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AN INITIAL BOUNDARY VALUE PROBLEM FOR MAXWELL'S EQUATIONS IN A PARABOLIC LIMIT CASE

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Dedicated to Professor Rolf Leis on the occasion of his 65th birthday

Abstract. This paper is an extension of joint work with A. Milani [5]. The so–called magnetohydrodynamic limit case of Maxwell's equations with a monotone and Lipschitz continuous material relation is considered in a bounded domain of arbitrary topological genus. The solution theory is presented in a space–time Hilbert space setting. Existence, uniqueness and continuous dependence results are obtained.

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0. Introduction. The starting point of our considerations is the system of Maxwell's equations:

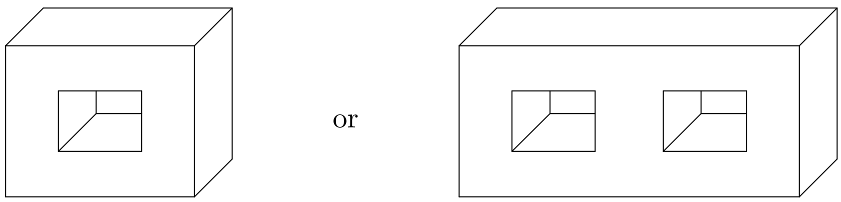
$$
\begin{aligned}
\operatorname{curl} H\,[-\dot D\,] &= j\,,\\
\operatorname{curl} E + \dot B &= 0\,,\\
\operatorname{div} B &= 0\,,\\
[\,\operatorname{div} D &= q\,]\,.
\end{aligned}
\tag{1}
$$

Here E denotes the electric field, H the magnetic field, D electric induction (displacement current), B the magnetic induction, q the charge density and finally $j = \sigma E$, with $\sigma > 0$ as conductivity, denotes the current density (for simplicity we assume absence of external current sources). The terms in brackets in (1) will be ignored based on the rationale that these terms are small in comparison with the others. The resulting system of equations plays an important role e.g. in magnetohydrodynamics and the analysis of transformer cores. It is occasionally referred to as the magnetohydrodynamic limit case of Maxwell's equations. Additionally to the above (reduced) equations one has the so-called material relation describing the material properties of the medium:

$$
H = \zeta(B)\,, \tag{2}
$$

where ζ is assumed to be monotone and Lipschitz continuous (the 'soft' iron case). As typical domains of interest one may consider e.g. transformer cores (see [3]):

(3)



As physically natural boundary conditions we have (in classical notation)

$$n \cdot j = \sigma\, n \cdot E = 0\,, \quad n \cdot B = 0 \quad \text{on} \quad \partial\,\Omega\,, \tag{4}$$

where n is the (exterior) normal to $\partial\,\Omega$. Moreover, the current density j is assumed to be divergence free and so

$$\sigma \operatorname{div} E = \operatorname{div} j = 0\,. \tag{5}$$

Thus we obtain as a first preliminary formulation of our problem:

$$\text{reduced Maxwell equations} \quad \begin{aligned} &\operatorname{curl} H(t) = \sigma\, E(t) \quad \text{in} \quad \Omega\,, t > 0\,, \\ &\operatorname{curl} E(t) + \dot{B} = 0 \quad \text{in} \quad \Omega\,, t > 0\,, \end{aligned} \tag{6}$$

$$\text{divergence conditions} \quad \begin{aligned} &\operatorname{div} B(t) = 0 \quad \text{in} \quad \Omega\,, t > 0\,, \\ &\operatorname{div} E(t) = 0 \quad \text{in} \quad \Omega\,, t > 0\,, \end{aligned} \tag{7}$$

$$\text{material relation} \quad H = \zeta(B)\,, \tag{8}$$

$$\text{boundary conditions} \quad \begin{aligned} &n \cdot E(t) = 0 \quad \text{on} \quad \partial\,\Omega\,, t > 0\,, \\ &n \cdot B(t) = 0 \quad \text{on} \quad \partial\,\Omega\,, t > 0\,, \end{aligned} \tag{9}$$

$$\text{initial condition} \quad B(0+) = B_0 \quad \text{in} \quad \Omega\,. \tag{10}$$

Remark 1. With $\operatorname{div} B_0 = 0$, the divergence conditions (7) are redundant.

As easy as it seems to formulate the above problem in heuristic terms, the central part of any solution theory is to produce a precise formulation of the problem yielding the basic requirements of a reasonable solution theory: existence, uniqueness, stability. Thus the formulation of a problem is inseparably linked to the solution theory. This fact leads to a considerable effort in formulating our problem, since it involves various vectorial differential operators which need to be generalized suitably. The paper is organized accordingly. In Section 1 we shall construct a variety of Hilbert spaces associated with the differential operators grad , curl , div , needed to formulate our problem in precise terms. Section 1 also provided the proper setting for formulating the time–dependence of the electro–magnetic field. In Section 2 we formulate the problem in the precise terminology provided in Section 1 and the main existence and uniqueness result is given. Section 3 provides the solution theory of parabolic evolution problems in the conceptual framework presented in this paper on which the main result of Section 2 was based. As a consequence of the development in Section 3 the following section gives a continuous dependence result for the electro–magnetic field. The last section provides for sake of completeness an elementary derivation of a Sobolev type estimate used in Section 4.

1. Hilbert spaces of vector fields. To define various Hilbert spaces needed to precisely formulate our problem we follow a familiar line of reasoning. As is common from the theory of Sobolev spaces we shall construct suitable Hilbert spaces as domains of certain vectorial differential operators. Let us first consider the differential operators

$$\begin{aligned} \operatorname{grad} &: \overset{\circ}{C}_\infty(\Omega) \subset L_2(\Omega) \longrightarrow L_2(\Omega) \\ \operatorname{curl} &: \overset{\circ}{C}_\infty(\Omega) \subset L_2(\Omega) \longrightarrow L_2(\Omega) \\ \operatorname{div} &: \overset{\circ}{C}_\infty(\Omega) \subset L_2(\Omega) \longrightarrow L_2(\Omega) \end{aligned} \tag{11}$$

defined on the linear subspace $\overset{\circ}{C}_\infty(\Omega) \subset L_2(\Omega)$ of C_∞-functions or C_∞-vector fields with compact support. We shall not notationally distinguish $\overset{\circ}{C}_\infty(\Omega)$ from $\overset{\circ}{C}_\infty(\Omega)^3$, since it will be clear from the context if we are referring to functions or fields. For the inner product of $L_2(\Omega)$ we assume that it is linear in the <u>second</u> factor. It can be seen easily that these well–defined differential operators are closable. We shall use the domains of their respective closures to introduce the Hilbert spaces

$$\overset{\circ}{H}(\operatorname{grad}, \Omega), \quad \overset{\circ}{H}(\operatorname{curl}, \Omega), \quad \overset{\circ}{H}(\operatorname{div}, \Omega).$$

In other words, these spaces are the completion of $\overset{\circ}{C}_\infty(\Omega)$ functions or vector fields with respect to the graph norms

$$\sqrt{|\cdot|^2 + |\operatorname{grad} \cdot|^2}, \ \sqrt{|\cdot|^2 + |\operatorname{curl} \cdot|^2}, \ \sqrt{|\cdot|^2 + |\operatorname{div} \cdot|^2}$$

of grad , curl , *div*, respectively.

It is well–known that

$$\varphi \in \overset{\circ}{H}(\operatorname{grad}, \Omega), \quad \Phi \in \overset{\circ}{H}(\operatorname{curl}, \Omega), \quad \Psi \in \overset{\circ}{H}(\operatorname{div}, \Omega)$$

generalize the classical boundary conditions

$$'\varphi = 0', \quad 'n \times \Phi = 0' \quad \text{and} \quad 'n \cdot \Psi = 0'.$$

We emphasize that no trace results with respect to $\partial\,\Omega$ and therefore no regularity assumptions on $\partial\,\Omega$ are needed for this generalization. The null spaces of the closures of the operators grad , curl , div are denoted by

$$\overset{\circ}{H}_0(\operatorname{grad}, \Omega), \quad \overset{\circ}{H}_0(\operatorname{curl}, \Omega), \quad \overset{\circ}{H}_0(\operatorname{div}, \Omega),$$

respectively. The first space is listed only for systematic reasons, since we find $\overset{\circ}{H}_0(\operatorname{grad}, \Omega) = \{0\}$.

