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OSCILLATION CRITERIA FOR SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH DAMPING

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Abstract. Some oscillation criteria are given for the second order linear differential equation with damping

$$[r(t)x'(t)]' + p(t)x'(t) + q(t)x(t) = 0, \quad t \geq t_0,$$

where $p(t)$ and $q(t)$ are allowed to change sign on $[t_0, \infty)$, and $r(t) > 0$. These results generalize and improve some known results for the differential equations

$$x''(t) + q(t)x(t) = 0,$$

and

$$x''(t) + p(t)x'(t) + q(t)x(t) = 0.$$

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1. Introduction. Consider the second order linear differential equation with damping

$$(E) \qquad [r(t)x'(t)]' + p(t)x'(t) + q(t)x(t) = 0, \quad t \geq t_0,$$

where $r, p, q \in C([t_0, \infty); \Re)$, $r > 0$,and p and q are allowed to take on negative values for arbitrarily large t.

A solution of equation (E) is called oscillatory if it has arbitrarily large zeros, and otherwise it is nonoscillatory. Equation (E) is oscillatory if all its solutions are oscillatory.

In the absence of damping, there is a very large body of literature devoted to the corresponding equations

$$(E_1) \qquad x''(t) + q(t)x(t) = 0,$$

and

$$(E_2) \qquad (r(t)x'(t))' + q(t)x(t) = 0.$$

Although (E) can be easily transformed to the forms (E_1) and (E_2) by multiplication by an integrating factor, there are advantages in obtaining direct oscillation theorems for (E): besides the obvious practical advantage of eliminating the need for the integrating factor, there is an incentive in developing methods which will generalize to more general equations.

Averaging function method is one of the most important techniques in the study of oscillation. By using this technique, many oscillation criteria have been found which involve the behavior of the integrals of the coefficients, see [**?**, **?**, **?**, **?**, **?**, **?**] .

In this article, by using general Riccati technique due to Yu [**?**] and averaging functions method and following the results of Yan [**?**] and Philos [**?**], we establish some oscillation criteria for equation (E).

For other results, we refer to [**?**, **?**, **?**, **?**].

2. Main results. Throughout this section, we assume that $a(t) \in C^2([t_0, \infty); (0, \infty))$ is a given function, $f(t) = -\frac{a'(t)}{2a(t)}$ and $\psi(t) = a(t)\{q(t) - p(t)f(t) + r(t)f^2(t) - [r(t)f(t)]'\}$.

The following theorem provides sufficient conditions for the oscillation of the equation (E).

Theorem 1. *Assume $D = \{(t, s) | t \geq s \geq t_0\}$. Let $H \in C(D; \Re)$ satisfy the following two conditions:*

(i) $H(t, t) = 0$ *for* $t \geq t_0$, $\quad H(t, s) > 0$ *for* $t > s \geq t_0$;

(ii) *H has a continuous and nonpositive partial derivative on D with respect to the second variable.*

Let $h : D \to \Re$ be a continuous function with

$$-\frac{\partial H}{\partial s}(t,s) = h(t,s)\sqrt{H(t,s)} \quad \textit{for all } (t,s) \in D.$$

If

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big\{ H(t,s)\psi(s) - \frac{1}{4}a(s)r(s)\Big(h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)}\Big)^2 \Big\} ds = \infty, \quad (C_1)$$

then equation (E) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (E). Without loss of generality, we may assume that $x(t) > 0$ on $[T_0, \infty)$, for some $T_0 \geq t_0$. Define

$$w(t) = a(t)r(t)\left(\frac{x'(t)}{x(t)} + f(t)\right), \quad t \geq T_0. \tag{1}$$

Then it follows from (E) that

$$w'(t) = -\frac{w^2(t)}{a(t)r(t)} - \frac{p(t)}{r(t)}w(t) - \psi(t), \quad t \geq T_0. \tag{2}$$

