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EPI/HYPO–CONVERGENCE: THE SLICE TOPOLOGY AND SADDLE POINTS APPROXIMATION

A. BAGH

Abstract. We show that the slice convergence of the convex parents of saddle functions implies the epi/hypo–convergence of these saddle functions and hence the convergence of their saddle points. We also obtain conditions for the slice convergence of sums of convex functions. We then apply these results to problems in convex programming, optimal control and Chebyshev approximations.

• **Introduction.** The general idea behind duality theory in optimization is to embed the optimization problem under consideration in a parametrized family of problems and to study the solutions of the family of problems as well as the solutions of their duals (see [9]). This approach allows us to gain more insight into the nature of the solution of our original problem, in particular about the stability of the solution with respect to certain perturbations. More specifically, we consider an abstract convex optimization problem (P) over an arbitrary Banach space X. We are interested in the minimum of $f_0(x)$ subject to the constraint $x \in C$, where f_0 is a convex

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real–valued function and C is a convex subset of X. This problem can be restated in the following form:

$$\inf_{x \in X} f(x),$$

where

$$f(x) = \begin{cases} f_0(x) & \text{if } x \in C; \\ +\infty & \text{otherwise.} \end{cases} \tag{P}$$

We introduce a perturbation space Y with a dual Y^*. We also introduce a convex perturbation function F defined on $X \times Y$ with values in the extended reals $\overline{I\!R}$ in such a way that

$$F(x, 0) = f(x).$$

We then form the convex–concave Lagrangian $K : X \times Y^* \to \overline{I\!R}$,

$$K(x, y^*) = \inf_{y \in Y} \{F(x, y) - < y, y^* >\},$$

and consider the saddle points of this Lagrangian, i.e. points $(\bar{x}, \bar{y}^*)$ in $X \times Y^*$ such that

$$K(\overline{x}, y^*) \leq K(\overline{x}, \overline{y}^*) \leq K(x, \overline{y}^*) \quad, \forall x \in X \ , \forall y^* \in Y^*.$$

These points play a very important role in determining necessary and sufficient optimality conditions. In many cases the Lagrangian K is quite complicated and we need to approximate it with a sequence K_n of Lagrangians of a simpler form in a way that guarantees the convergence of the saddle points of K_n to the saddle points of K. To that end, Attouch and Wets [2] introduced the concept of epi/hypo–convergence of Lagrangians which preserves the convergence of their saddle points. These authors, and later followed by Abdulfattah [1] and Soueycatt [15], studied the relationship between the Mosco convergence of the perturbation functions F_n and the epi/hypo–convergence of the Lagrangians induced by these functions. Their results, however, were valid only for reflexive spaces, and a substantial number of the applications in optimal control theory and convex optimization involve nonreflexive spaces. In this paper, we use the notion of slice convergence, introduced by Sonntag and Zalinescu [14] and later studied in great detail by Beer [6], as an alternative to Mosco convergence in nonreflexive spaces. We obtain results relating the slice convergence of perturbation functions to the epi/hypo–convergence of the corresponding Lagrangians for general Banach spaces X and Y. We also obtain results regarding the slice convergence of sums of convex functions that are different from the existing results due to Lahrache [10]. Finally, we apply our results to problems in convex programming and optimal control.

• **Preliminaries.** In this section we go through a brief introduction to the slice topology and the concept of epi/hypo–convergence. Let (X, τ) be a locally convex Hausdorff space and let (X^*, τ^*) be the topological dual of(X, τ). Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ be the set of extended reals. Let f be a function defined on X and taking values in $\overline{\mathbb{R}}$. We define the *epigraph*, the *strict epigraph*, and the *domain* of f as follows

$$\begin{aligned} \text{epi } f &= \{(x,\alpha) \in X \times \mathbb{R} \,|\, f(x) \le \alpha\}, \\ \text{epi}_s f &= \{(x,\alpha) \in X \times \mathbb{R} \,|\, f(x) < \alpha\}, \\ \text{dom } f &= \{x \in X \text{ such that } f(x) < +\infty\}. \end{aligned}$$

The *lower closure* of a function f is defined as the function $\text{cl}_\tau f$ that satisfies:

$$\text{epi cl}_\tau f = \text{cl}_\tau \text{epi } f,$$

where the right side of the above equality is the closure of the set epi f in the topology τ. We say f is *lower semicontinuous* (lsc) if its epigraph is closed in $X \times \mathbb{R}$ and we say f is proper if $f \not\equiv \infty$ and does not assume the value $-\infty$. We will use $\mathcal{E}(X)$ and $\mathcal{E}(X^*)$ to denote the spaces of convex proper lsc functions with values in $\overline{\mathbb{R}}$ that are defined on X and X^* respectively. We now define f^*, the conjugate of f, as

$$\forall x^* \in X^* \ , f^*(x^*) = \sup_{x \in X} \{\langle x, x^* \rangle - f(x)\}.$$

Clearly if $f \in \mathcal{E}(X)$, then f^* is a proper lsc convex function defined on X^*. For $f^* \in \mathcal{E}(X^*)$, and following a standard abuse of notataion, we write

$$f^{**}(x) = f(x) = \sup_{x^* \in X^*} \{\langle x^*, x \rangle - f^*(x^*)\}.$$

Therefore for functions in $\mathcal{E}(X^*)$, the conjugates will be understood to be defined on X rather than X^{**}. We also recall that for functions f and g in $\mathcal{E}(X)$, the epi–sum (inf–addition) $f \# g$ is define by the formula

$$f \# g = \inf_{z \in X} \{f(z) + g(x - z)\},$$

and we have (cf. [12])

$$(f + g)^* = \text{cl}_{\tau^*} (f^* \# g^*).$$

Furthermore, the strict epigraph of $f \# g$ is $\text{epi}_s f + \text{epi}_s g$ (the usual set addition of $\text{epi}_s f$ and $\text{epi}_s g$).

Now we follow Attouch and Wets [2] in defining the sequential epi/hypo–convergence of saddle functions. We consider a second pair of locally convex topological spaces (Y, σ) and its topological dual (Y^*, σ^*). A saddle function K defined on $X \times Y^*$ is a convex–concave bivariate function with values in $\overline{\mathbb{R}}$. In general, we would like K to be lower semicontinuous in x and upper semicontinuous in y^*. However, this is too much of a restriction since it prohibits the use of saddle functions that take both of the values ∞ and $-\infty$, which is the case when we have constraints on both x and y^*. Therefore, we define the *epi–closure*, $\mathrm{cl}_\tau K$, of K to be the lower semicontinuous regularization of K in x:

$$\mathrm{epi}\,(\mathrm{cl}_\tau K(\cdot, y^*)) = \mathrm{cl}_\tau(\mathrm{epi}\ K(\cdot, y^*)).$$

The *extended lower closure* is then defined as a function that for every y^* in Y^*, satisfies

$$\underline{\mathrm{cl}}_\tau K(\cdot, y^*) = \begin{cases} \mathrm{cl}_\tau K(\cdot, y^*) & \text{if } \mathrm{cl}_\tau K(\cdot, y^*) > -\infty; \\ -\infty & \text{otherwise.} \end{cases}$$

Similarly, we define the *hypo–closure*, $\mathrm{cl}_{\sigma^*} K$, of K by

$$\mathrm{hypo}\,(\mathrm{cl}_{\sigma^*} K(x, \cdot) = \mathrm{cl}_{\sigma^*}(\mathrm{hypo}\ K(x, \cdot)),$$

and the *extended upper closure* of K by

$$\overline{\mathrm{cl}}_{\sigma^*} K(x, \cdot) = \begin{cases} \mathrm{cl}_{\sigma^*} K(x, \cdot) & \text{if } \mathrm{cl}_{\sigma^*} K(x, \cdot) < +\infty; \\ +\infty & \text{otherwise.} \end{cases}$$

We say that two saddle functions K, L are *equivalent* if

$$\underline{\mathrm{cl}}_\tau K = \underline{\mathrm{cl}}_\tau L \text{ and } \overline{\mathrm{cl}}_{\sigma^*} K = \overline{\mathrm{cl}}_{\sigma^*} L$$

