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DOMAINS OF ATTRACTION WITH INNER NORMING ON STURM–LIOUVILLE HYPERGROUPS

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Abstract. In this article we study the convergence of convolution powers of normalized measures $(\theta_{c_n}\nu)^{*n}$ on a Sturm–Liouville hypergroup $(\mathbb{R}_+,*)$. It is shown that this sequence converges for a suitable choice of the normalizing constants $c_n>0$ if and only if the usual regular variation conditions of the tail of ν are valid. The possible limit distributions are described in terms of their Fourier transform; they form a two dimensional family of probability measures on $\mathbb{R}_+$.

1. Introduction and main results. Let $(X_n : n \geq 1)$ be a sequence of independent identically distributed symmetrical real random variables. Then it is well known that for the random walk $S_n := \sum_{j=1}^{n} X_j$ we have $c_nS_n = \sum_{j=1}^{n} c_nX_j \longrightarrow \mu$ in distribution for some $\mu \neq \varepsilon_0$ and suitable norming constants $c_n > 0$ if and only if the tail of the distribution of X_1 satisfies a regular variation condition (see [3], VIII.8 Theorem 1). A similar result $c_nS_n \xrightarrow{D} \mu$ for random walks $(S_n : n \geq 1)$ on Urbanik's generalized convolutions on $\mathbb{R}_+$ has been proved in [8] and [1]. For arbitrary hypergroups

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on $\mathbb{R}_+$ with polynomial growth the possible limit laws and the corresponding domains of attraction have been characterized in [6].

For the Bessel–Kingman hypergroups [5] (which are generalized convolutions in the sense of [8]) the dilatations $\theta_c : x \mapsto c \cdot x$ (for $c > 0$) are endomorphisms of the convolution and so the randomized sums $\Lambda \sum_{j=1}^n c_n X_j$ (see [10], Definition 6.4) have the same distribution as $c_n S_n$ and therefore converge to the same limit distribution. However the Bessel–Kingman hypergroups are the only cases where all θ_c are endomorphisms; in all other cases we have to expect different limit laws in general. This phenomenon is already known in the case of a finite second moment (compare [7] and [11]) where the randomized sums converge to a Gaussian distribution on the hypergroup which is different from the limit law of $c_n S_n$, a Rayleigh distribution.

In this article we will study the convergence in distribution of the internally normed sums, i.e. we will explore under which conditions on the common law ν of the X_j the distribution $\theta_{c_n}(\nu)^{*n}$ of the randomized sum $\Lambda \sum_{j=1}^n c_n X_j$ converges.

Definition 1.1. *Let $*$ be a convolution on $\mathbb{R}_+$ and $\mu \neq \varepsilon_0$ be a probability measure. Then $\nu \in \mathcal{M}^1(\mathbb{R}_+)$ is in the domain of attraction of μ with respect to inner norming* $\mathrm{DOA}^*(\mu)$ *if there exists a sequence of numbers $c_n > 0$ such that $\left(\theta_{c_n}(\nu)\right)^{*n}$ converges weakly to μ. If the c_n can be chosen in the form $c_n = c \cdot n^{-\kappa}$ for some $\kappa > 0$ then ν belongs to the domain of normal attraction* $\mathrm{DONA}^*(\mu)$.

The possible limit distributions μ can be described in terms of their Fourier transform. In this article we will consider Sturm–Liouville hypergroups on $\mathbb{R}_+$ in the sense of [2] and [10]; here the convolution $*$ is defined by the property that $\int \varphi_\lambda \, d\varepsilon_x * \varepsilon_y = \varphi_\lambda(x) \cdot \varphi_\lambda(y)$ for all $\lambda \in \mathbb{C}$ where $\varphi_\lambda : \mathbb{R}_+ \to \mathbb{C}$ is the solution of

$$L_A \varphi_\lambda = (\lambda^2 + \rho^2)\varphi_\lambda, \qquad \varphi_\lambda(0) = 1, \quad \varphi_\lambda'(0) = 0$$

with a given function $A : \mathbb{R}_+ \to \mathbb{R}_+$, the differential operator $L_A f := -f'' - \frac{A'}{A} f'$ and $\rho := \frac{1}{2} \lim_{x \to \infty} \frac{A'(x)}{A(x)} \geq 0$; for the precise assumptions on A see [2], (3.5.1) or [10], (2.1). Then the *Fourier transform* of a probability measure μ on $\mathbb{R}_+$ is the function $\mathcal{F}\mu : \hat{K} := i[0, \rho] \cup \mathbb{R}_+ \to \mathbb{R}$ given by $\mathcal{F}\mu(\lambda) := \int \varphi_\lambda(x) \, d\mu(x)$. The Haar measure ω_K of this hypergroup $(\mathbb{R}_+, *)$ has the Lebesgue density A and so the Fourier transform of a function $f \in \mathcal{L}^1(\omega_K)$ is defined as $\mathcal{F}f(\lambda) := \int \varphi_\lambda(x) f(x) A(x) \, dx$.