Hence, for all $t \geq T \geq T_0$, we have

$$\begin{aligned}
&\int_T^t H(t,s)\psi(s)ds \\
= \; & -\int_T^t H(t,s)w'(s)ds - \int_T^t \left(H(t,s)\frac{w^2(s)}{a(s)r(s)} + \frac{H(t,s)p(s)}{r(s)}w(s) \right) ds \\
= \; & H(t,T)w(T) - \int_T^t \Big\{ H(t,s)\frac{w^2(s)}{a(s)r(s)} \\
& \qquad + \left[\left(-\frac{\partial H}{\partial s}(t,s) \right) + \frac{H(t,s)p(s)}{r(s)} \right] w(s) \Big\} ds \\
= \; & H(t,T)w(T) - \int_T^t \Big\{ H(t,s)\frac{w^2(s)}{a(s)r(s)} \\
& \qquad + \left[h(t,s)\sqrt{H(t,s)} + \frac{H(t,s)p(s)}{r(s)} \right] w(s) \Big\} ds.
\end{aligned}$$

Then, for all $t \geq T \geq T_0$

$$\int_T^t \left\{ H(t,s)\psi(s) - \frac{1}{4}a(s)r(s)\left(h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)}\right)^2 \right\} ds \tag{3}$$

$$= H(t,T)w(T) - \int_T^t \left\{ \sqrt{\frac{H(t,s)}{a(s)r(s)}}w(s) + \frac{1}{2}\sqrt{\frac{a(s)r(s)}{H(t,s)}}\left(h(t,s)\sqrt{H(t,s)} + \frac{H(t,s)p(s)}{r(s)}\right) \right\}^2 ds.$$

This implies that for every $t \geq T_0$

$$\int_{T_0}^t \left\{ H(t,s)\psi(s) - \frac{1}{4}a(s)r(s)\left(h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)}\right)^2 \right\} ds$$

$$\leq H(t,T_0)w(T_0) \leq H(t,T_0)|w(T_0)| \leq H(t,t_0)|w(T_0)|.$$

Therefore,

$$\int_{t_0}^t \left\{ H(t,s)\psi(s) - \frac{1}{4}a(s)r(s)\left(h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)}\right)^2 \right\} ds$$

$$\leq H(t,t_0)\int_{t_0}^{T_0} |\psi(s)|ds + H(t,t_0)|w(T_0)|$$

$$= H(t,t_0)\left\{ \int_{t_0}^{T_0} |\psi(s)|ds + |w(T_0)| \right\}$$

for all $t \geq T_0$. This gives

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left\{ H(t,s)\psi(s) - \frac{1}{4}a(s)r(s)\left(h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)}\right)^2 \right\} ds$$

$$\leq \int_{t_0}^{T_0} |\psi(s)|ds + |w(T_0)|,$$

which contradicts (C_1). This completes the proof of the Theorem 1. $\square$

Remark 1. If $p(t) = 0$, then Theorem 1 reduces to a result in [**?**].

Remark 2. If $p(t) = 0$ and $a(t) = 1$, then Theorem 1 reduces to Theorem 1 in [**?**].

Theorem 2. *Let H(t,s) and h(t,s) be as in Theorem 1, and let*

$$0 < \inf_{s \geq t_0} \left\{ \liminf_{t \to \infty} \frac{H(t,s)}{H(t,t_0)} \right\} \leq \infty. \tag{C_2}$$

Suppose that there exists a function $A \in C[t_0, \infty)$ *satisfying*

$$\limsup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} a(s)r(s) \Big(h(t,s) + \frac{p(s)}{r(s)} \sqrt{H(t,s)} \Big)^2 ds < \infty, \tag{C_3}$$

$$\int_{t_0}^{\infty} \frac{A_+^2(s)}{a(s)r(s)} ds = \infty \tag{C_4}$$

and for every $T \geq t_0$

$$\limsup_{t \to \infty} \frac{1}{H(t,T)} \int_{T}^{t} \Big\{ H(t,s)\psi(s) - \frac{1}{4} a(s)r(s) \Big(h(t,s) + \frac{p(s)}{r(s)} \sqrt{H(t,s)} \Big)^2 \Big\} ds \geq A(T), \tag{C_5}$$

