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A NOTE ON THE SIERPIŃSKI PARTITION

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Abstract. Starting with the classical Sierpiński partition [7], we consider a question concerning the existence of a nonmeasurable set in the product measure space. Also, we discuss an analogous question on the existence of a set without the Baire property in the product topological space. In particular, it is shown that the situations of measure and category are essentially different.

Let ω be, as usual, the least infinite ordinal number and let ω_1 be the least uncountable ordinal number. It is a well known fact that Sierpiński was the first mathematician who considered, in his paper [7], a partition $\{A, B\}$ of the product set $\omega_1 \times \omega_1$, defined as follows:

$$A = \{(\xi, \zeta) : \xi \leq \zeta < \omega_1\}, \ B = \{(\xi, \zeta) : \omega_1 > \xi > \zeta\}.$$

He observed that, for any $\xi < \omega_1$ and $\zeta < \omega_1$, the inequalities

$$card(A^{\zeta}) \leq \omega, \ card(B_{\xi}) \leq \omega$$

are fulfilled, where

$$A^{\zeta} = \{\xi : (\xi, \zeta) \in A\}, \ B_{\xi} = \{\zeta : (\xi, \zeta) \in B\}.$$

In other words, each of the sets A and B can be represented as the union of a countable family of "curves" lying in $\omega_1 \times \omega_1$. This property of the partition

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$\{A, B\}$ has many interesting and important consequences. For instance, let us point out the following four results.

1. If $P(\omega_1)$ is the σ–algebra of all subsets of ω_1, then the product σ–algebra $P(\omega_1) \otimes P(\omega_1)$ coincides with the σ–algebra $P(\omega_1 \times \omega_1)$ of all subsets of $\omega_1 \times \omega_1$.

2. There does not exist a nonzero σ–finite diffused measure defined on the σ–algebra $P(\omega_1)$ (in fact, this result is due to Ulam [10] who established the nonexistence of such a measure in another way, applying a transfinite matrix of a special type).

3. If R denotes the real line, then, assuming the Continuum Hypothesis, there exists a function

$$\phi \ : \ R \to R$$

such that

$$R^2 = \cup_{n \in \omega} \{g_n(\phi) : n \in \omega\},$$

where g_n $(n \in \omega)$ are some motions of the plane R^2, each of which is either a translation or a rotation (about a point) whose angle is equal to $\pm\pi/2$.

4. The Continuum Hypothesis is equivalent to the statement that there exists a partition $\{X, Y, Z\}$ of the three-dimensional space R^3, such that

1) the set X intersects every line of R^3 parallel to $R \times \{0\} \times \{0\}$ in finitely many points;

2) the set Y intersects every line of R^3 parallel to $\{0\} \times R \times \{0\}$ in finitely many points;

3) the set Z intersects every line of R^3 parallel to $\{0\} \times \{0\} \times R$ in finitely many points.

There are also many other results which can be obtained starting with the above-mentioned Sierpiński partition (see, e.g., [2], [4], [8], [9]).

In this note we wish to discuss two natural questions of measure theory and general topology, respectively, the solutions of which also are closely connected with the Sierpiński partition.

In our further considerations we shall use the Axiom of Dependent Choice (DC) and the Axiom of Determinateness (AD). A detailed information about these axioms can be found, for instance, in [1] and [3]. We shall primarily work in theory (ZF) & (DC) which is basic for the classical domains of mathematics.

Let E be a nonempty set, S be a σ–algebra of subsets of E and let μ be a nonzero σ–finite diffused measure defined on S. In other words, we have a measure space (E, S, μ) where μ satisfies the conditions mentioned above.

As usual, we denote by $\mu \otimes \mu$ the product measure on $S \otimes S$. By the same symbol we also denote the completion of this product measure.

The following question arises naturally: is it possible that the existence of a μ–nonmeasurable subset of E cannot be proved in theory (ZF) & (DC), but the existence of a $(\mu \otimes \mu)$–nonmeasurable subset of $E \times E$ can be established within this theory?

Let us show that such a situation is possible. Indeed, we may take ω_1 as a set E. Equip ω_1 with the standard order topology and consider the σ–ideal I of all nonstationary subsets of ω_1 (we recall that a subset X of ω_1 is nonstationary if there exists a closed unbounded subset F of ω_1 such that $X \cap F = \emptyset$). Further, denote by S the σ–algebra of subsets of ω_1 generated by I. We have a canonical probability diffused measure λ on S, vanishing on all members of I (sometimes this measure λ is called the Dieudonné measure on ω_1). It is important for us that the existence of λ and its corresponding properties (as, e.g., countable additivity of λ) can be established within theory (ZF) & (DC).