Definition 1.2. *Let $\kappa \in]0,2[$ and $t > 0$. The measures $\mu_{\kappa,t}$ with Fourier transform*

$$\mathcal{F}\mu_{\kappa,t}(\lambda) \;=\; \exp\Bigl(t \cdot \int_0^\infty \varphi'_\lambda(x) \cdot x^{-\kappa}\, dx\Bigr) \qquad \lambda \in \hat{K}$$

are called the stable measures of index κ of the hypergroup $(\mathbb{R}_+, *)$. The stable measures of index $\kappa = 2$ are the Gauß measures $\mu_{2,t}$ with Fourier transform $\mathcal{F}\mu_{2,t}(\lambda) = \exp(-\frac{t}{2}(\lambda^2 + \rho^2))$.

The integral on the right hand side of the definition has singularities at 0 and ∞. The singularity at 0 does not pose a problem since $|\varphi'_\lambda(x)| \le Kx$ for some $K > 0$ and since $\kappa < 2$. At ∞ we observe $\lim_{y\to\infty} \int_1^y \varphi'_\lambda(x) \cdot x^{-\kappa}\, dx = \lim_{y\to\infty} [\varphi_\lambda(x) x^{-\kappa}]_1^y + \kappa \int_1^y \varphi_\lambda(x) \cdot x^{-\kappa-1}\, dx = -\varphi_\lambda(1) + \kappa \int_1^\infty \varphi_\lambda(x) \cdot x^{-\kappa-1}\, dx$ which exists because of $|\varphi_\lambda(x)| \le 1$ and $-\kappa - 1 < -1$.

We will later see in Theorem 2.1 and Remark 2.2 that such a probability measure $\mu_{\kappa,t}$ exists for every $\kappa \in]0,2]$ and $t \ge 0$. Since the characters of the hypergroup $(\mathbb{R}_+, *)$ appear in the definition of $\mu_{\kappa,t}$, it is clear that the stable measures of different hypergroups are different in general.

Remark 1.3. It follows from the shape of the Fourier transform that we have $\mu_{\kappa,s} * \mu_{\kappa,t} = \mu_{\kappa,s+t}$ for all $s, t \ge 0$ and therefore for each $\kappa \in]0,2]$ the family of probability measures $(\mu_{\kappa,t} : t \ge 0)$ forms a continuous convolution semigroup. Also for every ν with $(\theta_{c_n}\nu)^{*n} \to \mu_{\kappa,t}$ it follows that $(\theta_{c_n}\nu)^{*\lfloor sn \rfloor}$ converges to $\mu_{\kappa,st}$ for every $s > 0$ and hence $\mathrm{DOA}^*(\mu_{\kappa,t})$ is the same for all $t > 0$. We will therefore consider $\mathrm{DOA}^*(\mu_{\kappa,1})$ only.

2. Sufficient conditions.

Theorem 2.1. *Let $\kappa \in]0,2[$ and $\nu \in \mathcal{M}^1(\mathbb{R}_+)$ be such that $t \mapsto \nu[t,\infty[$ is varying regularly with index $-\kappa$. Then $\nu \in \mathrm{DOA}^*(\mu_{\kappa,1})$.*

Proof. We define $H(x) := \nu[x,\infty[$, choose positive numbers c_n such that $\lim_{n\to\infty} n \cdot H(1/c_n) = 1$ and put $H_n(x) := H(x/c_n)$. In the following we will show that

$$\lim_{n\to\infty} \int_0^\infty \varphi'_\lambda(x) \frac{H_n(x)}{H_n(1)}\, dx = \int_0^\infty \varphi'_\lambda(x) \cdot x^{-\kappa}\, dx \tag{1}$$