where $A_+(s) = max\{A(s), 0\}$. *Then equation (E) is oscillatory.*

Proof. Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (E) such that $x(t) > 0$ on $[T_0, \infty)$, for some $T_0 \geq t_0$. As in the proof of Theorem 1, (3) holds for all $t \geq T \geq T_0$. Hence, for $t > T \geq T_0$, we have

$$\begin{aligned} & \frac{1}{H(t,T)} \int_{T}^{t} \left\{ H(t,s)\psi(s) - \frac{1}{4} a(s)r(s) \left(h(t,s) + \frac{p(s)}{r(s)} \sqrt{H(t,s)} \right)^2 \right\} ds \\ = \quad & w(T) - \frac{1}{H(t,T)} \int_{T}^{t} \left\{ \sqrt{\frac{H(t,s)}{a(s)r(s)}} w(s) \right. \\ & \left. + \frac{1}{2} \sqrt{\frac{a(s)r(s)}{H(t,s)}} \left(h(t,s)\sqrt{H(t,s)} + \frac{H(t,s)p(s)}{r(s)} \right) \right\}^2 ds. \end{aligned} \tag{4}$$

Consequently,

$$\limsup_{t \to \infty} \frac{1}{H(t,T)} \int_{T}^{t} \Big\{ H(t,s)\psi(s) - \frac{1}{4} a(s)r(s) \left(h(t,s) + \frac{p(s)}{r(s)} \sqrt{H(t,s)} \right)^2 \Big\} ds$$

$$\leq w(T) - \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left\{ \sqrt{\frac{H(t,s)}{a(s)r(s)}} w(s) \right.$$
$$\left. + \frac{1}{2}\sqrt{\frac{a(s)r(s)}{H(t,s)}} \left(h(t,s)\sqrt{H(t,s)} + \frac{H(t,s)p(s)}{r(s)} \right) \right\}^2 ds.$$

for all $T \geq T_0$. Thus, by (C_5),

$$w(T) \geq A(T) + \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left\{ \sqrt{\frac{H(t,s)}{a(s)r(s)}} w(s) \right.$$
$$\left. + \frac{1}{2}\sqrt{\frac{a(s)r(s)}{H(t,s)}} \left(h(t,s)\sqrt{H(t,s)} + \frac{H(t,s)p(s)}{r(s)} \right) \right\}^2 ds$$

for all $T \geq T_0$. This shows that

$$w(T) \geq A(T) \tag{5}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left\{ \sqrt{\frac{H(t,s)}{a(s)r(s)}} w(s) \right.$$
$$\left. + \frac{1}{2}\sqrt{\frac{a(s)r(s)}{H(t,s)}} \left(h(t,s)\sqrt{H(t,s)} + \frac{H(t,s)p(s)}{r(s)} \right) \right\}^2 ds < \infty$$

for all $T \geq T_0$. Then

$$\liminf_{t\to\infty} \left\{ \frac{1}{H(t,T_0)} \int_{T_0}^t \frac{H(t,s)}{a(s)r(s)} w^2(s) ds \right. \tag{6}$$
$$\left. + \frac{1}{H(t,T_0)} \int_{T_0}^t \left(h(t,s)\sqrt{H(t,s)} + \frac{H(t,s)p(s)}{r(s)} \right) w(s) ds \right\}$$
$$\leq \liminf_{t\to\infty} \frac{1}{H(t,T_0)} \int_{T_0}^t \left\{ \sqrt{\frac{H(t,s)}{a(s)r(s)}} w(s) \right.$$
$$\left. + \frac{1}{2}\sqrt{\frac{a(s)r(s)}{H(t,s)}} \left(h(t,s)\sqrt{H(t,s)} + \frac{H(t,s)p(s)}{r(s)} \right) \right\}^2 ds < \infty.$$