The saddle function K is called (τ, σ^*)–closed if it is equivalent to $\underline{\mathrm{cl}}_\tau K$ and $\overline{\mathrm{cl}}_{\sigma^*} K$. We also recall the definition of the domain of a saddle function $K : X \times Y^* \to \overline{I\!R}$:

$$\mathrm{dom}\ K = \{x | K(x, \cdot) < +\infty\} \times \{y^* | K(\cdot, y^*) > -\infty\}.$$

The saddle function K is proper when $\mathrm{dom}\ K \neq \emptyset$. We now define the *convex parent* F of K:

$$F(x, y) : X \times Y \to \overline{I\!R},$$

$$F(x, y) = \sup_{y^* \in Y^*} \{K(x, y^*) + \langle y, y^* \rangle\}.$$

We also define the concave parent G of K:

$$G(x^*, y^*) : X^* \times Y^* \to \overline{I\!R},$$

$$G(x^*, y^*) = \inf_{x \in X} \{K(x, y^*) - \langle x, x^* \rangle\}.$$

We note that two equivalent saddle functions will have the same parents (cf. [4]). Moreover, we have

$$\underline{\mathrm{cl}}_\tau K(x, y^*) = \sup_{x^* \in X^*} \{G(x^*, y^*) + \langle x, x^* \rangle\},$$

and

$$\overline{\mathrm{cl}}_{\sigma^*} K(x,y^*) = \inf_{y \in Y} \{F(x,y) - \langle y, y^* \rangle\}.$$

We also know that if K is (τ, σ^*)–closed, then $-G = F^*$ and $(-G)^* = F$, where

$$F^*(x^*, y^*) = \sup_{x \in X, y \in Y} \{\langle x, x^* \rangle + \langle y, y^* \rangle - F(x,y)\}.$$

We will use $\underline{K}(x,y^*)$ and $\overline{K}(x,y^*)$ to denote $\underline{\mathrm{cl}}_\tau K(x,y^*)$ and $\overline{\mathrm{cl}}_{\sigma^*} K(x,y^*)$ respectively.

Let $K_n(\cdot,\cdot)$ be a sequence of saddle functions defined on $X \times Y^*$. Let

$$h_{\sigma^*}/e_\tau\text{- li } \ K_n(x,y^*) = \inf_{x_n \xrightarrow{\tau} x} \ \sup_{y_n^* \xrightarrow{\sigma^*} y} \ \liminf_n K_n(x_n, y_n^*),$$

where the infimum and the supremum are taken over all sequences converging to x and y^* respectively. Let

$$e_\tau/h_{\sigma^*}\text{- ls } \ K_n(x,y^*) = \sup_{y_n^* \xrightarrow{\sigma^*} y^*} \ \inf_{x_n \xrightarrow{\tau} x} \ \limsup_n K_n(x_n, y_n^*),$$

where again the infimum and the supremum are taken over all sequences converging to x, y^* respectively. We say K_n converges to K in the extended epi/hypo–sense if

$$\underline{\mathrm{cl}}_\tau(e_\tau/h_{\sigma^*}\text{- ls } K_n) \le K \le \overline{\mathrm{cl}}_{\sigma^*}(h_{\sigma^*}/e_\tau\text{- li } K_n).$$

We note that this definition of convergence is sequential only and we may not be able to find a topology that is compatible with it. Also the epi/hypo limit is not unique. The following theorem is the main connection between epi/hypo–convergence and the convergence of saddle points.

Theorem 0.1. [4] *Let (X,τ), (Y^*,σ^*) be two linear topological spaces. Let $K_n(\cdot,\cdot)$ be a sequence of saddle functions defined on $X \times Y^*$ such that*

$$\underline{\mathrm{cl}}_\tau(e_\tau/h_{\sigma^*}\text{- ls } K_n) \le K \le \overline{\mathrm{cl}}_{\sigma^*}(h_{\sigma^*}/e_\tau\text{- li } K_n).$$

Let $(\overline{x}_{n_k}, \overline{y}^*_{n_k})$ be a subsequence of saddle points of K_n such that $\overline{x}_{n_k} \xrightarrow{\tau} \bar{x}$ and $\overline{y}^*_{n_k} \xrightarrow{\sigma^*} \bar{y}^*$. Then, $(\bar{x}, \bar{y}^*)$ is a saddle point of K and

$$\lim_k K_{n_k}(\overline{x}_{n_k}, \overline{y}^*_{n_k}) = K(\bar{x}, \bar{y}^*).$$

In this paper we will use a result due to Steven Wright [16] that is slightly more general than Theorem 0.1. Let τ_1 and τ_2 be two topologies on the space X and let σ_1^* and σ_2^* be two topologies on the space Y^*.

Theorem 0.2. [16] *Let K be a (τ_2, σ_2^*)–closed bivariate function. Suppose (x_n, y_n^*) is a saddle point of K_n and that $x_n \xrightarrow{\tau_1} x$ and $y_n^* \xrightarrow{\sigma_1^*} y^*$. Assume that*

$$\underline{\mathrm{cl}}_{\tau_2}(e_{\tau_2}/h_{\sigma_1^*}\text{- ls } K_n) \leq K \leq \overline{\mathrm{cl}}_{\sigma_2^*}(h_{\sigma_2^*}/e_{\tau_1}\text{- li } K_n).$$

Then (x, y^*) is a saddle point of K and

$$\lim_n K_n(x_n, y_n^*) = K(x, y^*).$$

Now we go through a short review of the definition and the main properties of the slice topology. We restrict ourselves to a normed linear space X with a dual X^*. Let $C(X)$ be the space of nonempty closed convex subsets of X. The slice topology on $C(X)$ is defined as the weakest topology such that for every closed convex and bounded subset B of X the "gap" functionals

$$A \mapsto \inf_{a\in A, b\in B} \|a - b\|$$

are continuous. The slice topology on $C(X)$ can also be represented as a hit and miss topology. We introduce the following subsets of $C(X)$:

$$E^- = \{A \in C(X) : A \cap E \neq \emptyset\},$$

$$E^{++} = \{A \in C(X) : \exists\, \varepsilon > 0 \text{ such that } A + \varepsilon\mathcal{B} \subset E\},$$

where $\mathcal{B}$ is the unit ball in X. The slice topology has as a subbase consisting of all sets of the form V^-, where V is a norm open subset of X, plus all sets of the form W^{++} where W is the complement of a closed, bounded, convex subset of X. It is shown in chapter 8 of [7] that the following are equivalent:

1 – $(C(X), \mathit{slice})$ is metrizable;
2 – $(C(X), \mathit{slice}\ )$ has a countable base;
3 – $(C(X), \mathit{slice})$ has a countable local base;
4 – X^* is strongly separable.

The slice convergence of nets of proper convex lsc functions is defined as the slice convergence of their epigraphs in $C(X \times \mathbb{R})$. Moreover, if we let δ_A be the indicator function of the nonempty convex closed subset A of X:

$$\delta_A(x) = \begin{cases} 0 & \text{if } x \in A; \\ +\infty & \text{otherwise.} \end{cases}$$

Then δ is an embedding of $(C(X), \mathit{slice})$ in $(\mathcal{E}(X), \mathit{slice})$ (cf. [6]), and we have the following theorem.

Theorem 0.3. [6] *Let A_ν be a collection of sets in C(X). Then the following are equivalent:*

$$(i) \qquad A_\nu \text{ slice converges to } A \text{ in } C(X)$$
$$(ii) \qquad \delta_{A_\nu} \text{ slice converges to } \delta_A \text{in } \mathcal{E}(X).$$

We also define the dual slice topology on $C^*(X^*)$, the space of nonempty, weak* closed, convex subsets of X^*. The dual slice topology is the weakest topology on X^* such that the gap functionals

$$A \mapsto \inf_{a\in A, b\in B} \|a-b\|$$

are continuous for every weak* closed, bounded, convex subset B of X^*. And the dual slice topology has as a subbase, all the sets of the form V^- where V is norm open in X^* plus all the sets of the form W^{++} where W is the complement of a weak* closed, bounded, convex subset of X^*. Now we list some of the most important properties of the slice and the dual slice topology.