Let $\varepsilon > 0$ be arbitrary. It follows from [3], Theorem VIII.9.1 b) that

$$\begin{aligned} \Bigl|\int_0^\delta \varphi'_\lambda(x) \frac{H_n(x)}{H_n(1)}\, dx\Bigr| &\le \frac{Kc_n^2}{H_n(1)} \int_0^{\delta/c_n} xH(x)\, dx \\ &\to \lim_{n\to\infty} \frac{Kc_n^2 (\delta/c_n)^2 H(\delta/c_n)}{(2-\kappa) H(1/c_n)} = \frac{K\delta^{2-\kappa}}{2-\kappa}. \end{aligned}$$

We can choose $\delta > 0$ in such a way that $\frac{K\delta^{2-\kappa}}{2-\kappa} < \varepsilon$ and therefore $|\int_0^\delta \varphi'_\lambda(x) \cdot \frac{H_n(x)}{H_n(1)}\, dx| < \varepsilon$ for $n \geq n_0$. Furthermore we also have $|\int_0^\delta \varphi'_\lambda(x) \cdot x^{-\kappa}\, dx| < \varepsilon$.

Now let $\gamma > 0$ be chosen such that $2\gamma^{-\kappa} < \varepsilon$. Then as in the proof of Theorem 5.2 in [6] we see by the second mean value theorem that $|\int_\gamma^\infty \varphi'_\lambda(x) \cdot \frac{H_n(x)}{H_n(1)}\, dx| = |\varphi_\lambda(\gamma') - \varphi_\lambda(\gamma)| \frac{H_n(\gamma)}{H_n(1)}$ is $< \varepsilon$ for all $n \geq n_1 \geq n_0$; furthermore $|\int_\gamma^\infty \varphi'_\lambda(x) x^{-\kappa}\, dx| < \varepsilon$.

Finally, the integrand in $\int_\delta^\gamma \varphi'_\lambda(x) \frac{H_n(x)}{H_n(1)}\, dx$ is bounded by $\|\varphi'_\lambda\|_{[\delta,\gamma]} \cdot \frac{H_n(\delta)}{H_n(1)}$ and by bounded convergence the integral differs from $\int_\delta^\gamma \varphi'_\lambda(x) x^{-\kappa}\, dx$ by less than ε if $n \geq n_2 \geq n_1$ and hence $|\int_0^\infty \varphi'_\lambda(x) \frac{H_n(x)}{H_n(1)}\, dx - \int_0^\infty \varphi'_\lambda(x) x^{-\kappa}\, dx| < 5\varepsilon$. Thus (1) is proved.

From (1) we conclude that

$$
\begin{aligned}
n\Big[1 - \mathcal{F}\theta_{c_n}\nu(\lambda)\Big] &= n\Big[1 + \int_0^\infty \varphi_\lambda c_n x\, dH(x)\Big] \\
&= -nc_n \cdot \int_0^\infty \varphi'_\lambda(c_n x) \cdot H(x)\, dx \\
&= -nH(1/c_n) \cdot \int_0^\infty \varphi'_\lambda(x) \frac{H_n(x)}{H_n(1)}\, dx \\
&\to -\int_0^\infty \varphi'_\lambda(x) \cdot x^{-\kappa}\, dx
\end{aligned}
$$

and therefore we obtain that $\lim_{n\to\infty} \mathcal{F}\big(\theta_{c_n}\nu\big)^{*n}(\lambda) = \lim_{n\to\infty}[\mathcal{F}\theta_{c_n}\nu(\lambda)]^n = \exp(\int_0^\infty \varphi'_\lambda(x) \cdot x^{-\kappa}\, dx) = \mathcal{F}\mu_{\kappa,1}(\lambda)$ for all $\lambda \in \hat{K}$. The assertion of the theorem now follows from the continuity theorem. In particular a probability measure $\mu_{\kappa,t}$ with Fourier transform given in Definition 1.1 exists. □

Remark 2.2. If $\nu \in \mathcal{M}^1(\mathbb{R}_+)$ has a finite second moment, then ν belongs to the domain of normal attraction DONA$^*(\mu_{2,1})$ of the Gaussian measure.

Proof. This follows from [7], Théorème 2 or [13], Lemma 3.2. □

3. Necessary conditions. We will now show that the condition given in Theorem 2.1 (resp. a similar condition in the Gaussian case) is also necessary for ν being in the domain of attraction of some probability μ.