Define

$$u(t) = \frac{1}{H(t,T_0)} \int_{T_0}^t \frac{H(t,s)}{a(s)r(s)} w^2(s) ds$$

and

$$v(t) = \frac{1}{H(t,T_0)} \int_{T_0}^t \left(h(t,s)\sqrt{H(t,s)} + \frac{H(t,s)p(s)}{r(s)} \right) w(s) ds$$

for all $t > T_0$. Then (6) implies that

$$\liminf_{t\to\infty}[u(t)+v(t)] < \infty. \tag{7}$$

Now, we claim that

$$\int_{T_0}^{\infty} \frac{w^2(s)}{a(s)r(s)} ds < \infty. \tag{8}$$

Suppose to the contrary that

$$\int_{T_0}^{\infty} \frac{w^2(s)}{a(s)r(s)} ds = \infty. \tag{9}$$

By (C_2), there is a positive constant M_1 satisfying

$$\inf_{s\geq t_0}\left\{\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)}\right\} > M_1 > 0. \tag{10}$$

Let M_2 be any arbitrary positive number. Then it follows from (9) that there exists a $T_1 > T_0$ such that

$$\int_{T_0}^{t} \frac{w^2(s)}{a(s)r(s)} ds \geq \frac{M_2}{M_1} \qquad \text{for all } t \geq T_1.$$

Therefore,

$$\begin{aligned}
u(t) &= \frac{1}{H(t,T_0)}\int_{T_0}^{t} H(t,s) d\left\{\int_{T_0}^{s} \frac{w^2(\xi)}{a(\xi)r(\xi)} d\xi\right\} \\
&= \frac{1}{H(t,T_0)}\int_{T_0}^{t} \left(-\frac{\partial H}{\partial s}(t,s)\right)\left\{\int_{T_0}^{s} \frac{w^2(\xi)}{a(\xi)r(\xi)} d\xi\right\} ds \\
&\geq \frac{1}{H(t,T_0)}\int_{T_1}^{t} \left(-\frac{\partial H}{\partial s}(t,s)\right)\left\{\int_{T_0}^{s} \frac{w^2(\xi)}{a(\xi)r(\xi)} d\xi\right\} ds \\
&\geq \frac{M_2}{M_1 H(t,T_0)}\int_{T_1}^{t} \left(-\frac{\partial H}{\partial s}(t,s)\right) ds \\
&= \frac{M_2 H(t,T_1)}{M_1 H(t,T_0)}
\end{aligned}$$

for all $t \geq T_1$. By (10), there is a $T_2 \geq T_1$ such that

$$\frac{H(t,T_1)}{H(t,t_0)} \geq M_1 \qquad \text{for all } t \geq T_2,$$

this implies

$$u(t) \geq M_2 \qquad \text{for all } t \geq T_2.$$

Since M_2 is arbitrary,

$$\lim_{t\to\infty} u(t) = \infty. \tag{11}$$

Next, consider a sequence $\{t_n\}_{n=1}^{\infty}$ in (T_0, ∞) with $\lim_{n\to\infty} t_n = \infty$ satisfying

$$\lim_{n\to\infty} [u(t_n) + v(t_n)] = \liminf_{t\to\infty} [u(t) + v(t)].$$

It follows from (7) that there exists a number M such that

$$u(t_n) + v(t_n) \le M \qquad \text{for } n = 1, 2, 3, \cdots . \tag{12}$$

It follows from (11) that

$$\lim_{n\to\infty} u(t_n) = \infty. \tag{13}$$

This and (12) give

$$\lim_{n\to\infty} v(t_n) = -\infty. \tag{14}$$

Then, by (12) and (13),

$$1 + \frac{v(t_n)}{u(t_n)} \le \frac{M}{u(t_n)} < \frac{1}{2} \qquad \text{for } n \text{ large enough.}$$

