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ON AFFINE SELECTIONS OF SET–VALUED FUNCTIONS

SZ. WĄSOWICZ

Abstract. The main result of this paper is the theorem stating that every convex set–valued function $F : I \mapsto c(Y)$, where $I \subset \mathbf{R}$ is an interval and Y is a locally convex space, possesses an affine selection. In the case if $Y = \mathbf{R}$ and values of F are closed real intervals we can replace the assumption of convexity of F by the more general condition.

Introduction. K. Nikodem and Sz. Wąsowicz have proved (cf. [3, Theorem 1]) that if functions $f, g : I \mapsto \boldsymbol{R}$, where $I \subset \boldsymbol{R}$ is an interval, fulfil for every $x, y \in I$, $t \in [0,1]$ the following condition

$$\begin{array}{rcl} f(tx+(1-t)y) & \leq & tg(x)+(1-t)g(y) \\ & \text{and} & \\ g(tx+(1-t)y) & \geq & tf(x)+(1-t)f(y) \end{array} \tag{1}$$

then there exists an affine function $h : I \mapsto \boldsymbol{R}$ such that $f \leq h \leq g$ on I. The simple consequence of this fact is Theorem 1 which we prove at the beginning of this paper. Next we prove that every convex set–valued function $F : I \mapsto c(Y)$, where Y is a locally convex space and $c(Y)$ is the family of all compact non–empty subsets of Y, possesses an affine selection.

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In the case $Y = \boldsymbol{R}^n$, $n \in \boldsymbol{N}$, we also present a direct inductive proof of this theorem.

Notation. By I we will denote any fixed real interval.

If X is a topological vector space then we admit the following notation:

$$\begin{aligned} n(X) &= \{A \subset X : A \neq \emptyset\}; \\ c(X) &= \{A \in n(X) : A \text{ is a compact set}\}; \\ cc(X) &= \{A \in c(X) : A \text{ is a convex set}\}. \end{aligned}$$

The term *set–valued function* will be abbreviated in the form *s.v. function*.

If X, Y are vector spaces and $D \subset X$ is a convex set then we say that a s.v. function $F : D \mapsto n(Y)$ is

a) *convex*, iff for every $x, y \in D$, $t \in [0,1]$

$$tF(x) + (1-t)F(y) \subset F(tx + (1-t)y);$$

b) *concave*, iff for every $x, y \in D$, $t \in [0,1]$

$$F(tx + (1-t)y) \subset tF(x) + (1-t)F(y).$$

A function $f : D \mapsto Y$ is called the *selection* of the s.v. function $F : D \mapsto n(Y)$ iff $f(x) \in F(x)$ for every $x \in D$.

Let us start with the translation of the theorem mentioned in Introduction into the s.v. functions language (cf. [5, Remark 14]).

Theorem 1. *A s.v. function $F : I \mapsto cc(\boldsymbol{R})$ possesses an affine selection iff for every $x, y \in I$, $t \in [0,1]$ the following condition holds*

$$F(tx + (1-t)y) \cap [tF(x) + (1-t)F(y)] \neq \emptyset. \tag{2}$$

Before we start the proof let us observe that (2) is the weakest condition guaranteeing the existence of an affine selection for the s.v. function F.

Proof of Theorem 1. If there exists an affine selection $f : I \mapsto \boldsymbol{R}$ of the s.v. function F then the condition (2) is obvious.

Let us assume that the condition (2) holds for every $x, y \in I$, $t \in [0,1]$. Let

$$f(x) := \inf F(x), \quad g(x) := \sup F(x), \quad x \in I.$$

Then

$$F(x) = [f(x), g(x)], \quad x \in I. \tag{3}$$

We will show that $f, g : I \mapsto \boldsymbol{R}$ fulfil the condition (1) for any fixed $x, y \in I$, $t \in [0,1]$.

Let $z \in F(tx + (1-t)y) \cap [tF(x) + (1-t)F(y)]$. There exist $z_1 \in F(x)$, $z_2 \in F(y)$ such that

$$z = tz_1 + (1-t)z_2.$$

Using definitions of f and g we get

$$f(tx + (1-t)y) \leq \quad z = tz_1 + (1-t)z_2 \quad \leq tg(x) + (1-t)g(y)$$

and

$$g(tx + (1-t)y) \geq \quad z = tz_1 + (1-t)z_2 \quad \geq tf(x) + (1-t)f(y).$$

Then there exists an affine function $h : I \mapsto \boldsymbol{R}$ such that

$$f(x) \leq h(x) \leq g(x), \quad x \in I \tag{4}$$

(cf. [3, Theorem 1]). Conditions (3), (4) imply that $h(x) \in F(x)$, $x \in I$, which completes the proof. $\square$

Remark 1. It is well known that if a s.v. function $F : I \mapsto cc(\boldsymbol{R})$ is convex (or concave) then F has an affine selection. Applying the above theorem we get the new proof of this fact.