Lemma 3.1. *We have*

$$
1 - \varphi_\lambda(x) \;\leq\; \frac{\lambda^2 + \rho^2}{2} \cdot x^2 \qquad \textit{for all } x \geq 0.
$$

Proof. It follows from the differential equation for φ_λ and $A' \geq 0$ that $\varphi'_\lambda(x) \geq 0$ or $\varphi''_\lambda(x) \geq -(\lambda^2 + \rho^2)\varphi_\lambda \geq -(\lambda^2 + \rho^2)$ and hence $\varphi'_\lambda(x) \geq -(\lambda^2 + \rho^2) \cdot x$. □

In the next lemma we check the equality of two Lévy measures by using test functions connected to the *inverse Fourier transform* of functions ϕ on the dual $\hat{K}$: $\bar{\mathcal{F}}\phi(x) := \int \varphi_\lambda(x) f(\lambda)\, d\pi_K(\lambda)$ where $\pi_K \in \mathcal{M}_+(\hat{K})$ is the Plancherel measure (see [4]). As a *Hunt function* we use $h(x) := \frac{x^2}{x^2+1}$.

Lemma 3.2. *Let $\tilde{\eta}, \hat{\eta} \in \mathcal{M}^b(K)$ satisfy $\int g\, d\tilde{\eta} = \int g\, d\hat{\eta}$ for all functions $g := \bar{\mathcal{F}}\phi/h$ with $\phi \in \mathcal{K}(\hat{K})$, $\int \phi\, d\pi_K = 0$. Then $\tilde{\eta} = \hat{\eta}$.*

Proof. The assumption is valid for *all* $\phi \in \mathcal{C}(\hat{K}) \cap \mathcal{L}^1(\pi_K)$ with $\int \phi\, d\pi_K = 0$ and $\int (\lambda^2 + \rho^2)|\phi(\lambda)|\, d\pi_K(\lambda) < \infty$ by approximation with $\psi_n \in \mathcal{K}(K)$ dominated in modulus by $|\phi|$. Here we use that $L_A f(0) = -(\alpha_0 + 1)f''(0)$ if $f'(0) = 0$ and $\alpha_0 := \lim_{x\to 0} x\frac{A'(x)}{A(x)}$ and therefore $\frac{\bar{\mathcal{F}}\phi}{h}(0) = \frac{(\bar{\mathcal{F}}\phi)''(0)}{h''(0)} = \frac{-L_A\bar{\mathcal{F}}\phi(0)}{(\alpha_0+1)h''(0)} = \frac{-1}{(\alpha_0+1)h''(0)} \int (\lambda^2 + \rho^2)\phi(\lambda)\, d\pi_K(\lambda)$ to establish pointwise convergence of $\bar{\mathcal{F}}\psi_n$ in 0 and Lemma 3.1 for a bounded convergence argument.

Now let $f \in \mathcal{C}^2(K) \cap \mathcal{K}(K)$ with $f'(0) = 0$ and $\tilde{f} \in \mathcal{L}^1(\omega_K) \cap \mathcal{L}^2(\omega_K)$ with $f * \tilde{f}(0) = 0$. Then $\phi := \mathcal{F}(f * \tilde{f})$ satisfies the above conditions $\int \phi\, d\pi_K = f * \tilde{f}(0) = 0$ and $\int (\lambda^2 + \rho^2)|\phi(\lambda)|\, d\pi_K(\lambda) = \int |\mathcal{F}(L_A f) \cdot \mathcal{F}\tilde{f}|\, d\pi_K(\lambda) < \infty$ (see [4], 12.1A,B). Because of $\bar{\mathcal{F}}\phi = f * \tilde{f}$ ([4], 12.2C) we obtain $\int \frac{f*\tilde{f}}{h}\, d\tilde{\eta} = \int \frac{f*\tilde{f}}{h}\, d\hat{\eta}$.