The domains of the adjoints of the operators grad, curl, div as defined above are denoted by

$$H(\operatorname{div}, \Omega), \quad H(\operatorname{curl}, \Omega), \quad H(\operatorname{grad}, \Omega).$$

These are again Hilbert spaces with respect to the induced graph norm. In harmony with the rules of integration by parts, we shall use again '$-$ div', 'curl' and '$-$ grad' to denote these adjoints. This is well–justified since these are extensions of $-$ div, curl, $-$ grad as previously defined. As a consequence we have generalized the differential operators div, curl, grad.

Moreover, we have

$$\text{(12)} \quad \overset{\circ}{H}(\operatorname{div}, \Omega) \subset H(\operatorname{div}, \Omega), \; \overset{\circ}{H}(\operatorname{curl}, \Omega) \subset H(\operatorname{curl}, \Omega), \; \overset{\circ}{H}(\operatorname{grad}, \Omega) \subset H(\operatorname{grad}, \Omega),$$

Consequently, the restrictions $\operatorname{grad}\big|_{\overset{\circ}{H}(\operatorname{grad}, \Omega)}$, $\operatorname{curl}\big|_{\overset{\circ}{H}(\operatorname{curl}, \Omega)}$, $\operatorname{div}\big|_{\overset{\circ}{H}(\operatorname{div}, \Omega)}$ are indeed the above mentioned closures of the initially defined operators $\operatorname{grad} \equiv \operatorname{grad}\big|_{\overset{\circ}{C}_\infty(\Omega)}$, $\operatorname{curl} \equiv \operatorname{curl}\big|_{\overset{\circ}{C}_\infty(\Omega)}$, $\operatorname{div} \equiv \operatorname{div}\big|_{\overset{\circ}{C}_\infty(\Omega)}$. We will denote the null spaces of div, curl, grad by

$$H_0(\operatorname{div}, \Omega), \quad H_0(\operatorname{curl}, \Omega), \quad H_0(\operatorname{grad}, \Omega).$$

Again the latter space is of little interest here and has only been listed for systematic reasons. Indeed, $H_0(\operatorname{grad}, \Omega)$ is spanned by the characteristic functions of the connected components of Ω. Note that $\overset{\circ}{H}(\operatorname{grad}, \Omega)$, $H(\operatorname{grad}, \Omega)$ are the well-known Sobolev spaces $\overset{\circ}{H}_1(\Omega)$, $H_1(\Omega)$ (see e.g. [1]).

We are now able to formulate the spatial part of our problem more precisely. We are seeking E, H, B with $E(t)$, $H(t) \in H(\operatorname{curl}, \Omega)$, $B(t) \in L_2(\Omega)$ for $t > 0$ satisfying

$$\text{(13)} \quad \begin{aligned} \operatorname{curl} H(t) &= \sigma E(t) \quad \text{in} \quad \Omega, \, t > 0, \\ \operatorname{curl} E(t) + \dot{B}(t) &= 0 \quad \text{in} \quad \Omega, \, t > 0, \end{aligned}$$

$$\text{(14)} \quad H = \zeta(B),$$

(15) boundary conditions

$$\begin{aligned} E(t) &\in \overset{\circ}{H}_0(\operatorname{div}, \Omega), \\ B(t) &\in \overset{\circ}{H}_0(\operatorname{div}, \Omega), \end{aligned}$$

$$\text{(16)} \quad B(0+) = B_0 \quad \text{in} \quad \Omega.$$

Here we specify $\zeta : L_2(\Omega) \longrightarrow L_2(\Omega)$ now as a monotone, Lipschitz continuous mapping. Monotonicity means here

$$\operatorname{Re}(\varphi_1 - \varphi_2 \,|\, \zeta(\varphi_1) - \zeta(\varphi_2))_0 \geq \varepsilon_0 \,\|\, \varphi_1 - \varphi_2 \,\|_0 \,, \tag{17}$$

for all φ_1, $\varphi_2 \in L_2(\Omega)$ $(\varepsilon_0 > 0)$.

The formulation generated so far is still heuristic since we have not made precise our understanding of the t-dependence appearing in the list of requirements. But even the spatial part of the formulation is still incomplete as will become clear from the final formulation of our problem. We need further conditions involving the space of harmonic Neumann vector fields:

$$\overset{\circ}{H}_0(\operatorname{div}, \Omega) \cap H_0(\operatorname{curl}, \Omega)\,.$$

Let $\mathcal{B}$ denote an orthonormal basis of the (usually finite-dimensional) space of harmonic vector fields. Then we require additionally that E, B satisfy the conditions

$$(B(t), h)_0 = \omega_\beta(t)\,,\ (E(t), h)_0 = 0 \quad \text{for} \quad \beta \in \mathcal{B} \quad \text{and} \quad t \in \overline{\mathbb{R}^+}\,, \tag{18}$$

where ω_β are given functions. The latter set of conditions implies by an elementary application of the projection theorem (see [9])

$$E(t) \in \overline{\operatorname{curl} \overset{\circ}{H}(\operatorname{curl}, \Omega)} \quad \text{for} \quad t \in \overline{\mathbb{R}^+}\,. \tag{19}$$

As it will turn out the space $\mathcal{H}_0 := \overline{\operatorname{curl} \overset{\circ}{H}(\operatorname{curl}, \Omega)}$ will play a central role in solving the above problem. It is only now, that in order to go on with our deliberations that we need further assumptions on Ω. We shall assume that the domain Ω is such that

$$\boxed{\begin{array}{l} \overset{\circ}{H}(\operatorname{curl}, \Omega) \cap H(\operatorname{div}, \Omega), \quad \overset{\circ}{H}(\operatorname{div}, \Omega) \cap H(\operatorname{curl}, \Omega) \\ \hfill \text{are compactly imbedded in } L_2(\Omega). \end{array}} \tag{20}$$

This is the case if e.g. Ω has a Lipschitz boundary which is certainly the case for the above example configurations (see [8], for other characterizations see [12], [11], for more singular boundaries compare [13]).

Our assumption has as important consequences (see [4], [6], [7], [8], [9]):

$$\mathcal{B} \quad \text{is finite,} \tag{21}$$

$$\operatorname{curl} H(\operatorname{curl}, \Omega)\,, \quad \operatorname{curl} \overset{\circ}{H}(\operatorname{curl}, \Omega) \quad \text{are closed in } L_2(\Omega)\,. \tag{22}$$

The solution theory developed here (as well as the one in [5]) hinges on the following proposition.

Proposition 1. *Let N denote the orthogonal projection N onto the ortho–complement $\mathcal{H}_0$ of the subspace $H_0(\mathrm{curl}\,,\Omega)$*

$$N \,:\, L_2(\Omega) \longrightarrow \mathcal{H}_0 \equiv \mathrm{curl}\,\overset{\circ}{H}\,(\mathrm{curl}\,,\Omega)\,.$$

Then

$$N\,(\,\mathrm{curl}\,H(\mathrm{curl}\,,\Omega)\cap \overset{\circ}{H}\,(\mathrm{curl}\,,\Omega)\,) = \{\,\psi\in\mathcal{H}_0\,|\,\mathrm{curl}\;\psi\in\mathcal{H}_0\,\}\;. \tag{23}$$

Proof.

