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ON WEAKLY DARBOUX FUNCTIONS AND SOME PROBLEM CONNECTED WITH THE MORREY MONOTONICITY

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Abstract. In this paper we investigate relationships between the families of weakly Darboux, quasi-Darboux and Darboux functions which are quasi-continuous, and analyse the problem connected with the Morrey monotonicity of the restriction of a weakly Darboux function with the property $(\mathcal{P})$ to the closure of the union of levels which are continua.

In analysis, a very important role is played by Darboux functions as well as monotone ones. This has caused that generalizations of these functions are also considered. The problems of monotone transformations defined on spaces other than the real line have for many years now been intensively investigated by a large circle of mathematicians (e.g. [2], [3], [6], [4], [8], [13]). In a lot of papers, these investigations are connected with additional assumption concerning the transformations considered. In particular, there were analysed functions mapping some connected sets into connected sets with reference to the weak or Morrey monotonicity of the restriction of a function to the union of some of its levels. The results of papers [2], [3] together with those of paper [12] show how difficult it is to replace the assumption of the connectedness of the mappings under consideration by, for

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instance, the Darboux property. However, it turns out (Theorem 2) that, confining oneself to Morrey monotonicity instead of connected mappings, one may consider weakly Darboux functions possessing the property $(\mathcal{P})$. At the same time, Theorem 1 shows that the class of weakly Darboux functions is (in the topological sense) a class intermediate between the family of Darboux functions and the class of quasi-Darboux transformations.

A function $f : X \to R$ is a **Darboux function** if the image of an arbitrary arc $L \subset X$ is a connected set ([10]). We say ([11]) that $f : X \to R$ (X is an arcwise connected space) is a **quasi-Darboux** function if, for any $x, y \in X$ $(x \neq y)$, there exists an arc L with endpoints at x and y, such that $f_{|L}$ is a Darboux function. Now, we introduce the notion of a weakly Darboux function. We say that $f : X \to R$ is a **weakly Darboux** function if, for any open and arcwise connected set V and $x, y \in V$ $(x \neq y)$, there exists an arc $L \subset V$ with endpoints at x, y, such that $f_{|L}$ is a Darboux function.

The family (as well as the metric space with the metric ϱ^* of uniform convergence) of all bounded Darboux (quasi-Darboux, weakly Darboux) functions $f : R^2 \to R$ will be denoted by the symbol $\mathcal{D}$ ($\mathcal{D}_q$, $\mathcal{D}_w$).

Of course:

$$\mathcal{D} \subset \mathcal{D}_w \subset \mathcal{D}_q.$$

Our investigation (which leads to Theorem 1) intends a stronger inclusion even if we restricted the considerations to the case of quasi-continuous functions.

We shall use the standard notions and notations. By R (respectively, I) we denote the set of all real numbers (the unit interval). The bilateral closed (bilateral open, etc.) segment with endpoints a and b in R or R^2 is denoted by $[a, b]$ ((a, b), etc.). A subset $L \subset X$ is called an *arc* if there exists a homeomorphism $h : [0, 1] \overset{onto}{\to} L$. The elements $h(0)$ and $h(1)$ will be called endpoints of L. An arc with endpoints x and y is denoted by $L(x, y)$. If L is an arc and $a, b \in L$, then the symbol $L_L(a, b)$ denotes the arc with endpoints at a and b, contained in L.

The open ball with centre at x and radius $r > 0$ will be denoted by $B(x, r)$. The symbols $\overline{A}$ and $Int(A)$ stand for the closure and the interior of A, respectively.

A function $f : X \to Y$ is said to be **quasi-continuous** ([5]) at x if, for each neighbourhood W of $f(x)$ and each neighbourhood U of x, the set $Int(U \cap f^{-1}(W))$ is nonempty. The function f is said to be quasi-continuous if it is quasi-continuous at each point of its domain. The set of all bounded quasi-continuous functions $f : R^2 \to R$ will be denoted by $\mathcal{Q}$.

To simplify the notation, we shall write $\mathcal{DQ}$ ($\mathcal{D}_q\mathcal{Q}$, $\mathcal{D}_w\mathcal{Q}$) instead of $\mathcal{D}\cap\mathcal{Q}$ ($\mathcal{D}_q \cap \mathcal{Q}$, $\mathcal{D}_w \cap \mathcal{Q}$).