Theorem 0.4. *[Ch.8 in [7]] Let f_ν be a net of proper lsc convex functions, then*

$$f_\nu \text{ slice converges to } f \iff f_\nu^* \text{ converges in the dual slice topology to } f^*.$$

In this paper we will be dealing only with the slice convergence of sequences of proper convex lsc functions and therefore will need the following results:

Theorem 0.5. *[Ch.8 in [7]] A sequence $\{f_n\}$ of proper convex lsc functions slice converges to f if and only if the following conditions hold:*

$$(i) \quad \forall \text{ bounded sequence } x_n, \forall (x^*,\eta) \in \operatorname{epi}_s f^*,$$
$$f_n(x_n) \geq \langle x_n, x^*\rangle - \eta \text{ eventually},$$

and

$$(ii) \qquad \forall\, x \in X,\ \exists\, x_n \to x \text{ such that } \limsup_n f_n(x_n) \leq f(x).$$

Theorem 0.6. *[Ch.8 in [7]] A sequence $\{f_n\}$ of proper convex lsc functions slice converges to f if and only if the following hold:*

$$(i) \qquad \forall x \in X,\ \exists\, x_n \to x \text{ such that } \limsup_n f_n(x_n) \leq f(x),$$
$$(ii) \qquad \forall x^* \in X^*,\ \exists\, x_n^* \to x^* \text{ such that } \limsup_n f_n^*(x_n^*) \leq f^*(x^*).$$

Theorem 0.7. *[Ch.8 in [7]] If the unit ball in X^* is weak* sequentially compact, then a sequence f_n of proper convex lsc functions slice converges to f if and only if the following conditions hold:*

$$(i) \qquad \forall x^* \in X^*,\ \forall x_n^* \overset{\omega^*}{\to} x^*,\ \liminf_n f_n^*(x_n^*) \geq f^*(x^*),$$

$$(ii) \qquad \forall x^* \in X^*\ , \exists x_n^* \to x^* \textit{ such that} \limsup_n f_n^*(x_n^*) \leq f^*(x^*).$$

We remark here that due to the Banach–Alaoglu theorem, the unit ball in X^* is always weak* compact but it may or may not be weak* sequentially compact. For example, the unit ball in $(l^\infty)^*$ is not sequentially weak* compact. In general, the separability of X is sufficient for the sequential weak* compactness of the unit ball in X^*.

1. Continuity results. In this section we relate the slice convergence of F_n, the perturbation functions of a general optimization problem, to the epi/hypo–convergence of the corresponding Lagrangians K_n. The results that we will obtain can be thought of as results regarding the continuity of the partial (with respect to one variable) Young–Fenchel transform that maps F_n to K_n.
Throughout this section, X and Y are Banach spaces with duals X^* and Y^*. Furthemore, we assume that X has a predual (i.e. it is possible to define a *weak** topology on X). We will use s and w^* to denote the strong and the *weak** topologies and we will use $\to$and $\overset{w^*}{\to}$ to denote the convergence in the strong and the *weak** topologies respectively. The functions F_n, F are proper, convex, lsc (with respect to the product norm topology on $X \times Y$) bivariate functions defined on $X \times Y$ and taking values in $\overline{I\!R}$. K_n, K are the corresponding Lagrangians defined on $X \times Y^*$. Due to a result by Rockafellar [13], K_n and K are automatically closed (with repected to the norm topologies on X and Y^*). We start with three lemmas.

Lemma 1.1. *Let X and Y be two Banach spaces as described in the begining of this section. Let F_n and F be collection of proper, convex and lsc functions defined on $X \times Y$. Let K_n, K be the corresponding closed Lagrangians defined on $X \times Y^*$. Suppose that*

$$\forall\,(x, y) \in X \times Y, \exists\, x_n \to x,\, y_n \to y$$
$$\textit{such that } \limsup_n F_n(x_n, y_n) \leq F(x, y).$$

Then,

$$\underline{\mathrm{cl}}_s(e_s/h_{w^*}\text{-}\operatorname{ls} \overline{K}_n) \leq \underline{K}.$$

Proof. Let $(x_0, y_0^*) \in X \times Y^*$ be such that $\overline{K}(x_0, y_0^*) < \alpha < +\infty$. Then by the definition of $\overline{K}$ we get:

$$\inf_{y \in Y} \{F(x_0, y) - \langle y, y_0^* \rangle\} = \overline{K}(x_0, y_0^*) < \alpha.$$

Hence, there is a y_0 such that

$$F(x_0, y_0) - \langle y_0, y_0^* \rangle \leq \alpha.$$

By our assumption, $\exists\, x_n \to x_0, \exists y_n \to y_0$ such that

$$\limsup_n F_n(x_n, y_n) \leq F(x_0, y_0).$$

Also $\forall\, y_n^* \stackrel{w^*}{\to} y_0^*$, we have $\langle y_n, y_n^* \rangle \to \langle y_0, y_0^* \rangle$ and thus

$$\limsup_n (F_n(x_n, y_n) - \langle y_n, y_n^* \rangle) \leq \alpha.$$

However,

$$\overline{K}_n(x_n, y_n^*) \leq F_n(x_n, y_n) - \langle y_n, y_n^* \rangle.$$

Therefore, we get

$$\limsup_n \overline{K}_n(x_n, y_n^*) \leq \limsup_n \{F_n(x_n, y_n) - \langle y_n, y_n^* \rangle\} < \alpha.$$

Taking the inf of the above inequality over all α such that $\alpha > \overline{K}(x_0, y_0^*)$, we get

$$\limsup_n \overline{K}_n(x_n, y_n^*) \leq \overline{K}(x_0, y_0^*).$$

Hence,

$$\inf_{x_n \to x_0} \limsup_n \overline{K}_n(x_n, y_n^*) \leq \overline{K}(x_0, y_0^*),$$

and since y_n^* was an arbitrary sequence *weak** converging to y_0^*, we get

$$\sup_{y_n^* \xrightarrow{w^*} y_0^*} \inf_{x_n \longrightarrow x} \limsup_n \overline{K}_n(x_n, y_n) \leq \overline{K}(x_0, y_0^*),$$

which by definition means

$$e_s/h_{w^*}\text{-}\operatorname{ls} \overline{K}_n \leq \overline{K}.$$

Since K is assumed to be closed, we get

$$\underline{\mathrm{cl}}_s(e_s/h_{w^*}\text{-}\operatorname{ls} \overline{K}_n) \leq \underline{\mathrm{cl}}_s \overline{K} = \underline{K},$$

and the proof is complete. $\square$

The following two lemmas will provide the "dual" counterpart to Lemma 1.1.

Lemma 1.2. *Let X and Y be two Banach spaces as described in the beginning of this section. Let F_n and F be proper, convex and lsc functions defined on $X \times Y$. Consider $\varphi_n : Y^* \times X^* \to \overline{I\!R}$, where $\varphi_n(y^*, x^*) = F_n^*(x^*, y^*)$ and F_n^* are the conjugates of F_n. Suppose that*

$$\forall (y^*, x^*) \in Y^* \times X^* \ , \exists\, x_n^* \to x^*,\ y_n^* \to y^*$$

such that

$$\limsup_n \varphi_n(y_n^*, x_n^*) \le \varphi(y^*, x^*).$$

Also consider $\overline{L}_n : X \times Y^* \to \overline{I\!R}$, where

$$\overline{L}_n(x, y^*) = \inf_{x^* \in X^*} \{\varphi_n(y^*, x^*) - \langle x, x^* \rangle\}.$$