Now let $f \in \mathcal{C}^2(K) \cap \mathcal{K}(K)$ with $f(0) = f'(0) = 0$. Then we can choose functions $\tilde{f}_n \in \mathcal{L}^1(\omega_K) \cap \mathcal{L}^2(\omega_K)$ with $\tilde{f}_n = 0$ on $[\frac{1}{n}, \infty[$, $\int \tilde{f}_n\, d\omega_K = 1$, $\|\tilde{f}_n\|_{\mathcal{L}^1} \le 2$ such that $\int f \tilde{f}_n\, d\omega_K = f * \tilde{f}_n(0) = 0$. It is clear that $f * \tilde{f}_n \to f$ uniformly as $n \to \infty$ and hence $\frac{f*\tilde{f}_n}{h} \to \frac{f}{h}$ uniformly on $[\varepsilon, \infty[$ for every $\varepsilon > 0$. But since $L_A f \in \mathcal{K}(K)$ we also have $L_A(f * \tilde{f}_n) = (L_A f) * \tilde{f}_n \to L_A f$ uniformly. It follows from

$$\begin{aligned}
|f(x) - f * \tilde{f}_n(x)| &= \Big| \int_0^x \frac{1}{A(y)} \int_0^y A(z) \cdot L_A(f - f * \tilde{f}_n)(z)\, dz\, dy \Big| \\
&\le \|L_A(f - f * \tilde{f}_n)\|_\infty \int_0^x \frac{1}{A(y)} \int_0^y A(z)\, dz\, dy \\
&\le \|L_A(f - f * \tilde{f}_n)\|_\infty \cdot Cx^2
\end{aligned}$$

(the last inequality for a suitable $C > 0$ is a consequence of the assumption on A that $\alpha_0 := \lim_{x\to\infty} x\frac{A'(x)}{A(x)}$ exists) that $\frac{f*\tilde{f}_n}{h} \to \frac{f}{h}$ uniformly on $[0, \varepsilon]$. Hence $\int \frac{f}{h}\, d\tilde{\eta} = \int \frac{f}{h}\, d\hat{\eta}$ for all $f \in \mathcal{C}^2(K) \cap \mathcal{K}(K)$ with $f(0) = f'(0) = 0$, and this implies $\tilde{\eta} = \hat{\eta}$. ∎

Before proceeding we note a simple property of the characters in the case of exponential growth:

Lemma 3.3. *Let $\rho > 0$. Then $\varphi_\lambda \in \mathcal{C}^0(\mathbb{R}_+)$ for all $\lambda \in \hat{K} \setminus \{i\rho\}$.*

Proof. Since all characters in the support of the Plancherel measure are bounded by ϕ_0 it is sufficient to consider $\lambda \in i[0,\rho[$ which implies $\varphi_\lambda > 0$ and $\varphi'_\lambda < 0$. The assumption $\varepsilon := \lim_{x\to\infty} \varphi_\lambda(x) > 0$ now leads to a contradiction: from the differential equation for φ_λ we conclude $(A\varphi'_\lambda)' \leq -\delta \cdot A$ with $\delta := \varepsilon(\rho^2 + \lambda^2) > 0$ and with $\mathcal{A}(x) := \int_0^x A(t)\,dt$ we obtain $\varphi'_\lambda(x) \leq -\delta \frac{\mathcal{A}(x)}{A(x)}$ which converges to $\frac{-\delta}{2\rho} < 0$ as $x \to \infty$ in contradiction to $\varphi_\lambda > 0$. □

Corollary 3.4. *Let $(\tilde{\eta}_n : n \in \mathbb{N})$ be a sequence in $\mathcal{M}^b_+(K)$ such that*

$$\int \frac{1-\varphi_\lambda}{h}\,d\tilde{\eta}_n \;\to\; \psi(\lambda) \qquad \textit{for all } \lambda \in \hat{K}$$

where ψ is continuous in $i\rho$ and $\psi(i\rho) = 0$. Then $\tilde{\eta} := \lim_{n\to\infty} \tilde{\eta}_n \in \mathcal{M}^b_+(K)$ exists weakly.

Proof. Since in the case $A(x) = 1$ of the ordinary convolution of symmetrical measures on $\mathbb{R}$ nothing is to prove, we assume that A is not a constant function. Let $\lambda > 0$. Then $\frac{1-\varphi_\lambda}{h}$ is bounded away from 0 in a neighborhood of 0 (by continuity and $\varphi''_\lambda(0) < 0$) and outside this neighborhood because of [12], (2.2). Therefore there exists $C > 0$ with $\|\tilde{\eta}_n\| \leq C$ for all n. In order to show that $\tilde{\eta}_n$ converges *vaguely*, by the relative compactness of this sequence it is sufficient to show that any two accumulation points $\tilde{\eta}$ and $\hat{\eta}$ are equal.