A) Let $\psi\in \mathrm{curl}\;\overset{\circ}{H}\,(\mathrm{curl}\,,\Omega)$, $\mathrm{curl}\;\psi\in\mathrm{curl}\;\overset{\circ}{H}\,(\mathrm{curl}\,,\Omega)$. Then

$$\mathrm{curl}\;\psi = \mathrm{curl}\;\theta\,,$$

with a unique $\theta\in\mathrm{curl}\;H(\mathrm{curl}\,,\Omega)\cap\overset{\circ}{H}\,(\mathrm{curl}\,,\Omega)$. Thus, as above

$$\psi = N\,\theta\,.$$

B) The converse is also clearly true:

$$N\,\theta\in N\,(\,\mathrm{curl}\;H(\mathrm{curl}\,,\Omega)\cap\overset{\circ}{H}\,(\mathrm{curl}\,,\Omega)\,)\subseteq\mathrm{curl}\;\overset{\circ}{H}\,(\mathrm{curl}\,,\Omega)\,,$$

and

$$\mathrm{curl}\;N\,\theta\equiv\mathrm{curl}\;\theta\in\mathrm{curl}\;\overset{\circ}{H}\,(\mathrm{curl}\,,\Omega)\,.$$

This concludes the proof of the proposition. □

This proposition eventually proves the selfadjointness of curl. We first observe that since $\mathrm{curl}\;\overset{\circ}{C}_\infty(\Omega)$ is dense in $\mathcal{H}_0$ and clearly

$$\mathrm{curl}\;\overset{\circ}{C}_\infty(\Omega)\subseteq\mathcal{H}_1 := \{\,\psi\in\mathcal{H}_0\,|\,\mathrm{curl}\;\psi\in\mathcal{H}_0\,\}\;,$$

we have

$$\mathcal{H}_1 \quad \text{dense in} \;\; \mathcal{H}_0\,, \tag{24}$$

and

$$\begin{aligned}\mathcal{H}_0 = \mathrm{curl}\;(\,\overset{\circ}{H}\,(\mathrm{curl}\,,\Omega)\,) \;&=\; \mathrm{curl}\;(\,\mathrm{curl}\;H(\mathrm{curl}\,,\Omega)\cap\overset{\circ}{H}\,(\mathrm{curl}\,,\Omega)\,)\\ &=\; \mathrm{curl}\;N\,(\,\mathrm{curl}\;H(\mathrm{curl}\,,\Omega)\cap\overset{\circ}{H}\,(\mathrm{curl}\,,\Omega)\,)\\ &=\; \mathrm{curl}\;\mathcal{H}_1\,,\end{aligned}$$

i.e.

$$\mathrm{curl}\;\mathcal{H}_1 \;=\; \mathcal{H}_0\,. \tag{25}$$

Moreover, we have indeed

Lemma 1. curl $: \mathcal{H}_1 \subset \mathcal{H}_0 \longrightarrow \mathcal{H}_0$ *is self–adjoint.*

Proof.
A) To see symmetry we note that according to the above proposition for ϕ, $\psi \in \mathcal{H}_1$ there are

$$\phi_0\,,\,\psi_0 \in \text{curl}\ H(\text{curl}\,,\Omega) \cap \overset{\circ}{H}\ (\text{curl}\,,\Omega)\,,$$

such that

$$N\,\phi_0 = \phi\,,\ N\,\psi_0 = \psi\,,$$

and

$$\begin{aligned}(\text{curl}\ \phi\,|\,\psi) &= (\text{curl}\ N\,\phi_0\,|\,N\,\psi_0) = (\text{curl}\ \phi_0\,|\,\psi_0)\,,\\ &= (\phi_0\,|\,\text{curl}\ \psi_0)\,,\\ &= (N\,\phi_0\,|\,\text{curl}\ N\,\psi_0) = (\phi\,|\,\text{curl}\ \psi)\,.\end{aligned}$$

B) For $\eta \in \mathcal{H}_0$ to be in the domain of the adjoint of this densely defined operator means

$$(\text{curl}\ \phi\,|\,\eta) = (\phi\,|\,f)\ \text{for some}\quad f \in \mathcal{H}_0 \quad \text{and all} \quad \phi \in \mathcal{H}_1\,.$$

Since $\mathcal{H}_0$ is orthogonal to $H_0(\text{curl}\,,\Omega)$ in $L_2(\Omega)$, this implies

$$(\text{curl}\ \phi\,|\,\eta) = (\phi\,|\,f) \quad \text{for all} \quad \phi \in H(\text{curl}\,,\Omega) \quad \text{such that} \quad \text{curl}\ \phi \in \mathcal{H}_0\,.$$

Specializing to $\phi \in \overset{\circ}{C}_\infty(\Omega)$ we see that $\eta \in H(\text{curl}\,,\Omega)$ and $f = \text{curl}\ \eta$, i.e.

$$(\text{curl}\ \phi\,|\,\eta) = (\phi\,|\,\text{curl}\ \eta) \quad \text{for all} \quad \phi \in H\,(\text{curl}\,,\Omega)$$

such that curl $\phi \in \mathcal{H}_0$.

Now specializing to $\phi \in H_0(\text{curl}\,,\Omega)$ we see that curl η is orthogonal to $H_0(\text{curl}\,,\Omega)$ and so curl $\eta \in \mathcal{H}_0$. This shows that the domain of the adjoint is in fact in $\mathcal{H}_1$. □

Remark 2. As consequences of the above reasoning we find

$$\text{curl}^{\#}\ :\ D^{\#} \subset \overset{\circ}{H}_0(\text{div}\,,\Omega)\ \longrightarrow\ \overset{\circ}{H}_0(\text{div}\,,\Omega)$$

with

$$D^{\#} = \left\{\ \varphi \in_H (\text{curl}\,,\Omega) \cap \overset{\circ}{H}_0(\text{div}\,,\Omega)\,|\,\text{curl}\ \varphi \in \mathcal{H}_0\ \right\}\,,$$

$$\text{curl}^{\#\#}\ :\ D^{\#\#} \subset \overset{\circ}{H}\ (\text{div}\,,\Omega) \longrightarrow \overset{\circ}{H}\ (\text{div}\,,\Omega)$$

with

$$D^{\#\#} = \left\{ \varphi \in H(\mathrm{curl}\,,\Omega) \cap \overset{\circ}{H}(\mathrm{div}\,,\Omega)\,|\,\mathrm{curl}\ \varphi \in \mathcal{H}_0 \right\},$$

$$\mathrm{curl}^{\#\#\#} : D^{\#\#\#} \subset H(\mathrm{div}\,,\Omega) \longrightarrow H(\mathrm{div}\,,\Omega)$$

with

$$D^{\#\#\#} = \{\, \varphi \in H(\mathrm{curl}\,,\Omega) \cap H(\mathrm{div}\,,\Omega)\,|\,\mathrm{curl}\ \varphi \in \mathcal{H}_0 \,\},$$

and

$$\mathrm{curl}^{\#\#\#\#} : \{\, \varphi \in H(\mathrm{curl}\,,\Omega)\,|\,\mathrm{curl}\ \varphi \in \mathcal{H}_0 \,\} \subset L_2(\Omega) \longrightarrow L_2(\Omega)$$

are all self–adjoint in their respective base spaces.

It is

$$\mathrm{curl} \subset \mathrm{curl}^{\#} \subset \mathrm{curl}^{\#\#} \subset \mathrm{curl}^{\#\#\#} \subset \mathrm{curl}^{\#\#\#\#}.$$

These operators differ only with respect to their null–spaces:

$$\begin{aligned}
N(\mathrm{curl}) &= \{0\}\,, \\
N(\mathrm{curl}^{\#}) &= \overset{\circ}{H}_0(\mathrm{div}\,,\mathrm{curl}\,,\Omega)\,, \\
N(\mathrm{curl}^{\#\#}) &= \overset{\circ}{H}(\mathrm{div}\,,\Omega) \cap H_0(\mathrm{curl}\,,\Omega)\,, \\
N(\mathrm{curl}^{\#\#\#}) &= H(\mathrm{div}\,,\Omega) \cap H_0(\mathrm{curl}\,,\Omega)\,, \\
N(\mathrm{curl}^{\#\#\#\#}) &= H_0(\mathrm{curl}\,,\Omega)
\end{aligned}$$

Note that by our general assumption on Ω we have

$$\mathcal{H}_1 \hookrightarrow \mathcal{H}_0 \quad \text{is compactly imbedded}\,,$$

with respect to the graph norm of curl. Since $\lambda = 0$ is clearly not in the point spectrum of the operator curl, we know that curl has a (completely) continuous inverse defined on $\mathcal{H}_0$. Thus we may equip $\mathcal{H}_1$ with the norm $\|\cdot\|_1 = \|\,\mathrm{curl}\ \cdot\|_0$ to obtain curl as a unitary mapping $\mathrm{curl} : \mathcal{H}_1 \longrightarrow \mathcal{H}_0$.