Let $f : X \to R$. The inverse image of the singleton (i.e. the set $f^{-1}(\alpha)$) will be called the **level** of f. According to the notation in [2], we write $Y_c(f) = \{\alpha \in f(X) : \ f^{-1}(\alpha) \textit{ is a connected set}\}$ ($Y_{cn}(f) = \{\alpha \in f(X) : f^{-1}(\alpha) \textit{ is a continuum}\}$) and $S_c(f) = f^{-1}(Y_c(f))$ ($S_{cn}(f) = f^{-1}(Y_{cn}(f))$). If $S_c(f) = X$ ($S_{cn}(f) = X$) then we say that f is **weakly monotone** (**Morrey monotone**) [8], [2], [3].

A function $f : X \to R$ possesses the **property** ($\mathcal{P}$) if, for any α, each component of $f^{-1}((-\infty, \alpha])$ is a closed set and each component of $f^{-1}([\alpha, +\infty))$ is a closed set.

The notions and symbols we use, dealing with porosity, come from papers [14] and [15]. Let X be a metric space. Let $M \subset X$, $x \in X$ and $S > 0$. Then we denote by $\gamma(x, S, M)$ the supremum of the set of all $r > 0$ for which there exists $z \in X$ such that $B(z, r) \subset B(x, S) \setminus M$. If $p(M, x) = 2 \cdot limsup_{S\to0+} \frac{\gamma(x,S,M)}{S} > 0$, then we say that M is porous at x.

Obviously, every porous set is nowhere dense. If $X = R$, every porous set is of Lebesgue measure zero.

If $A \subset B \subset X$ (where X is a metric space) and A is porous at any point $b \in B$ (B is understood as a subspace of X), then we shall write $A \sqsubset B$.

It is evident that $\mathcal{DQ} \subset \mathcal{D}_w\mathcal{Q} \subset \mathcal{D}_q\mathcal{Q}$ and there exist functions f, g such that $f \in \mathcal{D}_q\mathcal{Q} \setminus \mathcal{D}_w\mathcal{Q}$ and $g \in \mathcal{D}_w\mathcal{Q} \setminus \mathcal{DQ}$. In the theorem below we shall prove that (in the topological sense) this is a "frequent event" in the space of bounded function with the metric of uniform convergence.

Theorem 1. *$\mathcal{DQ} \sqsubset \mathcal{D}_w\mathcal{Q} \sqsubset \mathcal{D}_q\mathcal{Q}$.*

Proof. First, we shall show that $\mathcal{D}_w\mathcal{Q} \sqsubset \mathcal{D}_q\mathcal{Q}$. Let $f \in \mathcal{D}_q\mathcal{Q}$ and let $\varepsilon > 0$. There exist $x_o = (x_1, x_2) \in R^2$ and $\delta > 0$ such that

$$f(\overline{B(x_o, \delta)}) \subset (\alpha_o - \frac{\varepsilon}{3}, \alpha_o + \frac{\varepsilon}{3}) \tag{1}$$

where α_o is some real number. Denote $K_o = B(x_o, \delta)$. Let $K_1 = \overline{B(x_o, \frac{\delta}{2})}$ and put $K_1^* = \{(x, y) \in K_1 : \ x \leq x_1\}$, $K_1^{**} = \{(x, y) \in K_1 : \ x > x_1\}$. By $\mathcal{C}$ we denote the Cantor set for the segment $[\frac{\delta}{2}, \delta]$ (i.e. $\min \mathcal{C} = \frac{\delta}{2}, \max \mathcal{C} = \delta$). Let $\{(p_n, q_n)\}_{n=1}^{\infty}$ ($p_n < q_n$ for $n = 1, 2, \dots$) be the sequence of all components of the complement of $\mathcal{C}$. Moreover let $\hat{\mathcal{C}} = (\frac{\delta}{2}, \delta) \setminus \bigcup_{n=1}^{\infty}[p_n, q_n]$.