Let $\phi \in \mathcal{K}(\hat{K})$ be such that $\int \phi\,d\pi_K = 0$. Then by the assumption, Fubini and dominated convergence (using Lemma 3.1) we have $\lim_{n\to\infty} \int \frac{\bar{\mathcal{F}}\phi}{h}\,d\tilde{\eta}_n = -\lim_{n\to\infty} \int \phi(\lambda) \int \frac{1-\varphi_\lambda}{h}\,d\tilde{\eta}_n\,d\pi_K(\lambda) = -\int \phi(\lambda)\cdot\psi(\lambda)\,d\pi_K(\lambda)$. Now $\bar{\mathcal{F}}\phi \in \mathcal{C}^0(\mathbb{R}_+)$ and the same is true for $\bar{\mathcal{F}}\phi/h$ (the continuity at 0 follows again by dominated convergence from $\phi \in \mathcal{K}(\hat{K})$). Since $\tilde{\eta}$ and $\hat{\eta}$ are vague accumulation points of $(\tilde{\eta}_n)_n$ we obtain

$$\int \frac{\bar{\mathcal{F}}\phi}{h}\,d\tilde{\eta} \;=\; -\int \phi\cdot\psi\,d\pi_K \;=\; \int \frac{\bar{\mathcal{F}}\phi}{h}\,d\hat{\eta}$$

for all $\phi \in \mathcal{K}(\hat{K})$ with $\int \phi\,d\pi_K = 0$ and therefore $\tilde{\eta} = \hat{\eta}$ by Lemma 3.2. Hence we have proved that $\tilde{\eta} := \lim_{n\to\infty} \tilde{\eta}_n$ exists *vaguely*.

We conclude the proof by showing that $\{\tilde{\eta}_n : n \in \mathbb{N}\}$ is uniformly tight. In order to do this we have to treat the cases $\rho = 0$ and $\rho > 0$ separately; we consider $\rho = 0$ first: let $\varepsilon > 0$ be given. We choose a neighborhood of 0 where $|\psi| < \varepsilon$ and a non negative function $\phi \in \mathcal{K}(\hat{K})$ with support in that neighborhood such that $\int \phi \, d\pi_K = 1$. Then

$$\int \frac{1 - \bar{\mathcal{F}}\phi}{h} \, d\tilde{\eta}_n \;\to\; \int \psi\phi \, d\pi_K$$

which is of modulus $< \varepsilon$ by construction, and hence $\int \frac{1-\bar{\mathcal{F}}\phi}{h} \, d\tilde{\eta}_n < \varepsilon$ for all $n \geq N_\varepsilon$. Since $\bar{\mathcal{F}}\phi \in \mathcal{C}^0(\mathbb{R}_+)$ there exists $c_\varepsilon > 0$ with $\frac{1-\bar{\mathcal{F}}\phi}{h} > \frac{1}{2}$ on $[c_\varepsilon, \infty[$. This implies $\tilde{\eta}_n([c_\varepsilon, \infty[) < 2\varepsilon$ for all $n \geq N_\varepsilon$ and so the uniform tightness is proved in this case. In the case of exponential growth we use the fact that $\varphi_\lambda \in \mathcal{C}^0(\mathbb{R}_+)$ whenever $\lambda \in i[0, \rho[$ (see Lemma 3.3) for a similar but simpler argument. Hence in both cases $\{\tilde{\eta}_n : n \in \mathbb{N}\}$ is uniformly tight and $\tilde{\eta}_n \to \tilde{\eta}$ *weakly* follows. □

Proposition 3.5. *If* $\nu \in \mathrm{DOA}^*(\mu)$ *with* $\mu \neq \varepsilon_0$*, then either* $t \mapsto \nu[t, \infty[$ *is regularly varying at* ∞ *with index* $-\kappa$ *and* $\kappa \in]0, 2[$*, or* $t \mapsto \int_0^t x^2 \, d\nu(x)$ *is slowly varying.*