Thus,

$$\frac{v(t_n)}{u(t_n)} < -\frac{1}{2} \qquad \text{for all large } n.$$

This and (14) imply that

$$\lim_{n\to\infty} \frac{v^2(t_n)}{u(t_n)} = \infty. \tag{15}$$

On the other hand, by the Schwarz inequality, we have

$$\begin{aligned} v^2(t_n) &= \left\{ \frac{1}{H(t_n, T_0)} \int_{T_0}^{t_n} \left(h(t_n, s)\sqrt{H(t_n, s)} + \frac{H(t_n, s)p(s)}{r(s)} \right) w(s)ds \right\}^2 \\ &\le \left\{ \frac{1}{H(t_n, T_0)} \int_{T_0}^{t_n} \frac{H(t_n, s)}{a(s)r(s)} w^2(s)ds \right\} \\ &\quad \times \left\{ \frac{1}{H(t_n, T_0)} \int_{T_0}^{t_n} a(s)r(s) \left(h(t_n, s) + \frac{p(s)}{r(s)}\sqrt{H(t_n, s)} \right)^2 ds \right\} \\ &\le u(t_n) \left\{ \frac{1}{H(t_n, T_0)} \int_{t_0}^{t_n} a(s)r(s) \left(h(t_n, s) + \frac{p(s)}{r(s)}\sqrt{H(t_n, s)} \right)^2 ds \right\} \end{aligned}$$

for any positive integer n. Consequently,

$$\frac{v^2(t_n)}{u(t_n)} \le \frac{1}{H(t_n, T_0)} \int_{t_0}^{t_n} a(s)r(s) \left(h(t_n, s) + \frac{p(s)}{r(s)}\sqrt{H(t_n, s)} \right)^2 ds$$

for all large n. But, (10) guarantees that

$$\liminf_{t\to\infty} \frac{H(t, T_0)}{H(t, t_0)} > M_1.$$

This means that there exists a $T_3 \geq T_0$ such that

$$\frac{H(t,T_0)}{H(t,t_0)} \geq M_1 \qquad \text{for all } t \geq T_3.$$

Thus,

$$\frac{H(t_n,T_0)}{H(t_n,t_0)} \geq M_1 \qquad \text{for } n \text{ large enough}$$

and therefore

$$\frac{v^2(t_n)}{u(t_n)} \leq \frac{1}{M_1 H(t_n,t_0)} \int_{t_0}^{t_n} a(s)r(s)\left(h(t_n,s) + \frac{p(s)}{r(s)}\sqrt{H(t_n,s)}\right)^2 ds$$

for all large n. It follows from (15) that

$$\lim_{n\to\infty} \frac{1}{H(t_n,t_0)} \int_{t_0}^{t_n} a(s)r(s)\left(h(t_n,s) + \frac{p(s)}{r(s)}\sqrt{H(t_n,s)}\right)^2 ds = \infty. \tag{16}$$

This gives

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} a(s)r(s)\left(h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)}\right)^2 ds = \infty,$$

which contradicts (C_3). Then (8) holds. Hence, by (5),

$$\int_{T_0}^{\infty} \frac{A_+^2(s)}{a(s)r(s)} ds \leq \int_{T_0}^{\infty} \frac{w^2(s)}{a(s)r(s)} ds < \infty,$$

which contradicts (C_4). This completes our proof. □

Theorem 3. *Let H(t,s) and h(t,s) be as in Theorem 1, and let (C_2) hold. Suppose that there exists a function $A \in C[t_0,\infty)$ such that (C_4) and the following conditions hold:*

$$(C_6) \qquad \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\psi(s) ds < \infty,$$

and for every $T \geq t_0$

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left\{ H(t,s)\psi(s) - \frac{1}{4} a(s)r(s)\left(h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)}\right)^2 \right\} ds \geq A(T), \tag{C_7}$$

where $A_+(s) = max\{A(s), 0\}$. Then equation (E) is oscillatory.

Proof. Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (E) such that $x(t) > 0$ on $[T_0, \infty)$ for some $T_0 \geq t_0$. As in the proof of Theorem 2, (4) holds for all $t > T \geq T_0$. Then