In preparation of our final formulation of the problem we need to construct a suitable Gelfand triplet. For this purpose we introduce Hilbert spaces

$$\mathcal{H}_k := D\,(\,|\,\mathrm{curl}\,|^k\,) \equiv D\,(\,\mathrm{curl}^k\,)\,,\ k \in \mathbb{N}\,,$$

As norms we shall use

$$\|\,|\,\mathrm{curl}\,|^k\ \cdot\,\|_0 \equiv \|\,\mathrm{curl}^k\ \cdot\,\|_0\,.$$

Correspondingly, we have dual spaces

$$\mathcal{H}_k^* =: \mathcal{H}_{-k}\,,\ k \in \mathbb{Z}^+,$$

and we choose to identify

$$\mathcal{H}_0^* \equiv \mathcal{H}_0\,.$$

This way we obtain a chain of compactly imbedded Hilbert spaces

$$\cdots \overset{\operatorname{curl}}{\hookrightarrow} \mathcal{H}_2 \overset{\operatorname{curl}}{\hookrightarrow} \mathcal{H}_1 \overset{\operatorname{curl}}{\hookrightarrow} \mathcal{H}_0 \overset{\operatorname{curl}}{\hookrightarrow} \mathcal{H}_{-1} \overset{\operatorname{curl}}{\hookrightarrow} \mathcal{H}_{-2} \overset{\operatorname{curl}}{\hookrightarrow} \cdots,$$

joined by the now unitary operators $\operatorname{curl}\ : \mathcal{H}_k \longrightarrow \mathcal{H}_{k-1}$, $k \in \mathbb{Z}$. Here for $k \leq 0$ the operator curl is extended by duality. We shall use the suggestive notation to re-use $(\,\cdot\,|\,\cdot\,)_{\nu,0,0}$ for the duality pairing between $\mathcal{H}_k$ and $\mathcal{H}_{-k}$.

We are now ready to discuss the open question of specifying the t–dependence.

To formulate a suitable time–dependence we introduce (for some $\nu > 0$ to be specified later)

$$H_{\nu,0} := L_2\,(\mathbb{R}^+, e^{-2\nu t}\,\mathrm{d}t\,)\,. \tag{26}$$

To obtain a sufficiently weak solution concept we introduce analogous to $(\mathcal{H}_k)_k$ the Hilbert spaces

$$H_{\nu,k} := D\,(\,|\,\partial_0\,|^k\,) \equiv D\,(\,\partial_0^k)\,,\ k \in \mathbb{N}\,,$$

based on the natural self–adjoint realizations of

$$|\,\partial_0\,|\,,$$

derived from the essentially normal operator (see [10])

$$\partial_0\ :\ \overset{\circ}{C}_\infty(\Omega) \subset H_{\nu,0} \longrightarrow H_{\nu,0}\,,$$

in a completely analogous way as for $(\mathcal{H}_k)_k$ above.

As norms we shall use

$$\|\,|\,\partial_0\,|^k\ \cdot\ \|_{\nu,0} \equiv \|\,\partial_0^k\ \cdot\ \|_{\nu,0}\,.$$

Correspondingly, we have dual spaces

$$H_{\nu,-k}\,,\ k \in \mathbb{Z}^+\,,$$

and we identify

$$H_{\nu,0}^* \equiv H_{\nu,0}\,.$$

Since $\|\,\partial_0^{-1}\,\| \leq \nu^{-1}$, we obtain a chain of continuously and densely imbedded Hilbert spaces $(H_{\nu,k})_{k\in\mathbb{Z}}$. Thus we have constructed a lattice of Hilbert spaces

$$(H_{\nu,k} \otimes \mathcal{H}_j\,)_{k,j\in\mathbb{Z}}\,, \tag{27}$$

inheriting its lattice structure from $\mathbb{Z}^2$, i.e.

$$H_{\nu,k} \otimes \mathcal{H}_j \subset H_{\nu,n} \otimes \mathcal{H}_{m\,j} \Longleftrightarrow k \geq n \wedge j \geq m\,, \quad j,\, k,\, m,\, n \in \mathbb{Z}\,.$$

We shall utilize only a small portion of this lattice, namely

$$(H_{\nu,k} \otimes \mathcal{H}_j\,)_{k=-1,0,1,\, j=-2,-1,0,1}\,. \tag{28}$$

2. Formulation of the problem and the main result. We are now ready to formulate our initial value problem precisely. In terms of the spaces introduced in Section 1 we shall look for a solution pair (E, B) with

$$\begin{aligned} &E \in H_{\nu,-1} \otimes \mathcal{H}_0 \cap H_{\nu,0} \otimes \mathcal{H}_{-1}\,, \\ &B \in H_{\nu,0} \otimes \overset{\circ}{H}_0(\operatorname{div}, \Omega) \end{aligned} \tag{29}$$

of the following problem.

Problem (MX):

(30)

$\operatorname{curl} H = \sigma\, E \quad \text{in} \quad H_{\nu,-1} \otimes \mathcal{H}_0 \cap H_{\nu,0} \otimes \mathcal{H}_{-1}\,,$
$\operatorname{curl} E + \partial_0\, B = \delta \otimes B_0 \quad \text{in} \quad H_{\nu,-1} \otimes \overset{\circ}{H}_0(\operatorname{div}, \Omega)\,,$
$H = \zeta\,(B) \quad \text{in} \;\; H_{\nu,0} \otimes L_2(\Omega)\,,$
$E \in H_{\nu,-1} \otimes \overset{\circ}{H}_0(\operatorname{div}, \Omega) \cap H_{\nu,0} \otimes \mathcal{H}_{-1}\,,$
$B \in H_{\nu,0} \otimes \overset{\circ}{H}_0(\operatorname{div}, \Omega)\,,$
$B(0+) - B_0 = 0 \quad \text{in} \;\; \mathcal{H}_{-2}\,,$
$(\,B\,\vert\,\phi \otimes \beta\,)_{\nu,0,0} = (\omega_\beta\,\vert\,\phi)_{\nu,0}\,,\;\; \beta \in \mathcal{B},\;\; \phi \in H_{\nu,0}\,,$
$(\,E\,\vert\,\phi \otimes \beta\,)_{\nu,0,0} = 0\,,\;\; \beta \in \mathcal{B},\;\; \phi \in H_{\nu,0}\,.$

Here the time–dependence of ω_β is assumed to be of the type

$$\omega_\beta \in H_{\nu,0}\,,\;\; \omega_\beta \omega_{\beta,0}\, h \in H_{\nu,1}\,,\;\; \beta \in \mathcal{B},\;\; \operatorname{supp} \omega_\beta \subset \mathbb{R}\,, \tag{31}$$

for some $\omega_{\beta,0} \in \mathbb{C}$, $\beta \in \mathcal{B}$.

The details of this formulation are based on the usual philosophy that a problem is well–posed if a unique and stable solution exists. Thus the particular demands on the solution will become clear from the derivation of the main result.

Theorem 1. *Let* $\omega_\beta \in H_{\nu,0}$, $\omega_\beta - h \otimes \omega_{\beta,0} \in H_{\nu,1}$, $\omega_{\beta,0} \in \mathbb{C}$, $\beta \in \mathcal{B}$, *and* $B_0 \in \overset{\circ}{H}_0(\operatorname{div}, \Omega)$ *be such that*

$$(B_0 \,|\, \beta)_0 = \omega_{\beta,0}\,, \quad \beta \in \mathcal{B} \qquad \textit{(compatibility condition).}$$

Then there exists a unique solution of problem (MX):

$$E \in H_{\nu,-1} \otimes \mathcal{H}_0 \cap H_{\nu,0} \otimes \mathcal{H}_{-1}\,,\ B \in H_{\nu,0} \otimes \overset{\circ}{H}_0(\operatorname{div}, \Omega)\,.$$

Proof.