Let $\mathcal{F}_{\hat{\mathcal{C}}}$ be a family of power continuum which consists of dense subsets of $\hat{\mathcal{C}}$ such that $\bigcup_{A\in\mathcal{F}_{\hat{\mathcal{C}}}} A = \hat{\mathcal{C}}$ and if $A, B \in \mathcal{F}_{\hat{\mathcal{C}}}$ *and* $A \neq B$, then $A\cap B = \emptyset$. Let $\xi : \mathcal{F}_{\hat{\mathcal{C}}} \overset{onto}{\rightarrow} [\alpha_o - \frac{\varepsilon}{3}, \alpha_o + \frac{\varepsilon}{3}]$ be an arbitrary bijection. Moreover, let h_n denote some homeomorphism mapping $[p_n, q_n]$ onto $[\alpha_o - \frac{\varepsilon}{3}, \alpha_o + \frac{\varepsilon}{3}]$ $(n = 1, 2...)$ such that $h_n(p_n) = \alpha_o - \frac{\varepsilon}{3}$ and $h_n(q_n) = \alpha_o + \frac{\varepsilon}{3}$.

Now, we shall define a function $g : R^2 \to R$ in the following way:

$$g(x) = \begin{cases} f(x) & if \quad x \notin K_o; \\ \alpha_o - \frac{\varepsilon}{3} & if \quad x \in K_1^*; \\ \alpha_o + \frac{\varepsilon}{3} & if \quad x \in K_1^{**}; \\ h_n(\varrho(x, x_o)) & if \quad \varrho(x, x_o) \in [p_n, q_n] \; for\ some\ n; \\ \xi(\Xi_x) & if \quad \varrho(x, x_o) \in \Xi_x \in \mathcal{F}_{\hat{\mathcal{C}}}; \end{cases}$$

where ϱ denotes the Euclidean metric on the plane.

Of course, $g \notin \mathcal{D}_w$. Now, we shall show that

$$g \in \mathcal{D}_q. \tag{2}$$

First, we prove:

$$\begin{aligned} &if\ L\ is\ an\ arc\ such\ that\ there\ exist\ z_1, z_2 \in L \\ &for\ which\ \varrho(x_o, z_1) \neq \varrho(x_o, z_2)\ and\ \varrho(x_o, z_1), \varrho(x_o, z_2) \in \hat{C}, \\ &then\ [\alpha_o - \frac{\varepsilon}{3}, \alpha_o + \frac{\varepsilon}{3}] \subset g(L). \end{aligned} \tag{3}$$

In fact, let $\beta \in [\alpha_o - \frac{\varepsilon}{3}, \alpha_o + \frac{\varepsilon}{3}]$. Then there exists a set $A_\beta \in \mathcal{F}_{\hat{\mathcal{C}}}$ such that $\xi(A_\beta) = \beta$. Since A_β is a dense set in $\hat{\mathcal{C}}$ and $\varrho(x_o, z_1) \neq \varrho(x_o, z_2)$ we may infer that there exists $z_o \in L$ such that $\varrho(x_o, z_o) \in A_\beta$. Then $g(z_o) = \beta$.

For the proof of (2), we consider the following cases:

1) $x = (x_1, x_2) \in K_o,\ y = (y_1, y_2) \in K_o$. Let $x^* \in (\{x_1\} \times (R \setminus \{x_2\})) \cap (K_o \setminus K_1)$ and $y^* \in (\{y_1\} \times (R \setminus \{y_2\})) \cap (K_o \setminus K_1)$ and let $\hat{L} = L(x^*, y^*)$ be an arc such that $\hat{L} \subset (K_o \setminus K_1)$ and $\hat{L} \cap ([x, x^*) \cup (y^*, y]) = \emptyset$. Then we put $L^* = [x, x^*] \cup \hat{L} \cup [y, y^*]$. Let L_o be a subarc of L^*. If $L_o \subset K_1^*$ or $L_o \subset K_1^{**}$ or $L_o \subset (\overline{B(x, r)} \setminus B(x, r))$ for some $r \in (\frac{\delta}{2}, \delta)$ then $g_{|L_o}$ is constant. If $L_o \subset \{e : \varrho(x_0, e) \in [p_{n_1}, q_{n_1}]\}$ (for some $n_1 \in \{1, 2...\}$), then, of course, $g_{|L_o}$ is a Darboux function. In the opposite case, according to the definition of g, $g(L_o) \subset [\alpha_o - \frac{\varepsilon}{3}, \alpha_o + \frac{\varepsilon}{3}]$ and, by (3), $[\alpha_o - \frac{\varepsilon}{3}, \alpha_o + \frac{\varepsilon}{3}] \subset g(L_o)$. This means that, in this case, $g(L_o) = [\alpha_o - \frac{\varepsilon}{3}, \alpha_o + \frac{\varepsilon}{3}]$.

