set $A \cap (X \times \mathbf{R}^2)$. According to relation 1), all sections of this set are at most countable. Consequently, there exists a countable family of Borel functions

$$f_n \ : \ X \to S^2 \qquad (n < \omega)$$

such that

$$A \cap (X \times \mathbf{R}^2) = \cup\{f_n \ : \ n < \omega\}.$$

Applying an analog of Lemma 2 to the sphere S^2, to the function f_0 and to the countable family of functions $\{f_n \ : \ n < \omega\}$, we get the required result.

Thus the proof of Theorem 2 is complete. □

Of course, Theorem 2 can be generalized to the situation where we have the unit m–dimensional Euclidean sphere S^m $(m > 2)$ equipped with the group of all its rotations. In that situation the proof can be obtained by induction on m starting with the corresponding result for the sphere S^2.

Theorem 3. *Assume that the Continuum Hypothesis holds. Then there exists a subset Z of the Euclidean plane $\mathbf{R}^2$ such that*

1) $card(Z) = \mathbf{c}$;

2) for every Lebesgue measure zero subset L of $\mathbf{R}^2$, we have

$$card(L \cap Z) \leq \omega.$$

3) for each translation g of $\mathbf{R}^2$, we have

$$card(g(Z) \triangle Z) \leq \omega.$$

Proof. Let $\{G_\xi \ : \ \xi < \omega_1\}$ be an uncountable family of subgroups of $\mathbf{R}^2$ satisfying the following conditions:

(a) this family is increasing with respect to inclusion;

(b) the union of this family coincides with $\mathbf{R}^2$;

(c) $card(G_\xi) \leq \omega$, for all ordinals $\xi < \omega_1$.

Further, let us denote by $\{B_\xi \ : \ \xi < \omega_1\}$ the family of all Lebesgue measure zero subsets of $\mathbf{R}^2$ belonging to the Borel σ–algebra of $\mathbf{R}^2$. We can define, by the method of transfinite recursion, an injective family

$$\{z_\xi \ : \ \xi < \omega_1\}$$

of points of $\mathbf{R}^2$ such that the relation

$$(G_\xi + z_\xi) \cap (\cup\{B_\zeta \ : \ \zeta < \xi\}) = \emptyset$$

holds, for all ordinals $\xi < \omega_1$. Then we put

$$Z = \cup\{G_\xi + z_\xi \ : \ \xi < \omega_1\}.$$

Now, it is easy to check that Z is the required subset of the plane $\mathbf{R}^2$. $\square$

Remark 2. Let us recall that a subset A of the plane $\mathbf{R}^2$ is a Sierpiński set (in $\mathbf{R}^2$) if A is uncountable and, for every Lebesgue measure zero subset L of $\mathbf{R}^2$, we have

$$card(L \cap A) \leq \omega.$$

It is well known that the Continuum Hypothesis implies the existence of a Sierpiński set in $\mathbf{R}^2$ (see, for example, [5] or [4]). On the other hand, it is not difficult to prove that if Martin's Axiom and the negation of the Continuum Hypothesis hold, then there are no Sierpiński subsets of $\mathbf{R}^2$. Theorem 3 shows us, in particular, that if the Continuum Hypothesis is true, then there exists a Sierpiński subset of the Euclidean plane, almost invariant with respect to the group of all translations of this plane.

Now, we shall give an application of the Sierpiński set Z mentioned in Theorem 3.

First we shall formulate one simple fact of abstract measure theory. However, this fact is interesting from different points of view. Namely, we have the following

Theorem 4. *Let E_1 and E_2 be two basic sets equipped with two σ–finite diffused measures μ_1 and μ_2 respectively. Let μ be the completion of the product measure $\mu_1 \times \mu_2$. Finally, let Z be an arbitrary μ–measurable subset of the product space $E_1 \times E_2$. Then there exists a set Z' such that*

1) $Z' \subseteq Z$;

2) $pr_1(Z') = pr_1(Z)$;

3) Z' is a μ–measure zero set;

4) Z' is the graph of a partial function acting from E_1 into E_2.

Similarly, there exists a set Z'' such that

(1) $Z'' \subseteq Z$;

(2) $pr_2(Z'') = pr_2(Z)$;

(3) Z'' is a μ–measure zero set;

(4) Z'' is the graph of a partial function acting from E_2 into E_1.