Uniqueness:

Let E_i, B_i, $i = 1, 2$, be two solutions for the same data. Then with

$$E := E_1 - E_2 \quad \text{and} \quad B := B_1 - B_2$$

we have

$$E \in H_{\nu,-1} \otimes \mathcal{H}_0 \cap H_{\nu,0} \otimes \mathcal{H}_{-1}\,, \quad B \in H_{\nu,0} \otimes \mathcal{H}_0\,,$$
$$\operatorname{curl} \zeta\,(B_1) - \operatorname{curl} \zeta\,(B_2) = \sigma\, E\,,$$

and

$$\partial_0\, B_i = -\operatorname{curl}\, E_i + \delta \otimes B_0\,,\ i = 1, 2\,.$$

We obtain

$$\operatorname{curl}\,(\,\zeta\,(\,-\operatorname{curl}\,\partial_0^{-1}\,E_1 + h \otimes B_0\,) - \zeta\,(\,-\operatorname{curl}\,\partial_0^{-1}\,E_2 + h \otimes B_0\,) = \sigma\, E\,.$$

Multiplying this by $-\,\partial_0^{-1}\,E = \operatorname{curl}^{-1} B \in H_{\nu,0} \otimes \mathcal{H}_1$ we obtain

$$(-\partial_0^{-1}\,E\,|\operatorname{curl}\,(\zeta\,(-\operatorname{curl}\,\partial_0^{-1}\,E_1 + h \otimes B_0\,) - \zeta\,(-\operatorname{curl}\,\partial_0^{-1}\,E_2 + h \otimes B_0\,)\,)_{\nu,0,0}$$
$$= \ (\,-\operatorname{curl}\,\partial_0^{-1}\,E\,|\,\zeta\,(-\operatorname{curl}\,\partial_0^{-1}\,E_1 + h \otimes B_0\,)$$
$$-\zeta\,(\,-\operatorname{curl}\,\partial_0^{-1}\,E_2 + h \otimes B_0\,)\,)_{\nu,0,0}$$
$$= \ \sigma\,(-\partial_0^{-1}\,E\,|\,E\,)_{\nu,0,0} = \sigma\,(-\,E\,|\,\partial_0\,E\,)_{\nu,-1,0}\,,$$

By taking real parts and estimating:

$$\varepsilon_0\,\|\operatorname{curl}\,\partial_0^{-1}\,E\,\|^2_{\nu,0,0} \le -\,\sigma\,\nu\,\|\,E\,\|^2_{\nu,-1,0} \le 0\,,$$

Thus $0 = \operatorname{curl}\,\partial_0^{-1}\,E \equiv -B = 0$ and $E = \operatorname{curl}^{-1}\,\partial_0\,B = 0$, as well as $H = \zeta\,(B_1) - \zeta\,(B_2) = 0$. This proves the desired uniqueness. □

Existence:

Noting $\overset{\circ}{H}_0(\operatorname{div}, \Omega) = \operatorname{curl}\,\overset{\circ}{H}\,(\operatorname{curl}, \Omega) \oplus \operatorname{span}(\mathcal{B})$, we expect that

(32) $$B = \operatorname{curl}\,\psi + \beta_0\,,$$

where

$$\beta_0 \in H_{\nu,1} \otimes \operatorname{span}(\mathcal{B}), \tag{33}$$

so that

$$B - \beta_0 \in H_{\nu,0} \otimes \mathcal{H}_0 . \tag{34}$$

Then

$$\psi = \operatorname{curl}^{-1}(B - \beta_0) \in H_{\nu,0} \otimes \mathcal{H}_1 . \tag{35}$$

The harmonic field β_0 is uniquely determined by

$$(B \,|\, \varphi \otimes \beta)_{\nu,0,0} = (\beta_0 \,|\, \varphi \otimes \beta)_{\nu,0,0} = (\omega_\beta \,|\, \varphi)_{\nu,0} , \tag{36}$$

in terms of ω_β, $\beta \in \mathcal{B}$, alone. In fact, $\beta_0 = \sum_\beta \omega_\beta \otimes \beta$. β_0 has a vector potential

$$\chi_0 = \operatorname{curl} \eta_0 \in H_{\nu,0} \otimes \mathcal{H}_0 . \tag{37}$$

Indeed, by the Riesz representation theorem we have unique existence of a solution

$$\eta_0 \in H_{\nu,0} \otimes (\operatorname{curl} H(\operatorname{curl},\Omega) \cap \overset{\circ}{H}(\operatorname{curl},\Omega)) \tag{38}$$

of the equation

$$\operatorname{curl}\operatorname{curl} \eta_0 = \beta_0 \in H_{\nu,0} \otimes \mathcal{H}_{-1} . \tag{39}$$

Thus we have

$$B = \operatorname{curl}(\psi + \chi_0), \quad B_0 = \operatorname{curl}(\psi_0 + \chi_{0,0}) \tag{40}$$

and so

$$\operatorname{curl}(E + \partial_0 \psi + \partial_0 \chi_0 - \delta \otimes \psi_0 - \delta \otimes \chi_{0,0}) = 0 ,$$

in $H_{\nu,-1} \otimes \overset{\circ}{H}_0(\operatorname{div},\Omega)$, i.e. in particular

$$E + \partial_0 \psi + \partial_0 \chi_0 - \delta \otimes (\psi_0 + \chi_{0,0}) \in H_{\nu,-1} \otimes H_0(\operatorname{curl},\Omega) . \tag{41}$$

Since $\mathcal{H}_0 \perp H_0(\operatorname{curl},\Omega)$ and $E \in \mathcal{H}_0$

$$E + \partial_0 \psi + \partial_0 (\chi_0 - h \otimes + \chi_{0,0}) - \delta \otimes \psi_0 = 0 , \tag{42}$$

in the sense of $H_{\nu,-1} \otimes L_2(\Omega)$.

Thus, we will be looking for a solution

$$\psi \in H_{\nu,0} \otimes \mathcal{H}_1 ,$$

of the equation

$$\begin{aligned}
\operatorname{curl} \zeta\,(B) &= \operatorname{curl}\,(\,\zeta\,(\operatorname{curl}\,\psi + \beta_0) \\
&= -\,\sigma\,\partial_0\,\psi - \sigma\,\partial_0\,(\,\chi_0 - h \otimes \chi_{0,0}\,) + \sigma\,\delta \otimes \psi_0 \\
&= \sigma\,E\,.
\end{aligned}$$

This is an evolution equation of parabolic type for ψ (involving a monotone operator), which can be solved along essentially known lines of reasoning.

For this it is important to realize that

$$N\,\zeta\,(\,\cdot + \beta_0\,)\,:\,H_{\nu,0} \otimes \mathcal{H}_0 \longrightarrow H_{\nu,0} \otimes \mathcal{H}_0$$

remains a monotone, Lipschitz continuous operator.

The matching initial condition is given by

$$\psi\,(0+) = \psi_0 \equiv \operatorname{curl}^{-1}\,(\,B_0 - \beta_0\,(0+)\,) \in \mathcal{H}_1\,,$$

and is well-defined, since

$$B_0 - \beta_0\,(0+) \in \mathcal{H}_0\,. \qquad \text{(by compatibility condition)}$$

The desired existence result now follows from a corresponding result for the parabolic evolution problem:

Problem (PB):

$$A\,(\psi) + \partial_0\,\psi = j + \delta \otimes \psi_0\,, \tag{43}$$

where A is monotone (coercive would be sufficient) and Lipschitz continuous mapping from $H_{\nu,0} \otimes \mathcal{H}_1$ to $H_{\nu,0} \otimes \mathcal{H}_{-1}$, $\psi_0 \in \mathcal{H}_0$, $j \in H_{\nu,0} \otimes \mathcal{H}_{-1}$.