2) $x \notin K_o$ or $y \notin K_o$. Let $L = L(x, y) \subset R^2$ be an arc such that $f_{|L}$ is a Darboux function. If $L \cap K_o = \emptyset$, then $g_{|L} = f_{|L}$ is also a Darboux function.

If $L \cap K_o \neq \emptyset$ then we consider the subcases:

2a) $x, y \notin K_o$ Then let $L_x = L_L(x, z_x)$ (if $x \in \overline{K_o}$ then we put $L_x = \{x\}$), $L_y = L_L(y, z_y)$ (if $y \in \overline{K_o}$ then we put $L_y = \{y\}$) be arcs (or singletons) such that $L_x \cap \overline{K_o} = \{z_x\}$, $L_y \cap \overline{K_o} = \{z_y\}$. If $z_x = z_y$, then let $L^* = L_x \cup L_y$. In the opposite case let $L_{xy} = L(z_x, z_y) \subset \overline{K_o}$ be an arc such that $L_{xy} \cap K_1 = \emptyset$ and $L_{xy} \cap (\overline{K_o} \setminus K_o) = \{z_x, z_y\}$. Put $L^* = L_x \cup L_{xy} \cup L_y$. Let L_o be a subarc of L^*. If $L_o \subset L_x \cup L_y$ then it is obvious that $g_{|L_o}$ is a Darboux function. In the opposite case, according to (1), $f(L_o \cap \overline{K_o}) \subset (\alpha_o - \frac{\varepsilon}{3}, \alpha_o + \frac{\varepsilon}{3})$; and by (3), $[\alpha_o - \frac{\varepsilon}{3}, \alpha_o + \frac{\varepsilon}{3}] = g(L_o \cap K_o)$. So, we have that $g((L_o) = g(L_o \cap L_x) \cup [\alpha_o - \frac{\varepsilon}{3}, \alpha_o + \frac{\varepsilon}{3}] \cup g(L_o \cap L_y)$ is a connected set.

2b) $x \in K_o$ (if $y \in K_o$, the proof is similar). Let $x^* \in (\{x_1\} \times (R \setminus \{x_2\})) \cap (K_o \setminus K_1)$ and $L_1 = [x, x^*]$. Let $L_2 = L_L(y, z_y)$ be an arc such that $L_2 \cap \overline{K_o} = \{z_y\}$. Moreover, let $L_3 = L(x^*, z_y) \subset K_o \cup \{z_y\}$ be an arc such that $L_3 \cap (L_1 \cup L_2) = \{x^*, z_y\}$. It is not hard to verify that $g_{|L^*}$ is a Darboux function if $L^* = L_1 \cup L_2 \cup L_3$.

This ends the proof of (2).

Now, we shall show that

$$g \in \mathcal{Q}. \tag{4}$$

Let $t = (t_o, t^o) \in R^2$. For the proof, we consider the following cases:

(a) $t \notin \overline{K_o}$; then the quasi-continuity of g at t follows from the quasi-continuity of f at t.

(b) $t \in K_1$; then the quasi-continuity of g at t follows from the fact that $g_{|K_1^*}$, $g_{|K_1^{**}}$ are constant.

(c) $\varrho(x_o, t) \in [p_{n_o}, q_{n_o}]$ for some n_o. From the continuity of $g_{|\{z:\, \varrho(x_o,z) \in [p_{n_o}, q_{n_o}]\}}$ we may deduce that g is quasi-continuous at t.

(d) $\varrho(x_o, t) \in \hat{C}$ or $\varrho(x_o, t) = \delta$. Thus $g(t) \in [\alpha_o - \frac{\varepsilon}{3}, \alpha_o + \frac{\varepsilon}{3}]$. Put $\varrho(x_o, t) = \mu$ and let η, ζ be arbitrary positive real numbers. Choose a positive integer n_* such that $[p_{n_*}, q_{n_*}]$ is a segment for which the inclusion $[p_{n_*}, q_{n_*}] \subset (\mu - \zeta, \mu + \zeta)$ takes place. Finally, let λ be an arbitrary element from $(g(t) - \eta, g(t) + \eta) \cap (\alpha_o - \frac{\varepsilon}{3}, \alpha_o + \frac{\varepsilon}{3})$. By the definition of g there exists a point $d \in [x_o, t] \cap \{y : \ \varrho(x_o, y) \in (p_{n_*}, q_{n_*})\}$ such that $g(d) = \lambda$. It is easy to check that $\varrho(t, d) < \zeta$ and d is a continuity point of g, which ends the proof of (4).