Now, let us recall a remarkable theorem of Solovay, according to which the equality $dom(\lambda) = P(\omega_1)$ is valid in theory **ZF** & **DC** & **AD** (for the proof of this theorem, see, e.g., [3] where more general combinatorial results are also presented). Thus, we see that it is impossible to establish, in (ZF) & (DC), the existence of a subset of ω_1 nonmeasurable with respect to λ. On the other hand, let us consider the product measure space $(\omega_1 \times \omega_1, \lambda \otimes \lambda)$. Then, applying the classical Fubini theorem (which is a result of theory (ZF) & (DC)), we immediately obtain that both of the sets A and B of the Sierpiński partition $\{A, B\}$ are nonmeasurable with respect to $\lambda \otimes \lambda$. Hence, we can prove, in (ZF) & (DC), the existence of a $(\lambda \otimes \lambda)$–nonmeasurable subset of $\omega_1 \times \omega_1$.

Remark 1. The converse situation is impossible. More precisely, there is no measure space (E, S, μ) with a σ–finite measure μ such that, in theory (ZF) & (DC), there exists a μ–nonmeasurable subset of E, and it can not be proved, in the same theory, the existence of a $(\mu \otimes \mu)$–nonmeasurable subset of $E \times E$. Indeed, if X is an arbitrary μ–nonmeasurable subset of E, then, by the Fubini theorem, the set $X \times E$ is a $(\mu \otimes \mu)$–nonmeasurable subset of $E \times E$.

Remark 2. As mentioned above, the respective properties of the Sierpiński partition $\{A, B\}$ imply the equality

$$P(\omega_1 \times \omega_1) = P(\omega_1) \otimes P(\omega_1).$$

This equality cannot be established within theory (ZF) & (DC). Indeed, suppose otherwise, i.e. that the equality above is true in (ZF) & (DC). Then it must be true in theory (ZF) & (DC) & (AD) which is stronger than (ZF) & (DC). Consequently, we must have, in (ZF) & (DC) & (AD), the following relation:

$$dom(\lambda \otimes \lambda) = P(\omega_1) \otimes P(\omega_1) = P(\omega_1 \times \omega_1).$$

But this yields immediately a contradiction, because the sets A and B are nonmeasurable with respect to $\lambda \otimes \lambda$.

Taking into account a deep analogy between measure and category (see, for instance, [5]), we can formulate the following question.

Can one find, in theory (ZF) & (DC), a topological space (E, T) such that

a) it is impossible to prove, in (ZF) & (DC), the existence of a subset of (E, T) without the Baire property;

b) it is possible to prove, in (ZF) & (DC), the existence of a subset of $(E \times E, T \times T)$ without the Baire property?

Let us show that the Sierpiński partition $\{A, B\}$ also solves this question positively.

Notice first that we must change the usual order topology of ω_1 because the sets A and B are rather good subsets of the product space $\omega_1 \times \omega_1$ (namely, A is a closed set and B is an open set, hence they have the Baire property in $\omega_1 \times \omega_1$).

Let us put $\tau = \{\emptyset\} \cup I^*$, where I^* denotes the δ–filter dual to the σ–ideal I. Obviously, τ is a topology on ω_1, so we have a topological space (ω_1, τ).

Now, let us consider the following property of a general topological space (E, T):

(*) there exists a π–base of open sets in E such that, for every decreasing (with respect to inclusion) sequence $\{U_n : n \in \omega\}$ of nonempty sets belonging to the π–base, the intersection of $\{U_n : n \in \omega\}$ contains a nonempty set also belonging to the π–base.

It is not difficult to prove the following auxiliary proposition.

Lemma. *The next four assertions are true in theory (ZF) & (DC):*

1) if a space (E, T) satisfies condition (*), then (E, T) is a Baire space;

2) the product of an arbitrary finite family of spaces satisfying condition (*), satisfies this condition, too;

3) if a space (E, T) satisfies (*) and X is a second category subset of E having the Baire property, then X contains a nonempty open subset of E;

4) the space (ω_1, τ) satisfies condition (*).

Now, we can formulate and prove the following statement.

Theorem 1. *In theory (ZF) & (DC), both of the sets A and B do not have the Baire property in the product space $(\omega_1, \tau) \times (\omega_1, \tau)$. At the same time, the existence of a subset of (ω_1, τ) without the Baire property cannot be established in (ZF) & (DC).*

Proof. Notice first that we cannot apply here a direct analogue of the Fubini theorem — the so called Kuratowski–Ulam theorem (see., e.g., [5]) — because the Kuratowski–Ulam theorem is true only under certain assumptions on topological spaces, which do not hold in the case of (ω_1, τ).

Suppose now that A (consequently, B) has the Baire property in $(\omega_1, \tau) \times (\omega_1, \tau)$.

According to assertions 1), 2) and 4) of Lemma, the product space $(\omega_1, \tau) \times (\omega_1, \tau)$ is a Baire topological space. Hence, at least one of the sets A and B is a second category subset of this product space. We may assume, without loss of generality, that A is a second category set. According to assertion 3) of Lemma, the set A contains a nonempty open subset W. Obviously, we can suppose that $W = U \times V$, where U and V are some closed unbounded subsets of ω_1 equipped with its order topology. Let us take an arbitrary element ζ from V. Since U is unbounded, there exists an element ξ from U such that $\zeta < \xi$. We thus get

$$(\xi, \zeta) \in U \times V = W \subseteq A,$$

which contradicts the definition of the set A.