Theorem 2. *Problem (PB) has a unique solution $\psi \in H_{\nu,0} \otimes \mathcal{H}_1$. Moreover, this solution satisfies*

$$\psi - h \otimes \psi_0 \in H_{\nu,1} \otimes \mathcal{H}_{-1}\,,$$

and consequently

$$\psi\,(0+) = \psi_0 \ \text{in} \quad \mathcal{H}_{-1}\,.$$

We remark that the initial condition for ψ makes sense because $\psi \in C_0\,([0,\infty), \mathcal{H}_{-1})$ (by a Sobolev type imbedding theorem). Let us postpone the development of the solution theory for the parabolic equation for a while (see Section 3). Assuming this existence result for now we find the solution of the original problem by specifying $A(\varphi) := \sigma^{-1} \operatorname{curl} \zeta(\operatorname{curl} \varphi + \beta_0)$ for $\varphi \in \mathcal{H}_1$ and

$$\beta_0 := \sum_{\beta \in \mathcal{B}} \omega_\beta \otimes \beta \in H_{\nu,1} \otimes \mathcal{H}_0\,, \tag{44}$$

$$\psi_0 := \operatorname{curl}^{-1}(B_0 - \beta_0(0+))\,, \tag{45}$$

$$\chi_0 := \operatorname{curl}^{-1} \beta_0 \in H_{\nu,1} \otimes \mathcal{H}_1\,, \tag{46}$$

$$\chi_{0,0} := \operatorname{curl}^{-1} \beta_0(0+) \in \mathcal{H}_0\,, \tag{47}$$

$$j := -\partial_0(\chi_0 - h \otimes \chi_{0,0}) \in H_{\nu,0} \otimes \mathcal{H}_0\,, \tag{48}$$

With the solution ψ we define

$$\begin{aligned} E &:= -\partial_0 \psi - \partial_0(\chi_0 - h \otimes \chi_{0,0}) + \delta \otimes \psi_0 \in H_{\nu,-1} \otimes \mathcal{H}_1 \\ B &:= \operatorname{curl} \psi + \beta_0 \in H_{\nu,0} \otimes \overset{\circ}{H}_0(\operatorname{div}, \Omega)\,. \end{aligned} \tag{49}$$

This is the desired solution of problem (**MX**). We see that indeed

$$H := \zeta(B) \in H_{\nu,0} \otimes L_2(\Omega)\,, \tag{50}$$

$$\operatorname{curl} H = \sigma E \in H_{\nu,-1} \otimes \overset{\circ}{H}_0(\operatorname{div}, \Omega)\,, \tag{51}$$

$$(B \,|\, \varphi \otimes \beta) = (\omega_\beta \,|\, \varphi) \quad \text{for} \quad \beta \in \mathcal{B}\,,\ \varphi \in H_{\nu,0}\,, \tag{52}$$

$$\operatorname{curl} E + \partial_0 B = \delta \otimes B_0 \quad \text{in} \ \ H_{\nu,-1} \otimes \overset{\circ}{H}_0(\operatorname{div}, \Omega)\,. \tag{53}$$

Moreover, we have

$$E \in H_{\nu,0} \otimes \mathcal{H}_{-1}\,, \tag{54}$$

and

$$B - h \otimes B_0 = \partial_0^{-1} \operatorname{curl} E \in H_{\nu,1} \otimes \mathcal{H}_{-2}\,, \tag{55}$$

and so

$$B(0+) - B_0 = 0 \quad \text{in} \quad \mathcal{H}_{-2} \tag{56}$$

3. Solution theory of problem (PB). We shall now give the postponed proof of the above Theorem 2 about the parabolic problem to which the original question could be reduced.

Let us first consider the following evolution problem:

$$(\partial_0 + A)\,(u) = \delta \otimes u_0 + f\,, \tag{57}$$

here u_0 represents the so-called initial data for the solution u, $A : H_{\nu,0} \otimes H_1 \longrightarrow H_{\nu,0} \otimes H_{-1}$, $\nu > 0$, (which we may illustrate e.g. by the bounded operator induced by a bounded operator (denoted again by A) $A : H_1 \longrightarrow H_{-1}$ mapping a Hilbert space H_1 into its dual space H_{-1}) is assumed to be coercive, in the sense that (with ε_0, $\gamma_0 > 0$)

$$\operatorname{Re}\,(\,\varphi - \psi\,|\,A\,(\varphi) - A\,(\psi)\,)_{\nu,0,0} \geq \varepsilon_0\,\|\,\varphi - \psi\,\|^2_{\nu,0,1} - \gamma_0\,\|\,\varphi - \psi\,\|^2_{\nu,0,0}\,,$$

and we also assume

$$\|\,A\,(\varphi) - A\,(\psi)\|_{\nu,0,-1} \leq M\,\|\,\varphi - \psi\,\|_{\nu,0,1}\,.$$

Here H_1, H_0, H_{-1} refer to the Hilbert spaces of a Gelfand triple $H_1 \subset H_0 \subset H_{-1}$. We identify the inner product of H_0 with the duality pairing between H_1 and H_{-1}. We assume $u_0 \in H_0$, $f \in H_{\nu,0} \otimes H_{-1}$ and expect by analogy to the linear case (i.e. A linear) $u \in H_{\nu,0} \otimes H_1 \cap H_{\nu,1} \otimes H_{-1}$. It is natural to assume that f, $A\,(\varphi)$ vanish on $\mathbb{R}^-$ for any $\varphi \in H_{\nu,0} \otimes H_1$ and we shall do so in the following. It will be clear from the existence part of the proof that in this case also the solution u will vanish on $\mathbb{R}^-$.

The initial equation

$$(\,\partial_0 + A\,)\,(u) = \delta \otimes u_0 + f\,,$$

may be considered as holding true (term by term) in the space

$$H_{\nu,-1} \otimes H_{-1} \subset H_{\nu,-1} \otimes H_1 + H_{\nu,0} \otimes H_{-1} + H_{\nu,-1} \otimes H_0 + H_{\nu,0} \otimes H_{-1}\,.$$

Remark 3. In this framework the initial condition needs to be considered in a suitably generalized sense.

A sufficiently weak way of looking at the initial condition is given by observing that (57) implies that

$$u - h \otimes u_0 \in H_{\nu,1} \otimes H_{-1}\,. \tag{58}$$

By the trace theorem and since u vanishes on $\mathbb{R}^-$ we have

$$(\,u - h \otimes u_0\,)\,(0) = 0 \text{ in } H_{-1}\,,$$

or

$$u\,(0+) = u_0 \text{ in } H_{-1}\,. \tag{59}$$

Assuming, as we shall do initially, that $u_0 \in H_1$, the evolution problem (57) simplifies to

$$(\partial_0 + A)(v) = f\,, \tag{60}$$

where $v = u - h \otimes u_0$, $\tilde{A}(\varphi) = A(\varphi + h \otimes u_0)$. We will write, for simplicity of notation, again A in place of $\tilde{A}$.

3.1. **Uniqueness.** Let now $u, v \in H_{\nu,0} \otimes H_1$ be two solutions, then we have

$$(\varphi \,|\, \partial_0 (u - v))_{\nu,0,0} + (\varphi \,|\, A(u) - A(v))_{\nu,0,0} = 0\,, \tag{61}$$

for all $\varphi \in H_{\nu,1} \otimes H_1$. Lacking regularity prevents us from substituting $(u - v)$ for φ.

However, recalling that ∂_0 is a normal operator and utilizing the spectral family $(\Pi(D_\nu\,;\lambda))_{\lambda \in \mathbb{R}}$ of the selfadjoint realization of the operator $D_\nu = \frac{1}{i}(\partial_0 - \nu) = \mathrm{Im}\,(\partial_0)$, we may regularize $(u - v)$. Let

$$\varphi \equiv \varphi_n = \Pi(D_\nu\,;(-n, n])(u - v) = u_n - v_n\,,$$

then

$$(\varphi_n \,|\, \partial_0 \varphi_n)_{\nu,0,0} + (\varphi_n \,|\, A(u) - A(v))_{\nu,0,0} = 0\,, \tag{62}$$

or

$$\begin{aligned}(\varphi_n \,|\, \partial_0 \varphi_n)_{\nu,0,0} &+ (\varphi_n \,|\, A(u_n) - A(v_n))_{\nu,0,0} \\ &= -(\varphi_n \,|\, A(u) - A(u_n))_{\nu,0,0} - (\varphi_n \,|\, A(v_n) - A(v))_{\nu,0,0}\,.\end{aligned} \tag{63}$$

From this we see that

$$\begin{aligned}(\varphi_n \,|\, \partial_0 \varphi_n)_{\nu,0,0} &+ (\varphi_n \,|\, A(u_n) - A(v_n))_{\nu,0,0} \\ &= O(\|\varphi_n\|_{\nu,0,1}(\|u - u_n\|_{\nu,0,1} + \|v - v_n\|_{\nu,0,1}))\,.\end{aligned} \tag{64}$$

Taking real parts on both sides we see

$$(\nu - \gamma_0)\|\varphi_n\|^2_{\nu,0,0} + \varepsilon_0 \|\varphi_n\|^2_{\nu,0,1} = o(\|\varphi_n\|_{\nu,0,1})\,,$$

as $n \to \infty$. Consequently,

$$(\nu - \gamma_0)\|\varphi_n\|^2_{\nu,0,0} + \varepsilon_0 \|\varphi_n\|^2_{\nu,0,1} = o(1)\,,$$

as $n \to \infty$. Since $\varphi_n \longrightarrow (u - v)$ as $n \to \infty$, this implies, for $\nu > \gamma_0$, $u \equiv v$.