From (2) and (4) we conclude that $g \in \mathcal{D}_q\mathcal{Q}$.

Now, we shall show that

$$B(g, \frac{\varepsilon}{4}) \cap \mathcal{D}_w = \emptyset.$$

Indeed, let $h \in B(g, \frac{\varepsilon}{4})$. Then $h(K_1^*) \subset (-\infty, \alpha_o)$ and $h(K_1^{**}) \subset (\alpha_o, +\infty)$, which means that $h \notin \mathcal{D}_w$.

It is clear that $B(g, \frac{\varepsilon}{4}) \subset B(f, \varepsilon)$. By the above, $p(\mathcal{D}_w\mathcal{Q}, f) \geq \frac{1}{2} > 0$. This completes, according to the arbitrariness of the choise of $f \in \mathcal{D}_q\mathcal{Q}$, the proof of the fact: $\mathcal{D}_w\mathcal{Q} \sqsubset \mathcal{D}_q\mathcal{Q}$.

In the next step of this proof we shall show that $\mathcal{DQ} \sqsubset \mathcal{D}_w\mathcal{Q}$.

Let $\varphi \in \mathcal{D}_w\mathcal{Q}$ and let $\varepsilon > 0$. Then there exist $x_o^{(\varphi)} \in R^2$ and $\delta^{(\varphi)} > 0$ such that $f(B(x_o^{(\varphi)}, \delta^{(\varphi)})) \subset (\alpha_o^{(\varphi)} - \frac{\varepsilon}{3}, \alpha_o^{(\varphi)} + \frac{\varepsilon}{3})$ where $\alpha_o^{(\varphi)}$ is some real number. Let $K = [k^1, k^2]$ be a closed segment with middle-point at $x_o^{(\varphi)}$ such that $K \subset B(x_o^{(\varphi)}, \frac{\delta^{(\varphi)}}{2})$. Denote $A_\alpha = \{x \in R^2 : \varrho(x, K) = \alpha\}$.

Let $\mathcal{C}^{(\varphi)}$ denote the Cantor set for the interval $[0, \frac{\delta^{(\varphi)}}{2}]$ (i.e. $\min \mathcal{C}^{(\varphi)} = 0, \max \mathcal{C}^{(\varphi)} = \frac{\delta^{(\varphi)}}{2}$) and let $\{(p_n^{(\varphi)}, q_n^{(\varphi)})\}_{n=1}^{\infty}$ be the sequence of all component of the complement of $\mathcal{C}^{(\varphi)}$. Moreover, put $\hat{\mathcal{C}^{(\varphi)}} = (0, \frac{\delta^{(\varphi)}}{2}) \setminus \bigcup_{n=1}^{\infty} [p_n^{(\varphi)}, q_n^{(\varphi)}]$. Let $\mathcal{F}^{(\varphi)}_{\hat{\mathcal{C}^{(\varphi)}}}$ be a family of power continuum consisting of the dense subsets of $\hat{\mathcal{C}^{(\varphi)}}$ such that $\bigcup_{A \in \mathcal{F}^{(\varphi)}_{\hat{\mathcal{C}^{(\varphi)}}}} A = \hat{\mathcal{C}^{(\varphi)}}$ and if $A, B \in \mathcal{F}^{(\varphi)}_{\hat{\mathcal{C}^{(\varphi)}}}$ are distinct sets, then they are disjoint. Let $\xi^{(\varphi)} : \mathcal{F}^{(\varphi)}_{\hat{\mathcal{C}^{(\varphi)}}} \overset{onto}{\to} [\alpha_o^{(\varphi)} - \frac{\varepsilon}{3}, \alpha_o^{(\varphi)} + \frac{\varepsilon}{3}]$ be an arbitrary bijection. Moreover, let $h_n^{(\varphi)}$ denote a homeomorphism $[p_n^{(\varphi)}, q_n^{(\varphi)}]$ onto $[\alpha_o^{(\varphi)} - \frac{\varepsilon}{3}, \alpha_o^{(\varphi)} + \frac{\varepsilon}{3}]$ (for $n = 1, 2...$).