Proof. We have $\left(\mathcal{F}\theta_{c_n}\nu(\lambda)\right)^n \to \mathcal{F}\mu(\lambda)$ for all $\lambda \in \hat{K}$ and since $\mathbb{R} \ni \mathcal{F}(\theta_{c_n}\nu) \to 1$ as $n \to \infty$ we obtain $\mathcal{F}\mu(\lambda) \geq 0$ with strict inequality on $[0, \lambda_0[$ where $\lambda_0 := \inf\{\lambda > 0 : \mathcal{F}\mu(\lambda) = 0\} \in]0, \infty]$ (we have $\lambda_0 > 0$ because of $\phi_0 > 0$). For all $\lambda \in [0, \lambda_0[$ we therefore have

$$\int \frac{1 - \varphi_\lambda}{h} \, d\tilde{\eta}_n \;=\; n\big(1 - \mathcal{F}\theta_{c_n}\nu(\lambda)\big) \;\to\; -\ln \mathcal{F}\mu(\lambda) \tag{2}$$

where $\tilde{\eta}_n := nh \cdot (\theta_{c_n}\nu)$ and h is a Hunt function. By the same argument as in the proof of Corollary 3.4 we obtain $\|\tilde{\eta}_n\| \leq C$ for all $n \in \mathbb{N}$.

We now show that the assumption $\lambda_0 < \infty$ leads to a contradiction. In this case there is also a constant $C' > 0$ with $\frac{1-\phi_{\lambda_0}}{h} \leq C'$ (Lemma 3.1) and this implies $-\ln \mathcal{F}\mu(\lambda_0) \leq CC'$ and $\mathcal{F}\mu(\lambda_0) \geq e^{-CC'} > 0$, a contradiction. Therefore the convergence in (2) holds for all $\lambda \in \hat{K}$ and it follows from Corollary 3.4 that $(\tilde{\eta}_n)$ has a weak limit $\tilde{\eta} \in \mathcal{M}_+^b(K)$.

Now the hypergroup structure plays no role any more and we are in the classical situation. We either have $\tilde{\eta} = c \cdot \varepsilon_0$ and therefore $t \mapsto \int_0^\infty x^2 \, d\nu(x)$ is slowly varying; or we have $\tilde{\eta}]0, \infty[> 0$ and hence $n\nu]c_n x, \infty[\to \eta]x, \infty[$ for all but countably many $x > 0$ where $\eta := \frac{1}{h} \cdot \tilde{\eta}$. Then by [3], VIII.8 Lemma 3 the function $t \mapsto \nu]t, \infty[$ is regularly $-\kappa$-varying for some $\kappa \in \mathbb{R}$ and $\eta]x, \infty[= cx^{-\kappa}$. Since η is a Lévy measure we must have $\kappa \in]0, 2[$. □

4. Conclusions.

We now summarize the necessary and sufficient conditions from Sections 3 and 2.

Theorem 4.1. *A measure* $\mu \neq \varepsilon_0$ *has a non empty domain of inner attraction* $\mathrm{DOA}^*(\mu)$ *if and only if* $\mu = \mu_{\kappa,c}$ *for some* $\kappa \in]0, 2]$ *and* $c > 0$.

Proof. This follows from Proposition 3.5, Theorem 2.1 and Remark 2.2. □

Theorem 4.2. *(a) For $\kappa \in \,]0,2[$, $c>0$ we have*

$$\mathrm{DOA}^*(\mu_{\kappa,c}) \;=\; \{\nu \in \mathcal{M}^1(K) : t \mapsto \nu[t,\infty[\text{ is regularly } -\kappa\text{-varying at } \infty\}.$$

(b) For $c>0$ we obtain

$$\mathrm{DOA}^*(\mu_{2,c}) \;=\; \{\nu \in \mathcal{M}^1(K) : t \mapsto \int_0^t x^2\, d\nu(x) \text{ is slowly varying at } \infty\}.$$

Proof. (a) It follows again from Theorem 2.1 and Proposition 3.5.

(b) The inclusion $\subseteq$ has been proved in Proposition 3.5.

If on the other side $t \mapsto \int_0^t x^2\, d\nu(x)$ is slowly varying we obtain by the classical result $\tilde{\eta}_n := nh\cdot(\theta_{c_n}\nu) \to 2c(\alpha_0+1)\varepsilon_0$ for a suitably chosen sequence of numbers $c_n > 0$. This implies

$$n\big(1-\mathcal{F}\theta_{c_n}\nu(\lambda)\big) \;=\; \int \frac{1-\varphi_\lambda}{h}\, d\tilde{\eta}_n \;\to\; \frac{-\varphi_\lambda''(0)}{h''(0)}\cdot 2c(\alpha_0+1) = c\,(\lambda^2+\rho^2)$$

and hence $\left(\theta_{c_n}\nu\right)^{*n} \to \mu_{2,c}$. □

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