3.2. **Existence.** As a starting point for our existence proof we approximate A by bounded operators to construct approximate solutions. For this purpose we use a family $(P_n)_n$ of finite–dimensional mappings defined by

$$P_n\,\varphi := \sum_{k=1}^{n} (\psi_k\,|\,\varphi)_{\nu,0,0}\,\psi_k\,,$$

where $(\psi_k)_k$ denotes an H_0-orthonormal system spanning $R\,(P_n)$, satisfying

$$R\,(P_n) \subset R\,(P_{n+1}) \subset H_1\,, \quad n \in \mathbb{Z}^+\,.$$

In other words, $(P_n)_n$ is a family of continuous extensions to H_{-1} of H_0–orthogonal projections. We assume $P_n \longrightarrow I$ in H_0 strongly and it follows $P_n \longrightarrow I$ in H_{-1} strongly by the density of H_1 in H_0. Writing P_n for $I \otimes P_n$ and letting

$$\begin{aligned} A_n &= P_n\,A\Big|_{R\,(P_n} \\ f_n &= P_n\,f\,, \end{aligned}$$

problem (57) will be replaced by

$$(\,\partial_0 + A_n\,)\,(u_n) = f_n\,. \tag{65}$$

Considering (65) in the form

$$\partial_0\,u_n = -\,A_n\,(u_n) + f_n\,, \tag{66}$$

we see that a solution $u_n \in H_{\gamma,1} \otimes H_1$ exists uniquely for $\gamma > 0$ sufficiently large. For this we note that the right-hand side of (66) is Lipschitz continuous from $R\,(P_n) \subset H_{\gamma,0} \otimes H_1$ into $R\,(P_n)$ and $\|\,\partial_0^{-1}\,\|_{\gamma,0,1} \leq \frac{1}{\gamma}$ for all $\gamma > 0$. So that for sufficiently large $\gamma > 0$

$$\partial_0^{-1}\,\{\,-\,A_n\,(\,\cdot\,) + f_n\,\}$$

is a contraction from $R\,(P_n) \subset H_{\gamma,0} \otimes H_1$ into itself. Clearly, $u_n = P_n\,u_n \in H_{\gamma,1} \otimes H_1$. We note that $u_n(t) \equiv 0$ for $t < 0$ and that by Sobolev's imbedding theorem u_n may be considered as a continuous mapping from $\mathbb{R}$ to H_0. Moreover, according to the above remark we have $u_n\,(0) = 0$. Unfortunately, constructing approximate solutions this way $\gamma > 0$ will have to be quite large as n increases. To see that the sequence $(u_n)_n$ (or initially rather a subsequence) converges to a solution u of (57) in some $H_{\nu,0} \otimes H_1$, with a fixed $\nu > 0$, we need a–priori estimates in suitable spaces and compactness properties to select suitably convergent subsequences.

Let

$$\psi_\varepsilon := \partial_0^{-1}\,(\chi_{[-1\,/\,\varepsilon\,,\,1-1\,/\,\varepsilon]} - \chi_{[1\,/\,\varepsilon\,,\,1+1\,/\,\varepsilon]}\,) \in H_{\nu,1}\,,$$

for $0 < \varepsilon < 1$, then $\psi_\varepsilon\,u_n \in H_{\nu,1} \otimes H_1$.

Since $A_n(u_n) = P_n A(u_n)$, we obtain initially

$$(\psi_\varepsilon u_n \mid \partial_0 u_n)_{\nu,0,0} + (\psi_\varepsilon u_n \mid A(u_n))_{\nu,0,0} = (\psi_\varepsilon u_n \mid f)_{\nu,0,0}.$$

Taking real parts on both sides, this yields

$$\nu(\psi_\varepsilon u_n \mid u_n)_{\nu,0,0} + (\chi_{[1/\varepsilon,1+1/\varepsilon]} u_n \mid u_n)_{\nu,0,0} + \mathrm{Re}\,(\psi_\varepsilon u_n \mid A(u_n))_{\nu,0,0} = -\,\mathrm{Re}\,(\psi_\varepsilon u_n \mid f)_{\nu,0,0}.$$

Now we see from the coercivity of A that

$$\begin{aligned}\mathrm{Re}\,(\psi_\varepsilon u_n \mid A(u_n))_{\nu,0,0} &= \mathrm{Re}\,(\psi_\varepsilon u_n - 0 \mid A(u_n) - A(0))_{\nu,0,0} \\ &\quad + \mathrm{Re}\,(\psi_\varepsilon u_n \mid A(0))_{\nu,0,0}, \\ &\geq \varepsilon_0 \| \sqrt{\psi_\varepsilon} u_n \|_{\nu,0,1}^2 - \gamma_0 \| \sqrt{\psi_\varepsilon} u_n \|_{\nu,0,0}^2 \\ &\quad - \| \sqrt{\psi_\varepsilon} u_n \|_{\nu,0,1} \| A(0) \|_{\nu,0,-1}.\end{aligned}$$

Thus we obtain

$$\begin{aligned}&(\nu - \gamma_0) \| \sqrt{\psi_\varepsilon} u_n \|_{\nu,0,0}^2 + \| \chi_{[1/\varepsilon,1+1/\varepsilon]} u_n \|_{\nu,0,0}^2 + \varepsilon_0 \| \sqrt{\psi_\varepsilon} u_n \|_{\nu,0,1}^2 \\ &\quad \leq \| \sqrt{\psi_\varepsilon} u_n \|_{\nu,0,1} \| A(0) \|_{\nu,0,-1} - \mathrm{Re}\,(\psi_\varepsilon u_n \mid f)_{\nu,0,0}, \\ &\quad \leq \| \sqrt{\psi_\varepsilon} u_n \|_{\nu,0,1} \| A(0) \|_{\nu,0,-1} + \| \sqrt{\psi_\varepsilon} u_n \|_{\nu,0,1} \| f \|_{\nu,0,-1}.\end{aligned}$$

Consequently,

$$4(\nu - \gamma_0) \| \sqrt{\psi_\varepsilon} u_n \|_{\nu,0,0}^2 + \varepsilon_0 \| \sqrt{\psi_\varepsilon} u_n \|_{\nu,0,1}^2 \leq \frac{1}{\varepsilon_0} (2 \| A(0) \|_{\nu,0,-1}^2 + \| f \|_{\nu,0,-1}^2),$$

for all $\varepsilon \in]0,1[$, $n \in \mathbb{Z}^+$. Letting now $\varepsilon \to 0+$ we see that

$$4(\nu - \gamma_0) \| u_n \|_{\nu,0,0}^2 + \varepsilon_0 \| u_n \|_{\nu,0,1}^2 \leq \frac{1}{\varepsilon_0} (2 \| A(0) \|_{\nu,0,-1}^2 + \| f \|_{\nu,0,-1}^2),$$

for all $n \in \mathbb{Z}^+$. Since

$$\begin{aligned}\| A(u_n) \|_{\nu,0,-1} &\leq \| A(u_n) - A(0) \|_{\nu,0,-1} + \| A(0) \|_{\nu,0,-1} \\ &\leq M \| u_n \|_{\nu,0,1} + \| A(0) \|_{\nu,0,-1},\end{aligned}$$

we obtain also the boundedness of $(A(u_n))_n$ in $H_{\nu,0} \otimes H_{-1}$. Therefore we may select a weakly convergent sub–sequence which we shall denote again by $(u_n)_n$ with

$$u_n \xrightarrow{w} u_\infty, \quad A(u_n) \xrightarrow{w} \zeta_\infty,$$

in $H_{\nu,0}\otimes H_1$ and $H_{\nu,0}\otimes H_{-1}$, respectively, as $n\to\infty$. Note that here '$\xrightarrow{w}$' stands for weak convergence. Thus, by letting $n\to\infty$ we obtain

$$(\,\partial_0^*\,\psi\,|\,u_\infty\,)_{\nu,0,0}+(\,\psi\,|\,\zeta_\infty\,)_{\nu,0,0}=(\,\psi\,|\,f\,)_{\nu,0,0}\,,$$

for all $\psi\in H_{\nu,1}\otimes H_1$. This shows that

$$\partial_0\,u_\infty+\zeta_\infty=f\,. \tag{67}$$

Noting that $u_\infty\in H_{\nu,0}\otimes H_1$, we are finished with the proof of our existence result if we can show that $A\,(u_\infty)=\zeta_\infty$.