Now, we shall define a function $g^{(\varphi)} : R^2 \to R$ in the following way:

$$g^{(\varphi)}(x) = \begin{cases} \varphi(x) & if \quad x \notin \bigcup_{0<\alpha<\frac{\delta^{(\varphi)}}{2}} A_\alpha; \\ \alpha_o^{(\varphi)} - \frac{\varepsilon}{3} & if \quad x \in [k^1, x_o]; \\ \alpha_o^{(\varphi)} + \frac{\varepsilon}{3} & if \quad x \in (x_o, k^2]; \\ h_n^{(\varphi)}(\alpha) & if \quad x \in A_\alpha \; and \; \alpha \in [p_n^{(\varphi)}, q_n^{(\varphi)}]; \\ \xi^{(\varphi)}(\Xi_x) & if \quad x \in A_\alpha \; and \; \alpha \in \Xi_x \in \mathcal{F}^{(\varphi)}_{\hat{\mathcal{C}^{(\varphi)}}}. \end{cases}$$

It follows immediately that $g^{(\varphi)} \notin \mathcal{D}$. The method of proof analogous to the method from the first part leads to the conclusion: $g^{(\varphi)} \in \mathcal{D}_w\mathcal{Q}$. Moreover, easy "computations" show that $B(g^{(\varphi)}, \frac{\varepsilon}{4}) \cap \mathcal{DQ} = \emptyset$ and $B(g^{(\varphi)}, \frac{\varepsilon}{4}) \subset B(\varphi, \varepsilon)$, which proves that $\mathcal{DQ} \sqsubset \mathcal{D}_w\mathcal{Q}$. □

In many papers the authors investigate the problem connected with the monotonicity of the restriction of connected functions[1] to the closure of the union of all connected levels ([2], [3], [9]). In paper [12] it was shown that there exists a Darboux function $f : I^2 \to R$ such that the restriction of it to the closure of the union of all connected levels is not weakly monotone. In the theorem below we investigate the analogous problem for the modification of assumptions (the Darboux property is replaced by the weakly Darboux property and the property $(\mathcal{P})$, and the connected levels by levels which are continua).

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[1]i.e. functions mapping connected sets onto connected sets.

Theorem 2. *Let X be a metrizable locally connected continuum and let $f : X \to R$ be a weakly Darboux function which possesses the property $(\mathcal{P})$. Then $f_{|\overline{S_{cn}(f)}}$ is Morrey monotone.*

Proof. Denote, for simplicity, $S = S_{cn}(f)$, $Y = Y_{cn}(f)$ and $g = f_{|\overline{S}}$. First, we prove that

$$\begin{aligned}&if\ \alpha \in Y,\ then\ f^{-1}([\alpha, +\infty))\ and\ f^{-1}((-\infty, \alpha])\ are\ continua\\ &and\\ &if\ \alpha, \beta \in Y,\ then\ f^{-1}([\alpha, \beta])\ is\ a\ continuum.\end{aligned} \tag{5}$$

In fact, first, we shall demonstrate that $f^{-1}([\alpha, +\infty))$ is a closed set. It suffices to show that $f^{-1}((-\infty, \alpha))$ is an open set. Let $z \in f^{-1}((-\infty, \alpha))$. Then there exists an open and connected set K_z such that $z \in K_z$ and $K_z \cap f^{-1}(\alpha) = \emptyset$. Since[2] $f(K_z)$ is a connected set, therefore $f(K_z) \subset (-\infty, \alpha)$, which ends the proof of the closedness of $f^{-1}([\alpha, +\infty))$. In a similar way we can prove that $f^{-1}((-\infty, \alpha])$ is a closed set. Observe that $f^{-1}((-\infty, \alpha]) \cup f^{-1}([\alpha, +\infty)) = R$ and $f^{-1}((-\infty, \alpha]) \cap f^{-1}([\alpha, +\infty)) = f^{-1}(\alpha)$. According to the fact that $f^{-1}(\alpha)$ is a continuum, we infer (see [7]) that $f^{-1}((-\infty, \alpha]), f^{-1}([\alpha, +\infty))$ are continua.