Then,

$$e_{w^*}/h_s\text{- ls } \overline{L}_n \le \overline{L}.$$

Proof. Let $(x_0, y_0^*) \in X \times Y^*$ such that $\overline{L}(x_0, y_0^*) < \alpha < +\infty$. Then $\exists x_0^* \in X^*$ such that

$$\varphi(y_0^*, x_0^*) - \langle x_0, x_0^* \rangle \le \alpha.$$

Hence, $\exists\, x_n^* \to x,\ y_n^* \to y_0^*$ such that

$$\limsup_n \varphi_n(y_n^*, x_n^*) \le \varphi(y_0^*, x_0^*)$$

and $\forall x_n \overset{w^*}{\to}\ x_0$, we have

$$\limsup_n (\varphi_n(y_n^*, x_n^*) - \langle x_n, x_n^* \rangle) \le \overline{L}(x_0, y_0^*)$$

and thus

$$\sup_{x_n \xrightarrow{w^*} x} \ \inf_{y_n^* \longrightarrow y_0^*} \ \limsup_n \overline{L}_n(x_n, y_n^*) \le \overline{L}(x_0, y_0^*),$$

$$e_{w^*}/h_s\text{- ls } \overline{L}_n \le \overline{L}.$$

$\square$

Lemma 1.3. *Let F, G be respectively the convex and concave parents of K. Let*

$$\underline{K}_n(x, y^*) = \sup_{x^* \in X^*} \{G_n(x^*, y^*) + \langle x, x^* \rangle\}$$

$$\overline{L}_n(x, y^*) = \inf_{x^* \in X^*} \{\varphi_n(y^*, x^*) - \langle x, x^* \rangle\}.$$

Then,

$$\overline{L}_n = -\underline{K}_n.$$

Proof. $\forall x \in X,\ \forall y^* \in Y^*$, we have

$$\overline{L}_n(x,y^*) = \inf_{x^*\in X^*}\{\varphi_n(y^*,x^*) - \langle x,x^*\rangle\}$$
$$= \inf_{x^*\in X^*}\{F_n^*(x^*,y^*) - \langle x,x^*\rangle\}.$$

Since K_n are proper and closed, we have $F_n^*(x^*,y^*) = -G_n(x^*,y^*)$. Thus,

$$\overline{L}_n(x,y^*) = \inf_{x^*\in X^*}\{-G_n(x^*,y^*) - \langle x,x^*\rangle\}$$
$$= -\sup_{x^*\in X^*}\{G_n(x^*,y^*) + \langle x,x^*\rangle\} = -\underline{K}_n(x,y^*).$$

□

Theorem 1.4. *Let $F_n : X\times Y \to \overline{I\!R}$ be a sequence of convex closed proper bivariate functions that are the convex parents of a sequence K_n of proper closed convex–concave functions. Assume F_n slice converges to F. Then,*

$$\underline{\mathrm{cl}}_s(e_s/h_{w^*}\text{-}\mathrm{ls}\ \overline{K}_n) \le \underline{K}$$

and

$$\overline{\mathrm{cl}}_s(h_s/e_{w^*}\text{-}\mathrm{li}\ \underline{K}_n) \ge \overline{K}.$$

Proof. The slice convergence of F_n implies that conditions of Lemmas 1.1 and 1.2 hold. Hence,

$$\underline{\mathrm{cl}}_s(e_s/h_{w^*}\text{-}\mathrm{ls}\ \overline{K}_n) \le \underline{K}$$

and since $\overline{L}_n = -\underline{K}_n$ and $\overline{L} = -\underline{K}$, we get

$$e_s/h_{w^*}\text{-}\mathrm{ls}\ \overline{L}_n = -h_s/e_{w^*}\text{-}\mathrm{li}\ \underline{K}_n,$$
$$\underline{K} \le h_s/e_{w^*}\text{-}\mathrm{li}\ \underline{K}_n.$$

And,

$$\overline{K} = \overline{\mathrm{cl}}_s\underline{K} \le \overline{\mathrm{cl}}_s(h_s/e_{w^*}\text{-}\mathrm{li}\ \underline{K}_n),$$

where the first equality follows from the fact that K is closed. □

Corollary 1.5. *If F_n slice converge to F and $(\overline{x}_{n_k},\overline{y}_{n_k})$ is a subsequence of saddle points of K_n, where K_n are the Lagrangians corresponding to F_n, and $\overline{x}_{n_k} \xrightarrow{w^*} \overline{x}$ and $\overline{y}_{n_k} \xrightarrow{w^*} \overline{y}$, then $(\overline{x},\overline{y})$ is a saddle point of K and*

$$\lim_k K_{n_k}(\overline{x}_{n_k},\overline{y}_{n_k}) = K(\overline{x},\overline{y}).$$

Proof. It is clear that $\underline{K}_n \le K_n \le \overline{K}_n$ and $\underline{K} \le K \le \overline{K}$. let τ_2, σ_2^* be the norm topologies on X and Y^* respectively. Let τ_1, σ_1^* be the *weak** topologies on X and Y^* respectively. Then, Theorem 0.2 and Theorem 1.4 complete the proof. □

Corollary 1.6. *If F_n slice converge to F and $(\overline{x}_{n_k}, \overline{y}_{n_k})$ is a bounded subsequence of saddle points of K_n, where K_n are the Lagrangians corresponding to F_n, then K ,the Lagrangian corresponding to F, has a saddle point. This point is a w^* cluster point of $(\overline{x}_{n_k}, \overline{y}_{n_k})$.*

Under certain conditions on F_n and F, it is possible to obtain a converse to Theorem 1.4 and show that the epi/hypo convergence of K_n implies the slice convergence of F_n. The proof, however, is left for a subsequent paper.

We end this section by briefly discussing an alternative approach to the problem of approxaimting saddle points. This approach involves the relationship between graph convergence and pointwise convergence of set valued mappings. Recall that the *graph* of a set-valued mapping $S : X \rightrightarrows Y$ is a subset of $X \times Y$:

$$\text{gph } S := \{ (x, y) \in X \times Y \mid y \in S(x) \}.$$

We also can define the inverse of such a mapping by

$$S^{-1}(y) := \{ x \in X \mid y \in S(x) \}. \tag{1.1}$$

One immediately has gph $S^{-1} =$ gph S. Recall also that for closed subsets A, A_1, $A_2, \dots$ of a Banach space X, Lim $A_n = A$ if Ls $A_n =$ Li $A_n = A$, where

$$\text{Li } A_n = \{x \in X : \ \exists x_n \to x \text{ and } x_n \in A_n\}.$$

Ls $A_n = \{x \in X : \ \exists\, n(1) < n(2) < n(3) < \dots \forall\, k,\ x_k \in A_{n(k)} \text{ and } x_k \to x\}$.

Given a sequence of set valued mappings $S_n : X \rightrightarrows Y$ we can define two types of convergence. We say S_n graph converges to S when

$$\text{Lim gph } S_n = \text{gph } S.$$

We say S_n pointwise converges to S if

$$\forall x \in X \quad \text{Lim } S_n(x) = S(x).$$

Now the subgradient of a convex function is an example of a set–valued mapping defined as follows: we say $x^* \in X^*$ is a subgradient of $f \in \mathcal{E}(X)$ at $x_0 \in X$, and we write $x^* \in \partial f(x_0)$ if for each $x \in X$ we have

$$f(x) \geq f(x_0) + \langle x - x_0, x^* \rangle.$$

The subdifferential of f is then the following subset of $X \times X^*$:

$$\partial f = \{(x, x^*) \in X \times X^* : x \in X \text{ and } x^* \in \partial f(x)\}$$

Similarly, for a function $f^* \in \mathcal{E}(X^*)$, we say $x \in X$ is a subgradient of f^* at $x_0^* \in X^*$ and we write $x \in \partial f^*(x_0^*)$ if for each $x^* \in X^*$

$$f^*(x^*) \geq f^*(x_0^*) + \langle x^* - x_0^*, x \rangle.$$

The subdifferential of f^* is the following subset of $X^* \times X$:

$$\partial f^* = \{(x^*, x) \in X^* \times X : x^* \in X^* \text{ and } x \in \partial f^*(x^*)\}.$$

Let $K : X \times Y^* \to \overline{I\!R}$ be a saddle function and let $F : X \times X \to I\!R$ is the convex parent of K. We define a subdifferential for K:

$$\partial K(x, y^*) = \partial_x K(x, y^*) \times (-\partial_{y^*}(-K)(x, y^*)),$$

where

$\partial_x K(x, y^*)$ is the subgradient set of the convex function $K(\cdot, y^*)$ at x, $\partial_{y^*}(-K)(x, y^*)$ is the subgradient set of the convex function $-K(x, \cdot)$ at y^*.