The contradiction obtained above shows us that both of the sets A and B do not have the Baire property in $(\omega_1, \tau) \times (\omega_1, \tau)$.

On the other hand, it is clear that a subset of (ω_1, τ) has the Baire property if and only if it is measurable with respect to the measure λ. Hence, by the result of Solovay, the existence of a subset of (ω_1, τ) without the Baire property cannot be proved in theory (ZF) & (DC). Moreover, it is easy to see that the Borel σ–algebra of the space (ω_1, τ) coincides with the σ–algebra of all subsets of (ω_1, τ) having the Baire property. Thus, we conclude that the existence of a subset of (ω_1, τ) not belonging to the Borel σ–algebra of (ω_1, τ) cannot be proved in theory (ZF) & (DC), too. □

In Remark 1, we have mentioned a simple fact that, if μ is an arbitrary σ–finite measure defined on some σ–algebra of subsets of a nonempty set E, then the implication

$$P(E) \neq dom(\mu) \rightarrow P(E \times E) \neq dom(\mu \otimes \mu)$$

is true in theory (ZF) & (DC) (here the symbol $P(E)$ denotes the family of all subsets of E).

Consider now an arbitrary topological space (E, T) and denote by $Br(E)$ the σ–algebra of all subsets of E having the Baire property.

The following question arises naturally: is the implication

$$P(E) \neq Br(E) \rightarrow P(E \times E) \neq Br(E \times E)$$

true in theory (ZF) & (DC)?

It turns out that the answer to the question just formulated is negative (so, in this aspect, the situation of category essentially differs from the situation of measure).

Theorem 2. *The implication*

$$P(E) \neq Br(E) \rightarrow P(E \times E) \neq Br(E \times E)$$

cannot be established in theory (ZF) & (DC).

Proof. Suppose otherwise, i.e. that the implication above is a theorem of theory (ZF) & (DC). Then, obviously, this implication is also a theorem of theory (ZFC) & (CH), where (ZFC) denotes the Zermelo-Fraenkel set theory with the Axiom of Choice and (CH) denotes, as usual, the Continuum Hypothesis.

Recall now that Oxtoby [6] constructed, in (ZFC) & (CH), a topological space Z such that

(1) $card(Z) = c = \omega_1$;

(2) Z does not contain isolated points;

(3) Z is a completely regular Baire space;

(4) $Z \times Z$ is a first category space.

It is not difficult to see that the space Z, constructed by Oxtoby, satisfies also the countable chain condition (i.e. any family of nonempty pairwise disjoint open subsets of Z is at most countable). Actually, Z is an everywhere dense subset of some compact topological space Z' equipped with a probability diffused measure ν such that ν is defined on the σ–algebra of subsets of Z' generated by a certain base of the topology of Z', and the values of ν are strictly positive, for all members of that base (the construction of Z' and ν is given in details in [6]). Evidently, the space Z' satisfies the countable chain condition. Hence, the space Z, being a dense subset of Z', satisfies this condition, too.

Now, applying the classical transfinite $(\omega \times \omega_1)$–matrix of Ulam [10], we see that the space Z contains a subset without the Baire property. At the same time, the product space $Z \times Z$, being a first category topological space, does not contain a subset without the Baire property. Thus, we obtained a contradiction, which shows that the implication mentioned above is not a theorem of theory (ZF) & (DC). □

Remark 3. Let E be an arbitrary uncountable Polish topological space equipped with a probability diffused Borel measure μ. It is well known that there exists (in theory (ZF) & (DC)) a Borel isomorphism $f : E \to E \times E$, which simultaneously is an isomorphism between measures μ and $\mu \otimes \mu$. Consequently, in (ZF) & (DC), the existence of a subset of E nonmeasurable with respect to the completion of μ is equivalent to the existence of a subset of $E \times E$ nonmeasurable with respect to the completion of $\mu \otimes \mu$.

Similarly, if E is an uncountable Polish topological space without isolated points, then there exists (in theory (ZF) & (DC)) a Borel isomorphism $g : E \to E \times E$ such that g and g^{-1} preserve the first category sets. In particular, g and g^{-1} preserve the sets having the Baire property. Consequently, in (ZF) & (DC), the existence of a subset of E without the Baire property is equivalent to the existence of a subset of $E \times E$ without the Baire property. We recall also that, in theory (ZFC), there exists a subset of E nonmeasurable with respect to the completion of μ and, if E does not contain isolated points, there exists a subset of E without the Baire property. Namely, any Bernstein subset of E is nonmeasurable with respect to the completion of μ; in addition, if E does not contain isolated points, then any Bernstein subset of E does not have the Baire property in E (a detailed information about Bernstein sets can be found, e.g., in [5]).

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