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left\{ H(t,s)\psi(s) - \frac{1}{4}a(s)r(s)\left(h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)}\right)^2 \right\} ds$$
$$\leq w(T) - \limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left\{ \sqrt{\frac{H(t,s)}{a(s)r(s)}} w(s) + \frac{1}{2}\sqrt{\frac{a(s)r(s)}{H(t,s)}}\left(h(t,s)\sqrt{H(t,s)} + \frac{H(t,s)p(s)}{r(s)}\right) \right\}^2 ds.$$

for all $T \geq T_0$. It follows from (C_7) that

$$w(T) \geq A(T) + \limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left\{ \sqrt{\frac{H(t,s)}{a(s)r(s)}} w(s) + \frac{1}{2}\sqrt{\frac{a(s)r(s)}{H(t,s)}}\left(h(t,s)\sqrt{H(t,s)} + \frac{H(t,s)p(s)}{r(s)}\right) \right\}^2 ds$$

for all $T \geq T_0$. Hence, (5) holds and

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left\{ \sqrt{\frac{H(t,s)}{a(s)r(s)}} w(s) + \frac{1}{2}\sqrt{\frac{a(s)r(s)}{H(t,s)}}\left(h(t,s)\sqrt{H(t,s)} + \frac{H(t,s)p(s)}{r(s)}\right) \right\}^2 ds < \infty$$

for all $T \geq T_0$. This implies that

$$\begin{aligned} &\limsup_{t\to\infty} [u(t) + v(t)] \qquad (17) \\ = \; &\limsup_{t\to\infty} \frac{1}{H(t,T_0)} \int_{T_0}^t \left\{ \frac{H(t,s)}{a(s)r(s)} w^2(s) + \left(h(t,s)\sqrt{H(t,s)} + \frac{H(t,s)p(s)}{r(s)}\right) w(s) \right\} ds \\ \leq \; &\limsup_{t\to\infty} \frac{1}{H(t,T_0)} \int_{T_0}^t \left\{ \sqrt{\frac{H(t,s)}{a(s)r(s)}} w(s) + \frac{1}{2}\sqrt{\frac{a(s)r(s)}{H(t,s)}}\left(h(t,s)\sqrt{H(t,s)} + \frac{H(t,s)p(s)}{r(s)}\right) \right\}^2 ds < \infty, \end{aligned}$$

where $u(t)$ and $v(t)$ are defined as in the proof of Theorem 2. By (C_7),

$$\begin{aligned}A(t_0) \le \liminf_{t\to\infty} \ & \frac{1}{H(t,t_0)}\int_{t_0}^{t}\Big\{H(t,s)\psi(s)\\ & \qquad -\frac{1}{4}a(s)r(s)\Big(h(t,s)+\frac{p(s)}{r(s)}\sqrt{H(t,s)}\Big)^2\Big\}ds\\ \le \liminf_{t\to\infty} \ & \frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\psi(s)ds\\ & -\frac{1}{4}\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}a(s)r(s)\Big(h(t,s)+\frac{p(s)}{r(s)}\sqrt{H(t,s)}\Big)^2 ds\end{aligned}$$

This and (C_6) imply that

$$\liminf_{t\to\infty} \ \frac{1}{H(t,t_0)}\int_{t_0}^{t}a(s)r(s)\Big(h(t,s)+\frac{p(s)}{r(s)}\sqrt{H(t,s)}\Big)^2 ds<\infty.$$

Then there exists a sequence $\{t_n\}_{n=1}^{\infty}$ in (t_0,∞) with $\lim_{n\to\infty} t_n=\infty$ satisfying

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{H(t_n,t_0)}\int_{t_0}^{t_n}a(s)r(s)\left(h(t_n,s)+\frac{p(s)}{r(s)}\sqrt{H(t_n,s)}\right)^2 ds \qquad (18)\\ = \ &\liminf_{n\to\infty} \ \frac{1}{H(t,t_0)}\int_{t_0}^{t}a(s)r(s)\left(h(t,s)+\frac{p(s)}{r(s)}\sqrt{H(t,s)}\right)^2 ds<\infty.\end{aligned}$$