We note that

$$(x^*, y) \in \partial K(x, y^*) \iff (x^*, y^*) \in \partial F(x, y) \tag{1.2}$$

and

$$(0, 0) \in \partial K(x, y^*) \iff (x, y^*) \text{ is a saddle point of } K. \tag{1.3}$$

It is known that the slice convergence of f_n in $\mathcal{E}(X)$ implies the Painlevé–Kuratowski convergence of ∂f_n to ∂f (see chapter 8 in [7]). Now we can relate the slice convergence of F_n, the convex parents a sequence K_n of saddle functions, to the graph convergence of ∂K_n.

Proposition 1.7. *If F_n slice converges to F, then*

$$\text{gph } \partial K = \text{Lim gph } \partial K_n.$$

Proof. The proof follows from the above remarks and from (1.2). □

Now (1.1) and (1.3) imply the following :

$$(\bar{x}_n, \bar{y}_n^*) \text{ is a saddle point of } K_n \iff (\bar{x}_n, \bar{y}_n^*) \in (\partial K_n)^{-1}(0, 0).$$

Therefore the question of the convergence of the saddle points of K_n is a question about the pointwise convergence of $(\partial K_n)^{-1}$ at the point $(0, 0)$. Since we already have $\text{Lim gph}\,(\partial K_n)^{-1} = \text{gph } \partial K$, the concept of outer semicontinuity of set valued maps studied by Bagh and Wets [5] can be used to obtain results about the pointwise convergence of these mappings.

2. More on the slice topology. Before we can proceed to apply the results of the previous section to some classical optimization problems, we need more results regarding the slice convergence of functions in $\mathcal{E}(X)$. We start with a simple observation about the slice convergence of a sequence of monotone functions.

Proposition 2.1. *Let f_n be a monotone decreasing sequence in $\mathcal{E}(X)$ that converges pointwise to f_0 in $\mathcal{E}(X)$. Then f_n slice converge to f_0.*

Proof. For any bounded sequence x_n in X and $\forall (x^*, \eta) \in \operatorname{epi}_s f^*$ and $\forall n$,

$$f_n(x_n) \geq f(x_n) \geq \langle x_n, x^* \rangle - \eta.$$

Furthermore, we have pointwise convergence and thus we can apply Theorem 0.5. ☐

Now Proposition 2.2 provides a dual statement:

Proposition 2.2. *Let f_n be a monotone increasing sequence in $\mathcal{E}(X)$ that converges pointwise to f_0 in $\mathcal{E}(X)$. Then, f_n slice converge to f_0.*

Proof. The fact that f_n is a monotone increasing sequence in $\mathcal{E}(X)$ implies that f_n^* is a monotone decreasing sequence in $\mathcal{E}(X^*)$. ☐

REMARK. The two propositions above can be used to prove slice convergence of nested subsets in $C(X)$ by appealing to the fact that the indicator function is an embedding from $(C(X), slice)$ into $(\mathcal{E}(X), slice)$.

Our next result is about the slice convergence of the sum of two sequences in $\mathcal{E}(X)$. First we state a result by Lahrache [10] that is valid for any Banach space X. Then we prove a result that is valid only for a separable Banach space but under weaker assumptions on the sequences themselves.

Theorem 2.3. [10] *Let f_n and g_n be two sequences in $\mathcal{E}(X)$ where X is a normed linear space and assume*

(i) *f_n slice converges to f and g_n slice converges to g;*
(ii) $\exists\, x_0 \in \operatorname{dom} f, \rho \geq 0$ *such that* $\sup_n g_n(x) < M < +\infty$ $, \forall x \in B(x_0, \rho)$.

Then $f_n + g_n$slice converge to $f + g$.

We immediately get the following corollary.

Corollary 2.4. *Let A_n be a sequence of sets in C(X) that slice converges to A. Assume that* $\operatorname{int}(A \cap B) \neq \emptyset$ *for some set* $B \in C(X)$ *. Then* $A_n \cap B$ *slice converge to* $A \cap B$*.*

Proof. The proof follows from the fact that $\delta_{A_n \cap B} = \delta_{A_n} + \delta_B$, Theorem 0.1 and Propositon 2.1. □

For our applications, condition (ii) of Theorem 2.1 is too restrictive. We therefore need the following theorem which is a generalization of Theorem 4.1 in [4] to nonreflexive Banach spaces:

Theorem 2.5. *Let f_n, g_n, f, g be functions in $\mathcal{E}(X)$ where X is a separable Banach space X. Assume that f_n slice converges to f, g_n slice converges to g and* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$*. Furthermore, assume*

$$\exists\, r \textit{ such that } \forall z \in B(0,r),$$

$$\exists \textit{ bounded } \; x_n,\; y_n \textit{ such that } x_n - y_n = z,$$

$$\limsup_n f_n(x_n) < +\infty \textit{ and } \limsup_n g_n(y_n) < +\infty.$$

Then, $\exists\, N$ such that $f_n + g_n$ is proper for $n > N$ and $f_n + g_n$ slice converges to $f + g$.

Proof. We shall prove this theorem in three steps.
Step 1: We show that $(f_n + g_n)$ are proper for $n > N$. Clearly $f + g$ is proper by our assumptions. Now take $z = 0$, $x_n = y_n$, then

$$\limsup_n (f_n + g_n)(x_n) < +\infty.$$

Hence, $f_n + g_n$ are proper for large enough n.
Step 2: We show that condition (ii) of Theorem 0.7. holds.
Let V^* be an open set in X^*. Let $\beta \in \mathbb{R}$ be such that

$$\{V^* \times (-\infty, \beta)\} \cap \operatorname{epi}(f+g)^* \neq \emptyset.$$

Then

$$\{V^* \times (-\infty, \beta)\} \cap \operatorname{cl} \;\; \operatorname{epi}(f^* \# g^*) \neq \emptyset,$$

where the closure is with respect to the norm topology on X^*. Hence,

$$\{V^* \times (-\infty, \beta)\} \cap \operatorname{epi}(f^* \# g^*) \neq \emptyset,$$

and

$$\{V^* \times (-\infty, \beta)\} \cap \operatorname{cl}(\operatorname{epi} f^* + \operatorname{epi} g^*) \neq \emptyset,$$

$$\{V^* \times (-\infty, \beta)\} \cap (\operatorname{epi} f^* + \operatorname{epi} g^*) \neq \emptyset.$$

Therefore, there is a $(x^*,\eta) \in X^* \times I\!R$ such that
$$(x^*,\eta) \in \{V^* \times (-\infty,\beta)\} \cap (\text{epi } f^* + \text{epi } g^*),$$
and
$$(x^*,\eta) = (x_1^*,\eta_1) + (x_2^*,\eta_2),$$
where $(x_1^*,\eta_1) \in \text{epi } f^*$ and $(x_2^*,\eta_2) \in \text{epi } g^*$. By the continuity of addition in a linear topological space, we know that there is V_1^* and V_2^* open in X^* and β_1 and β_2 in $I\!R$ such that
$$(x_1^*,\eta_1) \in V_1^* \times (-\infty,\beta_1),$$
$$(x_2^*,\eta_2) \in V_2^* \times (-\infty,\beta_2)$$
and
$$\{V_1^* \times (-\infty,\beta_1)\} + \{V_2^* \times (-\infty,\beta_2)\} \subset V^* \times (-\infty,\beta).$$
Furthermore, f_n^* slice* converges to f^* and g_n^* slice* converges to g^*, which imply that $\exists N_1$ such that $\forall n > N_1$, we have
$$V_1^* \times (-\infty,\beta_1) \cap \text{epi } f_n^* \neq \emptyset$$
$$V_2^* \times (-\infty,\beta_2) \cap \text{epi } g_n^* \neq \emptyset.$$
Hence,
$$\Big(\{(V_1^* \times (-\infty,\beta_1)\} + \{V_2^* \times (-\infty,\beta_2)\}\Big) \bigcap (\text{epi } f_n^* + \text{epi } g_n^*) \neq \emptyset,$$
and thus,
$$\{V^* \times (-\infty,\beta)\} \cap (\text{epi } f_n^* + \text{epi } g_n^*) \neq \emptyset$$
$$\{V^* \times (-\infty,\beta)\} \cap \text{epi cl}\,(f_n^* \# g_n^*) \neq \emptyset$$
and
$$\{V^* \times (-\infty,\beta)\} \cap \text{epi}\,(f_n + g_n)^* \neq \emptyset.$$
Therefore, $\forall\, z^* \in \text{epi}\,(f+g)^*$, $\exists\, z_n^* \in \text{epi}\,(f_n+g_n)^*$ such that $z_n^* \to z^*$ and condition (ii) of Theorem 0.7 follows immediately.