In other words, Theorem 4 states that every μ–measurable set in the product space $E_1 \times E_2$ admits a uniformization by a μ–measure zero set in this space.

One can prove Theorem 4 using the classical Fubini theorem and applying the countable chain condition to the given σ–finite measures μ_1 and μ_2 (notice that an analogous result in terms of the Baire property and category is true, too).

Also it can be shown that Theorem 4 does not hold, in general, for those σ–finite complete diffused measures μ which are defined on the product space $E_1 \times E_2$ and are not the completions of product measures. In this connection, we shall establish a much stronger result.

Let us denote by λ the standard two–dimensional Lebesgue measure defined on the plane $\mathbf{R}^2$. Since λ is the completion of the product of two one–dimensional Lebesgue measures, Theorem 4 is true for λ. Now, we are going to construct an extension μ of λ which is invariant under the group of all translations of the plane and for which an analogue of this theorem does not hold.

Let Z be the subset of $\mathbf{R}^2$ described in Theorem 3. From the properties of Z it immediately follows that

$$\lambda_*(\mathbf{R}^2 \setminus Z) = 0,$$

i.e. Z is a thick set with respect to λ. Furthermore, Z is an almost invariant set under the group of all translations of $\mathbf{R}^2$. Consequently, we can define, using the standard constructions of extensions of invariant measures (see, e.g., [3]), a measure μ on $\mathbf{R}^2$ satisfying the next three conditions:

a) μ is an extension of λ;

b) $Z \in dom(\mu)$ and $\mu(\mathbf{R}^2 \setminus Z) = 0$;

c) μ is complete and invariant under the group of all translations of $\mathbf{R}^2$.

Moreover, we may assume that μ is the smallest (with respect to inclusion) complete $\mathbf{R}^2$–invariant extension of λ satisfying condition b). Now, taking into account the fact that Z is a Sierpiński subset of the plane $\mathbf{R}^2$, it is easy to check that every μ–measure zero subset of Z is at most countable. On the other hand, the projection of Z on any straight line lying in $\mathbf{R}^2$ is uncountable. Hence, the set Z does not admit a uniformization by a μ–measure zero set.

Finally, let us notice that the difference between μ and λ, from the measure-theoretical point of view, is very slight. Namely, it is not difficult to see that, for every μ–measurable set X, there exists a λ–measurable set Y satisfying the equality $\mu(X \triangle Y) = 0$.

Remark 3. Let us apply to the invariant measure μ constructed above the well known theorem of von Neumann and Maharam (see, for example, [5]). According to this theorem, there exists a topology $T(\mu)$ on the plane $\mathbf{R}^2$ such that

1) $(\mathbf{R}^2, T(\mu))$ is a Baire topological space;

2) $T(\mu)$ satisfies the countable chain condition, i.e. every disjoint family of nonempty sets from $T(\mu)$ is at most countable;

3) the σ–algebra of all sets having the Baire property (with respect to $T(\mu)$) coincides with the σ–algebra of all μ–measurable sets;

4) the σ–ideal of all first category sets (with respect to $T(\mu)$) coincides with the σ–ideal of all μ–measure zero sets.

Furthermore, since the measure μ is invariant under the group of all translations of the plane $\mathbf{R}^2$, the σ–algebra of all sets having the Baire property (in the space $(\mathbf{R}^2, T(\mu))$) and the σ–ideal of all first category sets (in the same space) are invariant under the mentioned group. The Sierpiński set Z described in Theorem 3 is the complement of a first category subset of $\mathbf{R}^2$ (with respect to $T(\mu)$, of course). Thus we see that the set Z has the Baire property (with respect to $T(\mu)$) and does not admit a uniformization by a first category subset of the space $(\mathbf{R}^2, T(\mu))$.

Let us recall that the construction of the Sierpiński set Z is essentially based on the Continuum Hypothesis. If we assume Martin's Axiom (much weaker than the Continuum Hypothesis), then we are able to construct an almost invariant generalized Sierpiński subset of the plane $\mathbf{R}^2$. Starting with that subset we can obtain the results analogous to ones presented above. Notice that the corresponding result in terms of the Baire property and category can also be obtained (under Martin's Axiom) starting with an almost invariant generalized Luzin subset of the plane.

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