The proof of the fact that $f^{-1}([\alpha, \beta])$ is a continuum is analogous.

Now, we shall prove that

$$g\ is\ a\ continuous\ function. \tag{6}$$

So, let $x_o \in \overline{S}$, $\alpha_o = g(x_o)$ and let $\varepsilon > 0$. If $(\alpha_o - \varepsilon, \alpha_o) \cap Y \neq \emptyset$, then let a_o be an arbitrary point of this intersection. In the opposite case, let $a_o = sup\{\alpha \leq \alpha_o - \varepsilon :\ \alpha \in Y\}$ (of course, it is possible that $\{\alpha \leq \alpha_o - \varepsilon : \alpha \in Y\} = \emptyset$, then $a_o = -\infty$). In a similar way we can define a point $b_o \in (\alpha_o, +\infty)$.

Now, we shall show that

$$if\ a_o \neq -\infty,\ then\ \overline{f^{-1}((-\infty, a_o))} \subset f^{-1}((-\infty, a_o]). \tag{7}$$

Of course, if $a_o \in Y$, then (according to (5)) $f^{-1}((-\infty, a_o])$ is a closed set, and so, in this case, inclusion (7) takes place.

In the opposite case ($a_o \notin Y$), there exists a sequence $\{a_n\}_{n=1}^{\infty} \subset Y$ such that $a_n \nearrow a_o$. By (5), $f^{-1}((-\infty, a_o)) = \bigcup_{n=1}^{\infty} f^{-1}((-\infty, a_n])$ is a connected set contained in $f^{-1}((-\infty, a_o])$. Since f possesses the property $(\mathcal{P})$, therefore $f^{-1}((-\infty, a_o))$ is included in some closed component C of $f^{-1}((-\infty, a_o])$, which ends the proof of (7).

[2]Mazurkiewicz-Moor theorem.

If $c_o \neq +\infty$, then similarly as (7) we can prove

$$\overline{f^{-1}((c_o, +\infty))} \subset f^{-1}([c_o, +\infty)). \tag{8}$$

Now, we shall show that

$$\begin{aligned}&\textit{there exists an open set } K_a \textit{ containing } x_0\\ &\textit{such that } g(K_a \cap \overline{S}) \subset (\alpha_o - \varepsilon, +\infty).\end{aligned} \tag{9}$$

To prove this fact we consider the following cases:
1) $a_o = -\infty$. Then (9) is true for $K_a = X$.
2) $a_o \in (\alpha_o - \varepsilon, \alpha_o)$. Then (9) follows from (5).
3) $a_o \notin (\alpha_o - \varepsilon, \alpha_o)$. Let $c_o = inf\{\alpha \geq \alpha_o :\ \alpha \in Y\}$.

If $a_0 \in Y$, then by (5), $f^{-1}((a_0, +\infty))$ is an open set. Simultaneously, $f^{-1}((a_0, \alpha_o)) \cap S = \emptyset$. Then $S \subset f^{-1}((-\infty, a_0]) \cup f^{-1}([c_o, +\infty))$. If $c_o \in Y$, then by (5), $\overline{S} \subset f^{-1}((-\infty, a_0]) \cup f^{-1}([c_o, +\infty))$. If $c_o \notin Y$, then $\overline{S} \subset f^{-1}((-\infty, a_0]) \cup \overline{f^{-1}((c_o, +\infty))}$ and by (8), $\overline{S} \subset f^{-1}((-\infty, a_0]) \cup f^{-1}([c_o, +\infty))$. Consequently, $f^{-1}((a_0, \alpha_o)) \cap \overline{S} = \emptyset$. So, $g^{-1}((a_0, +\infty)) = f^{-1}((a_0, +\infty)) \cap \overline{S} = g^{-1}([\alpha_o, +\infty))$ is an open set in $\overline{S}$.