Step 3: We show that condition (i) of Theorem 0.7 hold.

Assume $\liminf_n f_n^*(x_n^*) < +\infty$. Otherwise there is nothing to prove. Then, by passing through a subsequence if necessary, we can assume that $(f_n + g_n)^*(x_n^*)$ is bounded above. Now we know that
$$(f_n + g_n)^* = \text{cl}\,(f_n^* \# g_n^*).$$
Hence, there exists a sequence z_n^* such that
$$[f_n^* \# g_n^*](z_n^*) \leq (f_n+g_n)^*(x_n^*) + \frac{1}{n},$$
and
$$\|z_n^* - x_n^*\| \leq \frac{1}{n}.$$

The definition of the epi–sum yields a sequence ζ_n^* in X^* such that

$$f_n^*(\zeta_n^*) + g_n^*(z_n^* - \zeta_n^*) \leq (f_n + g_n)^*(x_n^*) + \frac{1}{n}. \tag{2.1}$$

Consider $\xi \in B(0,r)$ with $r > 0$ and two bounded sequences y_n and w_n that satisfy our assumption. Then,

$$\langle \zeta_n^*, w_n \rangle \leq f_n(w_n) + f_n^*(\zeta_n^*),$$
$$\langle z_n^* - \zeta_n^*, y_n \rangle \leq g_n(y_n) + g_n^*(z_n^* - \zeta_n^*).$$

Since $\xi = w_n - y_n$, we get

$$\langle \zeta_n^*, \xi \rangle \leq f_n(w_n) + g_n(y_n) + g_n^*(z_n^* - \zeta_n^*) + f_n^*(\zeta_n^*) - \langle z_n^*, y_n \rangle.$$

From the above inequality and our assumption, we get

$$\limsup_n \langle \zeta_n^*, \xi \rangle < +\infty.$$

Since the above inequality holds for every $\xi \in B(0,r)$, ζ_n^* is bounded by the Banach–Steinhaus theorem, and hence it must have a weak* cluster point ζ^*. Now we use the continuity of the Young–Fenchel transform with respect to the slice topology. We take the lim inf of (2.1)

$$\liminf_n (f_n + g_n)^*(x_n^*) \geq \liminf_n [f_n^*(\zeta_n^*) + g_n^*(z_n^* - \zeta_n^*)],$$
$$\liminf_n (f_n + g_n)^*(x_n^*) \geq \liminf_n f_n^*(\zeta_n^*) + \liminf_n g_n^*(z_n^* - \zeta_n^*).$$

The slice convergence of f_n^* and g_n^* implies

$$\liminf_n f_n^*(\zeta_n^*) \geq f^*(\zeta^*) \text{and} \liminf_n g_n^*(z_n^* - \zeta_n^*) \geq g^*(x^* - \zeta^*).$$

Therefore,

$$\liminf_n (f_n + g_n)^*(x_n^*) \geq f^*(\zeta^*) + g^*(x^* - \zeta^*) \geq f^* \# g^*(x^*)$$

and hence

$$\liminf_n (f_n + g_n)^*(x_n^*) \geq (f+g)^*(x^*).$$

Now from the three parts of our proof and Theorem 0.7, we obtain the slice convergence of $f_n + g_n$ to $f + g$. □

Remarks. The condition dom $f \cap$ dom $g \neq \emptyset$ in Theorem 2.5 can be eliminated if we assume the existence of a sequence x_n contained in some weakly compact set, such that $\limsup f_n(x_n) < +\infty$ and $\limsup g_n(x_n) < +\infty$. In this case, we can extract a weakly convergent subsequence $x_k \xrightarrow{w} \bar{x}$ such that $\liminf_n (f_n + g_n)(x_n) \geq (f+g)(\bar{x})$ and hence dom $f \cap$ dom $g \neq \emptyset$. We also note that if X is separable, then the condition of Theorem 2.3 is stronger than the condition of Theorem 2.5. Let f_n be uniformly bounded over a neighborhood V of x_0 in dom g. Clearly $f + g$ is proper. Now let

$r \geq 0$ be such that $B(0, 2r) \subset V$. From the slice convergence of g_n we know that there exists a sequence z_n such that $\limsup_n g_n(z_n) < +\infty$. For large enough n, $f_n(z_n)$ is bounded above and $z_n \in B(0, r)$. For any $\zeta \in B(0, r)$, set

$$x_n = z_n + \zeta \text{ and } y_n = z_n.$$

Then, $x_n - y_n = \zeta$, the sequences x_n, y_n are bounded, and $\limsup_n g_n(z_n) < +\infty$ and $\limsup_n f_n(z_n) < +\infty$.

Furthermore, for a separable X we can use Theorem 2.5 to obtain a result about the slice convergence of the intersection of a collection of sets under conditions weaker than those of Corollary 2.1.

Corollary 2.6. *Let A_n, B_n be two sequences of sets in $C(X)$ that slice converge to A and B respectively and assume that $A \cap B \neq \emptyset$. Assume also that $\exists r \geq 0$ such that $\forall \zeta \in B(0, r)$, $\exists$ bounded$x_n \in A_n$, $y_n \in B_n$ such that $x_n - y_n = \zeta$. Then*

$$A_n \cap B_n \textit{ slice converges to } A \cap B.$$

Proof. The proof is identical to the proof of Corollary 2.4 with the exception of using Theorem 2.5 instead of Theorem 2.3. □

Furthermore, the separability of X was used only in part 3 to guarantee that the unit ball in X^* was weak* sequentially compact. Therefore we can still obtain some results for an nonseparable Banach Space X that cannot be obtained directly from Theorem 2.3.

Corollary 2.7. *If f_n and g_n converge in the slice topology to f and g respectively where* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *Suppose that $\forall x \in X$, $\exists x_n \to x$ such that* $\limsup_n (f_n + g_n)(x_n) \leq (f + g)(x)$, *then $f_n + g_n$ slice converges to $f + g$.*

Proof. Step 2 of the proof of Theorem 2.5 does not requir separability assumption on X. This step shows that condition (ii) of Theorem 0.6 holds. Furthemore, our assumption for $f_n + g_n$ is condition (i) of the same theorem. □

The following corollary is a direct result of the previous one.

Corollary 2.8. *If f_n and g_n converge in the slice topology and pointwise to f and g respectively and* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, *then $f_n + g_n$ slice converges to $f + g$.*

Finally, we recall the definition of weak equi–lower semicontinuity for convex lsc functions: We say the collection f_n is equi–lsc at x if there exists $\varepsilon_x > 0$ such that $\forall \varepsilon \in (0, \varepsilon_x)$, there exists a weak neighborhood V of x such that for all n,

$$\inf_{y \in V} f_n(y) \leq \min[f_n(x) - \varepsilon, \varepsilon^{-1}].$$

If $\{f_n\}$ are equi–lsc at every x, we say that f_n is an equi–lsc collection.