Remark 2. The assumption of compactness of the sets $F(x)$, $x \in I$ in Theorem 1 is essential. Consider two s.v. functions $F : \boldsymbol{R} \mapsto n(\boldsymbol{R})$ and $G : (-1,1) \mapsto n(\boldsymbol{R})$ defined by formulas

$$F(x) \quad = \quad [x^2, +\infty), \quad x \in \boldsymbol{R}$$

and

$$G(x) \quad = \quad (x^2, 1), \quad x \in (-1,1).$$

It is easy to see that F and G are convex, but they do not have any affine selection.

Remark 3. It is known (cf. [1, Remark 1], [3, Remark 2]) that a s.v. function $F : D \mapsto cc(\boldsymbol{R})$, where $D \subset \boldsymbol{R}^2$ is a convex set, need not possess any affine selection although F fulfils (2). We can also find the example of the s.v. function $F : I \mapsto cc(\boldsymbol{R}^2)$ which fulfils (2) and does not have any affine selection. So, Theorem 1 can not be generalized both for s.v. functions defined on the convex subset of the plane and for s.v. functions $F : I \mapsto cc(\boldsymbol{R}^2)$. Below we present an example which is due to E. Sadowska from Bielsko–Biala.

Let $I = [0,4]$ (only in this remark) and $F : I \mapsto cc(\boldsymbol{R}^2)$ be defined as follows

$$\begin{array}{rcl} F(0) & = & [-4,4] \times \{1\}; \\ F(1) & = & \{-1\} \times [-4,4]; \\ F(2) & = & [-4,4] \times \{-1\}; \\ F(3) & = & \{1\} \times [-4,4]; \\ F(4) & = & \{(x,x) : x \in [-4,4]\}; \\ F(x) & = & [-4,4] \times [-4,4] \quad \text{for all } x \in I \setminus \{0,1,2,3,4\}. \end{array}$$

One can prove that F fulfils (2). We will show that F does not possess any affine selection. An easy calculus shows that the straight line

$$l : \begin{cases} x = 1 - \xi, \\ y = -1 - \xi, \qquad \xi \in \boldsymbol{R}, \\ z = \xi, \end{cases}$$

is the only line which intersects four segments $F(0), F(1), F(2)$ and $F(3)$. But l does not intersect the segment $F(4)$. So the s.v. function F has not any affine selection.

It is well known that every continuous function $f : I \mapsto I$ has a fixed point (if I is the compact interval). On the second hand, every affine function $f : I \mapsto I$ is continuous. So, as a consequence of Theorem 1 we obtain the following

Corollary 1. *If the interval I is compact then every s.v. function $F : I \mapsto cc(I)$ fulfiling (2) has a fixed point (i.e. there exists a point $x \in I$ such that $x \in F(x)$).*

Now we shall prove the main theorem of this paper. We present two proofs. One of them is an application of K. Nikodem's results (cf. [1]) and it requires the Axiom of Choice. The second one is direct and inductive but it works for finite–dimensional spaces $\boldsymbol{R}^n$.

Theorem 2. *Let Y be a locally convex topological vector space. Every convex s.v. function $F : I \mapsto c(Y)$ possesses an affine selection $f : I \mapsto Y$.*

Proof. Let $F : I \mapsto c(Y)$ be a convex s.v. function. Then, in particular, F is a midconvex s.v. function (it fulfils the condition of convexity with

$t = 1/2$), and so F possesses a Jensen selection $f : I \mapsto Y$ (i.e. $f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2}$, $x, y \in I$) (cf. [1, Lemma 2]).

Since F is continuous on int$I$ as a convex s.v. function defined on a subset of $\boldsymbol{R}$ (cf. [2, Theorem 3.7]), also f is continuous (cf. [2, Theorem 4.3] with $K = \{0\}$, $G = F$ and $F = f$). Thus f as a continuous Jensen function is an affine function, which completes the proof. □

In the above proof we have used K. Nikodem's results which require the Lemma of Kuratowski–Zorn and one of versions of the Theorem of Hahn–Banach. Below we give an inductive proof in the case $Y = \boldsymbol{R}^n$, $n \in \boldsymbol{N}$.

Second proof (for $Y = \boldsymbol{R}^n$). Before we start an induction on n let us notice that if F is convex then the values of F are convex sets.

STEP 1. If $n = 1$ then our theorem follows directly from Remark 1.