Now, suppose that (9) holds. Using the procedure of the proof of Theorem 2, we conclude that (11) is satisfied. It follows from (17) that there exists a constant M such that (12) is fulfilled. Then, as in the proof of Theorem 2, we see that (16) holds, which contradicts (18). This contradiction proves that (9) fails. Since the remainder of the proof is similar to that of Theorem 2, so we omit the detail. Now, let the function $H(t,s)$ be defined by

$$H(t,s)=(t-s)^n, \qquad t\ge s\ge t_0,$$

where $n\ge 2$ is a constant. Then H is continuous on $D=\{(t,s): t\ge s\ge t_0\}$ and satisfies

$$H(t,t)=0 \quad \text{for } t\ge t_0, \qquad H(t,s)>0 \quad \text{for } t>s\ge t_0.$$

Moreover, H has a continuous and nonpositive partial derivative on D with respect to the second variable. Clearly, the function

$$h(t,s)=n(t-s)^{\frac{n}{2}-1}, \qquad t\ge s\ge t_0$$

is continuous and satisfies

$$-\frac{\partial H}{\partial s}(t,s)=h(t,s)\sqrt{H(t,s)}, \qquad t\ge s\ge t_0.$$

We see that (C_3) holds because for every $s \geq t_0$

$$\lim_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} = \lim_{t\to\infty} \frac{(t-s)^n}{(t-t_0)^n} = 1.$$

Then, by Theorems 1 and 2, we have following two corollaries. $\square$

Corollary 1. *Let $n \geq 2$ be a constant. If*

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_0}^t \left\{ (t-s)^n \psi(s) - \frac{(t-s)^{n-2}}{4} a(s) r(s) \left(n + \frac{p(s)}{r(s)}(t-s) \right)^2 \right\} ds = \infty,$$

Then equation (E) is oscillatory.

Corollary 2. *Let $n \geq 2$ be a constant and suppose that there exists a function $A \in C[t_0, \infty)$ satisfying (C_5) and*

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_0}^t (t-s)^{n-2} a(s) r(s) \left(n + \frac{p(s)}{r(s)}(t-s) \right)^2 \Bigg\} ds < \infty$$

and for every $T \geq t_0$

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_T^t \left\{ (t-s)^n \psi(s) - \frac{(t-s)^{n-2}}{4} a(s) r(s) \left[n + \frac{p(s)}{r(s)}(t-s) \right]^2 \right\} ds \geq A(T).$$

Then equation (E) is oscillatory.

Example 1. Consider the equation

$$(E_3) \qquad x''(t) + \frac{\alpha}{t} x'(t) + \frac{\beta}{t^2} x(t) = 0, \quad t \geq 1,$$

where α and β are two constants with $\beta \geq 0$. Let $a(t) = t$ and $n = 2$. Then $f(t) = -\frac{1}{2t}$, $h(t,s) = 2$ and

$$\psi(t) = a(t)\{q(t) - p(t)f(t) + r(t)f^2(t) - [r(t)f(t)]'\} = \frac{4\beta + 2\alpha - 1}{4t}.$$

This implies

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_0}^t \left\{ (t-s)^n \psi(s) - \frac{(t-s)^{n-2}}{4} a(s) r(s) \left(n + \frac{p(s)}{r(s)}(t-s) \right)^2 \right\} ds \\
&= \limsup_{t\to\infty} \frac{1}{t^2} \int_{t_0}^t \left\{ (t-s)^2 \cdot \frac{4\beta + 2\alpha - 1}{4s} - \frac{1}{4} s \left(2 + \frac{\alpha}{s}(t-s) \right)^2 \right\} ds \\
&= \limsup_{t\to\infty} \frac{1}{t^2} \int_{t_0}^t \left\{ (t-s)^2 \cdot \frac{4\beta - (\alpha-1)^2}{4s} - s - \frac{\alpha(t-s)}{s} \right\} ds \\
&= \infty, \quad \text{if } 1 - 2\sqrt{\beta} < \alpha < 1 + 2\sqrt{\beta}.
\end{aligned}$$

It follows from Corollary 1 that equation (E_3) is oscillatory if $1-2\sqrt{\beta} < \alpha < 1+2\sqrt{\beta}$. Moerover, we note that equation (E_3) has a solution $x(t) = t^{\frac{1-\alpha}{2}}$ if $(\alpha-1)^2 = 4\beta$.

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