Corollary 2.9. *If f_n and g_n are weak equi–lsc sequences that slice converge to f, g respectively and* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$*, then $f_n + g_n$ slice converges to $f + g$.*

Proof. Slice convergence implies Mosco convergence (cf. [6]), and weak equi–lower semicontinuity and Mosco convergence imply pointwise convergence (cf. [8]). Hence the Corollary 2.8 will complete the proof. □

Our last result shows that slice convergence holds under rather restrictive conditions. However, these conditions are satisfied in a number of applications as we shall see later. First we assume X is a Banach space with a separable predual. $L^\infty[0,1]$ with $L^1[0,1]$ as its predual, is an example of such space. We start with a lemma.

Lemma 2.10. *Let f be in $\mathcal{E}(X)$ such that $w^* - \operatorname{int}$ epi $f \neq \emptyset$. Let $(x^*, \eta) \in$* $\operatorname{epi}_s f^*$*. Then the graph of $A : x \to \langle x, x^* \rangle - \eta$ is w^*–closed in $X \times I\!R$.*

Proof. The graph of A is a hyperplane in $X \times I\!R$. Therefore, it is either w^*–closed or its w^*–closure is the entire space (a hyperplane is a translate of a maximal subspace). Let us denote the graph of A by K. Let K_1 be the graph of functional $< ., x^* > -\eta - 1$. If K is not w^*–closed, then K_1 is also not w^*–closed. Hence K_1 is w^*–dense in $X \times I\!R$. Therefore, w^*- int epi $f \cap K_1 \neq \emptyset$ and w^*- int epi $f \cap K \neq \emptyset$, which clearly is a contradiction. Thus K has to be w^*–closed. □

Now we are ready to prove our last result about slice convergence.

Proposition 2.11. *Let f_n and f be elements in $\mathcal{E}(X)$ where X is a Banach space with separable predual. Assume*

$$(i) \qquad w^*\text{- int} \;\; \text{epi } f \neq \emptyset$$

$$(ii) \qquad \forall x \in X,\ \forall x_n \overset{\omega^*}{\to}\ x,\ \liminf_n f_n(x_n) \geq f(x)$$

$$(iii) \qquad \forall x \in X,\ \exists\ x_n \to x, \limsup_n f_n(x_n) \leq f(x)$$

Then f_n slice converges to f.

Proof. In order to use Theorem 0.5, we only need to show that $\forall (x^*, \eta) \in \operatorname{epi}_s f^*$ and for all bounded x_n, we have

$$f_n(x_n) \geq \langle x_n, x^* \rangle - \eta \text{ eventually}.$$

Suppose not. Then, there exists a subsequence x_k such that $x_k \overset{\omega^*}{\to} x_0$ for some x_0 and

$$\forall\, k\ , \quad f_k(x_k) \leq \langle x_k, x^* \rangle - \eta\ .$$

We take the liminf of the above inequality and keeping in mind that the graph of $\langle \cdot, x^* \rangle - \eta$ is w^*–closed due to Lemma 2.1 we obtain

$$\liminf_k f_k(x_k) \leq \liminf_k \langle x_k, x^* \rangle - \eta \leq \langle x_0, x^* \rangle - \eta < f(x_0),$$

which clearly contradicts assumption (ii). □

Corollary 2.12. *Suppose f is weak* continuous at some point and conditions (ii) and (iii) of Proposition 2.11 hold, then f_n slice converge to f.*

Proof. The weak* continuity of f at any point implies that $w^*\text{-}\operatorname{int}\operatorname{epi} f \neq \emptyset$. □

3. Applications. In general, the perturbation space and the perturbation function for a given optimization problem are by no means unique. For the applications that we will discuss in this section, we will choose the classical (i.e., the one most commonly used) perturbation space and function associated with these problems (see [9]).

<u>Convex programming</u>
We consider the following minimization problem over a general Banach space X.

$$(P_0) \qquad \min_{x \in X} \{ f_0(x) + \delta_{D_0}(x) \},$$

where f_0 is a real–valued continuous convex function on X. δ_{D_0} is the indicator function of a closed convex set D_0 in X. Moreover, we assume that D_0 is given by the following constraints:

$$x \in D_0 \quad \Longleftrightarrow \quad g_0^i(x) \leq 0 \text{ for i=1, } \ldots \text{ , m.}$$

where g_0^i are real–valued, convex, continuous functions on X.
Following the general duality scheme, we perturb the problem (P) in the

following manner: we consider the perturbation function $F : X \times I\!R^m \to (-\infty, +\infty]$

$$F(x,u) = f_0(x) + \delta_{D_0^u}(x),$$

where $u \in I\!R^m$ and

$$x \in D_0^u \iff g_0^i(x) \leq u_i \text{ for i=1, \dots , m.}$$

Clearly F is a proper, convex, lsc function on $X \times I\!R^n$. A Lagrangian associated with F is:

$$K(x,y) = \begin{cases} f(x) - \Sigma_{i=1}^m y_i g_0^i(x) & \text{if } y_i \leq 0, \\ -\infty & \text{otherwise.} \end{cases}$$

Now we would like to approximate the saddle points of K with saddle points of K_n which are associated with the perturbation functions of the following problems:

$$(P_n) \qquad \min_{x \in X} \{f_0(x) + \delta_{D_n}(x)\},$$

where D_n are given by

$$x \in D_n \iff g_n^i(x) \leq 0 \text{ for i=1, \dots , m,}$$

where for every i, g_n^i are assumed to increase monotonically to g_0. For example, for a fixed i, g_n^i can be linear functionals that pointwise increase to g_0^i. The fact that g^i is convex, and hence is the supremum of a collection of linear functionals, makes such approximations possible. The perturbation functions associated with (P_n) are

$$F_n(x,u) = f_0(x) + \delta_{D_n^u}(x),$$

where

$$x \in D_n^u \iff g_n^i(x) \leq u_i \text{ for i=1, \dots , m.}$$

Clearly $\delta_{D_n^u}$ monotonically decrease to $\delta_{D_0^u}$. If (P_0) is feasible, then dom $F \neq \emptyset$. Hence, as a result of Proposition 2.1 and Theorem 2.5, we get the slice convergence of F_n to F. Now Theorem 1.4 and Corollary 1.5 can be used to approximate the saddle points of K.

The above problem can be generalized to a case with infinitely many constraints: minimize $f_0(x)$ over $C \subset X$, where C is given by

$$x \in C \iff h(x,s) \leq 0 \ \ , \forall s \in S,$$

where S is some indexing set (space) and h is real-valued convex lsc in the x argument. The perturbation space is $L^\infty(S)$ and the perturbation function is

$$F(x,u) = f_0(x) + \delta_D(x,u),$$

where

$$D = \{(x,u) | h(x,s) \leq u(s) \ , \forall s \in S\}.$$

We approximate h by a monotone sequence h_n which in turn will generate perturbation functions F_n that will slice converge to F. The results of the previous section will then hold regarding the convergence of saddle points of the associated Lagrangians.

Internal approximations

Let X, Y be two Banach spaces and let g and h be convex proper functions defined on X and Y respectively with values in the extended reals. We consider the following saddle function defined on $X \times Y^*$:

$$K(x, y^*) = g(x) - h^*(y^*) + \Gamma(x, y^*), \tag{3.1}$$

where h^* is the conjugate of h and Γ is a continuous biaffine functional on $X \times Y^*$. This type of Lagrangian appears in convex problems in optimal control and in multistage stochastic programming (see [16]). The idea of internal approximation is to approximate X and Y by increasing sequences X_n and Y_n of closed convex subsets of X and Y. The resulting Lagrangians are

$$K_n(x, y^*) = g_n(x) - h_n^*(y^*) + \Gamma(x, y^*), \tag{3.2}$$

where $g_n(x) = g(x) + \delta_{X_n}(x)$ and $h_n(y) = h(y) + \delta_{Y_n}(y)$. In [16], the epi/hypo convergence of K_n was studied in a reflexive setting using Mosco convergence of the convex parents of K_n. In this section we approximate the saddle points of K_n when X and Y are not necessarly reflexive. In order to use Corollary 1.5, we need to show that the convex parents of K_n slice converge to K.