STEP 2. Assume that every convex s.v. function $G : I \mapsto c(\boldsymbol{R}^n)$ has an affine selection $g : I \mapsto \boldsymbol{R}^n$.

STEP 3. Let $F : I \mapsto c(\boldsymbol{R}^{n+1})$ be a convex s.v. function. For any $x \in I$ we put

$$G(x) := \left\{ y \in \boldsymbol{R}^n : \exists_{z \in \boldsymbol{R}} (y, z) \in F(x) \right\}.$$

It is easy to verify that $G(x)$ is a compact and non–empty subset of $\boldsymbol{R}^n$ (i.e. $G : I \mapsto c(\boldsymbol{R}^n)$). We will check that G is a convex s.v. function. Fix any $x_1, x_2 \in I$, $t \in [0, 1]$. Let

$$y \in tG(x_1) + (1 - t)G(x_2).$$

There exist $y_i \in G(x_i)$, $i = 1, 2$ such that

$$y = ty_1 + (1 - t)y_2.$$

So there exist $z_i \in \boldsymbol{R}$, $i = 1, 2$ such that

$$(y_i, z_i) \in F(x_i),\ i = 1, 2.$$

Let $z = tz_1 + (1 - t)z_2$. Since F is convex we get

$$(y, z) = t(y_1, z_1) + (1 - t)(y_2, z_2) \in tF(x_1) + (1 - t)F(x_2) \subset$$

$$\subset F(tx_1 + (1 - t)x_2).$$

So we obtain

$$y \in G(tx_1 + (1 - t)x_2).$$

Let $g : I \mapsto \boldsymbol{R}^n$ be an affine selection of G. Let us define

$$H(x) := \{z \in \boldsymbol{R} : (g(x), z) \in F(x)\},\ \ x \in I.$$

$H(x)$ is a compact and convex subset of $\boldsymbol{R}$. From the fact that $g(x) \in G(x)$ we get that $H(x) \neq \emptyset$, $x \in I$. So $H : I \mapsto cc(\boldsymbol{R})$. It is not difficult to calculate that H is a convex s.v. function.

Let $h : I \mapsto \boldsymbol{R}$ be an affine selection of H. By putting

$$f(x) := (g(x), h(x)), \quad x \in I$$

we obtain an affine selection of the s.v. function F which completes the proof. □

Remark 4. The above proof was obtained as a result of investigations if s.v. functions $F : I \mapsto cc(\boldsymbol{R}^n)$ fulfiling the condition (2) possess an affine selection. First the author hoped that the method of decreasing of the dimension will give an effect in this case. But the example in Remark 3 gave the negative answer of the mentioned problem. However in the case of convex s.v. functions this direct method gives the simple way of constructing the affine selections.

Remark 5. It is known that convex s.v. functions $F : D \mapsto cc(\boldsymbol{R})$, where D is a convex subset of $\boldsymbol{R}^n$, $n \geq 2$, need not possess any affine selection (cf. [1, Remark 1]). However, if D is a convex cone with base in a real linear space, then every convex s.v. function defined on D with compact values in a real locally convex space has an affine selection. It was obtained recently by A. Smajdor and W. Smajdor [4].

Remark 6. It is known that the convex s.v. function $F : I \mapsto c(Y)$, where Y is a topological vector space are continuous. An application of our results gives another proof of this fact in the case when $Y = \boldsymbol{R}^n$. Using Theorem 2 (for $Y = \boldsymbol{R}^n$) we obtain the existence of an affine selection f of F. Since f is affine and $f : I \mapsto \boldsymbol{R}^n$, it must be continuous. Then F has a continuous selection, so F must be continuous on I (cf. [2, Theorem 3.3]).

References

[1] Nikodem, K., *A characterization of midconvex set–valued functions*, Acta Univ. Carolin.–Math. Phys. 30 (1989), 125–129.

[2] Nikodem, K., *K–convex and K–concave set–valued functions*, Zeszyty Naukowe Politechniki Lódzkiej No 559, Lódź (1989).

[3] Nikodem, K. and Wąsowicz, Sz., *A sandwich theorem and Hyers–Ulam stability of affine functions*, Aequationes Math. 49 (1995), 160–164 .

[4] Smajdor, A. and Smajdor, W., *Affine selections of convex set–valued functions* (to appear).

[5] *Report of Meeting (The 31st International Symposium on Functional Equations, Debrecen, Hungary)*, Aequationes Math. 47 (1994), 263–327.

Department of Mathematics
Technical University of Łódź
Branch in Bielsko–Biala
Willowa 2
43-309 Bielsko–Biala, Poland
e-mail: szw@merc.pb.bielsko.pl