Now, we assume that $a_0 \notin Y$. According to (7), we may infer that $f^{-1}(\alpha) \cap \overline{f^{-1}((-\infty, a_o))} = \emptyset$. So, let K_a be an open set containing x_0 such that

$$K_a \cap f^{-1}((-\infty, a_o)) = \emptyset. \tag{10}$$

Now, we shall show that

$$\overline{S} \cap K_a \cap f^{-1}(a_o) = \emptyset. \tag{11}$$

Suppose to the contrary that there exists w such that $w \in \overline{S} \cap K_a \cap f^{-1}(a_o)$. Of course, $w \notin S$ and so there exists a sequence $\{w_n\}_{n=1}^{\infty}$ such that $lim_{n\to\infty} w_n = w$ (we may assume that $w_n \in K_a$ for $n = 1, 2, ...$). Then (according to (10)), $f(x_n) \geq c_o$, but this fact, according to (5), (8) and the obvious inequality $c_o \geq \alpha_o$ means that $w \notin f^{-1}(a_o)$, which is impossible.

By (10) and (11), $g(K_a \cap \overline{S}) \subset (a_o, +\infty)$. One can prove that $(a_o, \alpha_o) \cap g(\overline{S}) = \emptyset$, which leads to the conclusion that (9) takes place.

When we replace a_0 by b_0, the considerations analogous to the above ones lead to the conclusion:

$$\begin{aligned}&\textit{there exists an open set } K_b \textit{ containing } x_0\\ &\textit{such that } g(K_a \cap \overline{S}) \subset (-\infty, \alpha_o + \varepsilon),\end{aligned}$$

which, according to (9), ends the proof of (6).

Now, we consider monotonicity.

Let $\gamma \in R$ be an arbitrary number such that $g^{-1}(\gamma) \neq \emptyset$. It is obvious that if $\gamma \in Y$ then $g^{-1}(\gamma) = f^{-1}(\gamma)$ is a continuum. Similarly, if γ is a bilateral accumulation point of Y, then, according to (5), $g^{-1}(\gamma)$ is a continuum.

Finally, we suppose that γ is a unilateral accumulation point of Y and denote by $\{\gamma_n\}_{n=1}^{\infty} \subset Y$ a monotone sequence which tends to γ. Let, for instance, $\gamma_n \nearrow \gamma$.

According to (6), $g^{-1}(\gamma)$ is compact.

Suppose that $g^{-1}(\gamma)$ is not a connected set, which means that $g^{-1}(\gamma) = A \cup B$ where A, B are nonempty separated sets. Let $x_A \in A$ and $x_B \in B$. Of course, $x_A, x_B \in \overline{S} \setminus S$. Since R^2 is hereditarily normal, then there exist ([1]) open sets U and V such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$. Let $\{K_n^A\}_{n=1}^{\infty}(\{K_n^B\}_{n=1}^{\infty})$ be a local base at x_A (x_B) such that K_n^A (K_n^B) is an open and connected set, for $n = 1, 2, \ldots$. Without loss of generality we may assume that $K_n^A \subset U$ and $K_n^B \subset V$ for any positive integer n.

Let n_o be a fixed positive integer. Then $f(K_n^A), f(K_n^B)$ are connected sets containing γ. From $\gamma \notin Y$ we conclude that $f(K_n^A), f(K_n^B)$ are nondegenerate intervals containing some elements of Y. According to the continuity of g and our assumption that γ is not a limit of a decreasing sequence of elements of Y, we may infer that there exists k_{n_o} such that $\gamma_{k_{n_o}} \in f(K_n^A) \cap f(K_n^B)$ and $\mid \gamma_{k_{n_o}} - \gamma \mid < \frac{1}{n_o}$.

Continuing this procedure we obtain a sequence $\{\gamma_{k_n}\}_{n=1}^{\infty} \subset Y$ such that

$$f^{-1}(\gamma_{k_n}) \cap U \neq \emptyset \neq f^{-1}(\gamma_{k_n}) \cap V \ (for\ n = 1, 2 \ldots)\ and\ \lim_{n \to \infty} \gamma_{k_n} = \gamma.$$

So, let $s_n \in f^{-1}(\gamma_{k_n}) \setminus (U \cup V)$ (for $n = 1, 2 \ldots$) and let s_o be an accumulation point of $\{s_n\}_{n=1}^{\infty}$. This means that $s_o \in \overline{S}$. According to (6), we infer that $g(s_o) = \gamma$, which contradicts our assumption (of course, $s_o \notin U \cup V$) that $g^{-1}(\gamma) = A \cup B$.

The contradiction obtained ends the proof of Theorem 2. $\square$

It can be demonstrated that, under the assumptions of Theorem 2, $S_{cn}(f)$ does not have to be a closed set. This will be shown in a forthcoming paper.

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