Theorem 3.1. *Let K_n, K be the saddle functions defined by equations (3.1) and (3.2) . Let F_n, F be the the convex parents of K_n and K respectively. Then F_n slice converge to F.*

Proof. From Proposition 2.1 and Corollary 2.8, we obtain the slice convergence of g_n and h_n to g and h respectively. Furthermore, there exists a continuous linear map $D : X \to Y$, elements $b^* \in X^*, a \in Y$ and a real number c such that

$$\Gamma(x, y^*) = < Dx, y^* > + < a, y^* > + < b, x > +c.$$

Thus, F_n can be written as

$$F_n(x, y) = g_n(x) - < b^*, x > -c + \sup_{y^*}\{-h_n^*(y^*) - < Dx + a, y^* > - < y^*, y >\},$$

or

$$F_n(x, y) = g_n(x) - < b^*, x > -c + h_n(-Dx - a - y),$$

and similarly

$$F(x,y) = g(x) - < b^*, x > -c + h(-Dx - a - y).$$

Since g and h are proper, dom $F \neq \emptyset$. For any $(x,y) \in X \times Y$, let $z = -Dx - a - y$. Because of the slice convergence of g_n and h_n, we can find sequences $x_n \to x$ and $z_n \to z$ such that

$$\limsup_n g_n(x_n) \leq g(x),$$

and

$$\limsup_n h_n(z_n) \leq h(z).$$

If we let $y_n = -Dx_n - z_n - a$, we get

$$\limsup_n F_n(x_n, y_n) \leq F(x,y).$$

Hence, F_n slice converges to F by Corollary 2.7. □

<u>Optimal control</u>

Let X, Y be two Banach spaces. Let Λ be a continuous linear map from X to Y. Let C be a closed convex set in Y and let f_0 be a convex continuous function on X. We are interested in the following optimization problem (see [9]):

$$(P_0) \qquad \min_{x \in X} f_0(x) + \delta_{\Lambda(x) \in C}(x),$$

where

$$\delta_{\Lambda(x)\in C}(x) = \begin{cases} 0 & \text{if } \Lambda(x) \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

Under the assumptions we have on C and Λ, the set $\{x | \Lambda(x) \in C\}$ is convex and closed in X. The standard perturbation function for this problem is

$$F(x,u) = f_0(x) + \delta_{\Lambda(x)-u \in C}(x,u),$$

where u is in Y and

$$\delta_{\Lambda(x)-u\in C}(x,u) = \begin{cases} 0 & \text{if } \Lambda(x) - u \in C; \\ +\infty & \text{otherwise.} \end{cases}$$

More specifically, we consider the following optimal control problem:

$$\min_{u \in L^\infty[0,1]} \int_0^1 g(x(t), t)dt,$$

subject to

$$u(t) = D(x(t)),$$
$$x \in C \subset C[0,1].$$

We assume that g is a convex continuous function from $\mathbb{R}^2$ to $\mathbb{R}$, and C is a closed convex subset of $C[0,1]$. We also assume that D is a differential operator from to $C[0,1]$ to L^∞ with an inverse Λ given by:

$$\Lambda(u(t)) = \int_0^1 H(t,s)u(s)ds,$$

where the kernel H is continuous on $[0,1] \times [0,1]$. This condition on D is satisfied by a large class of differential operators and it guarantees that Λ is weak* continuous. Now we can rewrite the problem as

$$\min_{u \in L^\infty[0,1]} \int_0^1 g(\Lambda(u(t)), t)dt + \delta_{\Lambda(u) \in C}(u).$$

We use $Y = C[0,1]$ as a perturbation space and we let the perturbation function defined on $L^\infty[0,1] \times C[0,1]$ be :

$$F(u,y) = \int_0^1 g(\Lambda(u(t)), t)dt + \delta_{\Lambda(u) - y \in C}(u,y).$$

This time we are interested in approximating F with:

$$F_n(u,y) = \int_0^1 g(\Lambda(u(t)), t)P_n(dt) + \delta_{\Lambda(u) - y \in C}(u,y),$$

where P_n is a sequence of discrete measures that approach the original measure dt in the following sense:

$$\int_0^1 h(t)P_n(dt) \to \int_0^1 h(t)dt, \quad \forall\ h(t) \text{ continuous and bounded on } [0,1].$$

This type of convergence of measures is often called the weak convergence of measures. An important property of weak convergence is the following (see [11]): If g_n are such that $\forall t_n \to t$, $g_n(t_n) \to g(t)$, then

$$\liminf_n \int_0^1 g_n(t)P_n(dt) \geq \int_0^1 g(t)dt. \tag{3.3}$$

Our goal is to show that when P_n converge weakly to the original measure on [0,1], F_n slice converge to F. For every u in $L^\infty[0,1]$, we define $I_n(u) = \int_0^1 g(\Lambda(u(t)), t)P_n(dt)$ and $I(u) = \int_0^1 g(\Lambda(u(t)), t)dt$. We want to show first that I_n slice converges to I.

Lemma 3.2. *Suppose $u_n \overset{w^*}{\to} u$ in $L^\infty[0,1]$. Then, $\forall t_n \to t$, we have*

$$\Lambda(u_n(t_n)) \to \Lambda(u(t)).$$

Proof. The collection $\{u_n\}$ is norm bounded. Thus for all $\varepsilon > 0$ there exists δ such that for all t_1, t_2 such that $|t_1 - t_2| < \delta$ and for all n, we have

$$|\Lambda(u_n(t_1)) - \Lambda(u_n(t_2))| \leq \int_0^1 |H(t_1, s) - H(t_2, s)||u_n(s)|ds \leq \varepsilon.$$

Furthermore, $\Lambda(u_n(t))$ pointwise converges to $\Lambda(u(t))$ because of the weak* convergence of u_n. Thus, for all t, $\varepsilon \geq 0$, $\exists$ a neighborhood W of t and $\exists n_0$ such that for all $n \geq n_0$ $\forall t' \in W$ we have

$$|\Lambda(u_n(t')) - \Lambda(u(t))| \leq \varepsilon$$

and hence

$$\forall\ t_n \to t, \text{ we have } \Lambda(u_n(t_n)) \to \Lambda(u(t)).$$

□

Now from property (3.3) of weak convergence of measures, we get

$$\forall u_n \overset{w^*}{\to} u,\ \liminf_n \int_0^1 g(\Lambda(u_n(t)), t)P_n(dt) \geq \int_0^1 g(\Lambda(u(t)), t)dt.$$

Furthermore,

$$\lim_n \int_0^1 g(\Lambda(u(t)), t)P_n(dt) = \int_0^1 g(\Lambda(u(t)), t)dt.$$

The function I is weak* continuous. Thus the conditions of Corollary 2.12 hold and I_n slice converges to I. If we assume that (P_0) is feasible, then dom $F \neq \emptyset$. By Theorem 2.5, F_n slice converges to F. We now can apply the continuity results of the previous sections to the Lagrangians and the saddle points of the approximating problems.

<u>Chebyshev approximations</u>

Let $h_i : [0,1] \to R$ be in $L^\infty[0,1]$ for $i = 0, \ldots, m$. Consider

$$(P) \qquad \min_{x \in R^m} f(x) = ||h_0 - \Sigma_{i=1}^m x_i h_i||_\infty.$$

A classical perturbation function for this problem F, defined on $\mathbb{R}^m \times L^\infty[0,1]$, is :

$$F(x, u) = ||h_0 - \Sigma_{i=1}^m x_i h_i - u||_\infty.$$

The approximations that we will use are

$$F_n(x, u) = ||h_0^n - \Sigma_{i=1}^m x_i h_i^n - u||_\infty,$$

where h_0^n and h_i^n are approximations that norm converge in L^∞ to h_0, h_i (simple functions, for example). The perturbation function F is convex, finite and continuous but nondifferentiable. We also note that in this case the space over which the original problem is defined is reflexive but the perturbation space is not and hence we still need to resort to slice convergence of the perturbation functions. It is a routine check to verify that F_n slice converges to F via Proposition 2.11 (the norm is w^*–lsc). Again this will allow us to use the results of Section 2 to approximate saddle points for this problem.

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ADIB BAGH
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA

DAVIS, CA 95616
USA