This follows by the standard variational argument (see e.g. [2]). We first recall the coerciveness of A i.e.

$$\mathrm{Re}\;((u_n-w)\,|\,A(u_n)-A(w))_{\nu,0,0}+\gamma_0\,\|\,u_n-w\,\|^2_{\nu,0,0}\geq\varepsilon_0\,\|\,u_n-w\,\|^2_{\nu,0,1}\geq 0\,,$$

for any $w\in H_{\nu,1}\otimes H_1$. On the other hand, from the definition of the solution $u_n\in H_{\nu,1}\otimes H_1$ we have (for all $w\in R\,(P_n)$)

$$(\,u_n-w\,|\,\partial_0\,u_n\,)_{\nu,0,0}+(u_n-w\,|\,A\,(u_n)\,)_{\nu,0,0}=(\,u_n-w\,|\,f\,)_{\nu,0,0}\,.$$

Taking real parts this becomes

$$\begin{aligned}\nu\,(\,u_n\,|\,u_n\,)_{\nu,0,0}+\mathrm{Re}\;(\,\partial_0^*\,w\,|\,u_n\,)_{\nu,0,0}+\mathrm{Re}\;(\,u_n-w\,|\,A\,(u_n)\,)_{\nu,0,0}&\\ =\mathrm{Re}\;(\,u_n-w\,|\,f\,)_{\nu,0,0}\,.&\end{aligned}$$

Letting $n\to\infty$, we obtain, observing

$$(\,u_\infty\,|\,u_\infty\,)_{\nu,0,0}\leq\lim_{n\to\infty}\sup(\,u_n\,|\,u_n\,)_{\nu,0,0}\,,$$

that (for $\nu>\gamma_0$)

$$\begin{aligned}&\nu\,(u_\infty\,|\,u_\infty)_{\nu,0,0}+\mathrm{Re}\;(\partial_0^*\,w\,|\,u_\infty)_{\nu,0,0}-\gamma_0\,\|\,u_\infty-w\,\|^2_{\nu,0,0}\\&\quad+\lim_{n\to\infty}\sup(\mathrm{Re}\;(\,u_n-w\,|\,A\,(u_n)-A\,(w)\,)_{\nu,0,0}+\gamma_0\,\|\,u_n-w\,\|^2_{\nu,0,0}\,)\\&\leq\mathrm{Re}\;(\,u_\infty-w\,|\,f\,)_{\nu,0,0}-\mathrm{Re}\;(\,u_\infty-w\,|\,A\,(w)\,)_{\nu,0,0}\,,\end{aligned}$$

and so

$$\begin{aligned}&\nu\,(\,u_\infty\,|\,u_\infty\,)_{\nu,0,0}+\mathrm{Re}\;(\,\partial_0^*\,w\,|\,u_\infty\,)_{\nu,0,0}-\gamma_0\,\|\,u_\infty-w\,\|^2_{\nu,0,0}\\&\leq\mathrm{Re}\;(\,u_\infty-w\,|\,f\,)_{\nu,0,0}-\mathrm{Re}\;(\,u_\infty-w\,|\,A\,(w)\,)_{\nu,0,0}\,.\end{aligned} \tag{68}$$

On the other hand we have from (67)

$$(\,\partial_0^*\,(\,u_{\infty,n}-w\,)\,|\,u_\infty\,)_{\nu,0,0}+(\,u_{\infty,n}-w\,|\,\zeta_\infty\,)_{\nu,0,0}=(\,u_{\infty,n}-w\,|\,f\,)_{\nu,0,0}\,.$$

Taking real parts here yields

$$\begin{aligned}\nu\,(\,u_{\infty,n}\,|\,u_{\infty,n}\,)_{\nu,0,0}+\mathrm{Re}\;(\,\partial_0^*\,w\,|\,u_\infty\,)_{\nu,0,0}+\mathrm{Re}\;(u_{\infty,n}-w\,|\,\zeta_\infty\,)_{\nu,0,0}&\\ =\mathrm{Re}\;(u_{\infty,n}-w\,|\,f\,)_{\nu,0,0}\,.&\end{aligned}$$

For $n \to \infty$ this leads to

$$\nu\,(\,u_\infty\,|\,u_\infty\,)_{\nu,0,0} + \mathrm{Re}\,(\,\partial_0^*\,w\,|\,u_\infty\,)_{\nu,0,0} + \mathrm{Re}\,(\,u_\infty - w\,|\,\zeta_\infty\,)_{\nu,0,0} = \mathrm{Re}\,(\,u_\infty - w\,|\,f\,)_{\nu,0,0}\,.$$

Subtracting this from (68) yields

$$\mathrm{Re}\,(u_\infty - w\,|\,\zeta_\infty - A\,(w)\,)_{\nu,0,0} + \gamma_0\,\|\,u_\infty - w\,\|^2_{\nu,0,0} \geq 0\,, \tag{69}$$

for all $w \in \bigcup_{n=1}^{\infty} R\,(P_n) \cap H_{\nu,1} \otimes H_1$ and also, by density, for all $w \in H_{\nu,0} \otimes H_1$.

Let now $w = u_\infty - \lambda\,\varphi$, $\lambda \in \mathbb{R}^+$, $\varphi \in H_{\nu,0} \otimes H_1$. After dividing by λ and letting $\lambda \to 0+$ we obtain from (69)

$$\mathrm{Re}\,(\varphi\,|\,\zeta_\infty - A\,(u_\infty)\,)_{\nu,0,0}\,) \geq 0\,. \tag{70}$$

Since φ is arbitrary, this implies, as desired,

$$\zeta_\infty = A\,(u_\infty)\,.$$

Remark 4. We have in particular

$$\mathrm{Re}\,(\,(\,\partial_0 + A\,)\,(u) - (\,\partial_0 + A\,)\,(v)\,|\,u - v\,)_{\nu,0,0} \geq \varepsilon_0\,\|\,u - v\,\|^2_{\nu,0,1} - (\,\gamma_0 - \nu\,)\,\|\,u - v\,\|^2_{\nu,0,0}$$

for $u,\ v \in H_{\nu,0} \otimes H_1 \cap H_{\nu,1} \otimes H_{-1}$.

Remark 5. (Non–linear right–hand side) From the above we see that

$$(\partial_0 + A)^{-1}\ :\ H_{\nu,0} \otimes H_{-1} \longrightarrow H_{\nu,0} \otimes H_1\,, \tag{71}$$

is a well-defined operator. Moreover, we have for $\nu > \gamma_0$

$$\|\,(\,\partial_0 + A\,)^{-1}\,(\varphi) - (\,\partial_0 + A\,)^{-1}\,(\psi)\,\|_{\nu,0,0} \leq \frac{1}{\nu - \gamma_0}\,\|\,\varphi - \psi\,\|_{\nu,0,0}\,, \tag{72}$$

and

$$\|\,(\,\partial_0 + A\,)^{-1}\,(\varphi) - (\,\partial_0 + A\,)^{-1}\,(\psi)\,\|_{\nu,0,1} \leq \frac{1}{\min(\,\varepsilon_0\,,\ \nu - \gamma_0\,)}\,\|\,\varphi - \psi\,\|_{\nu,0,-1}\,. \tag{73}$$

In particular, $(\,\partial_0 + A\,)^{-1}\ :\ H_{\nu,0} \otimes H_0 \longrightarrow H_{\nu,0} \otimes H_0$ is a contraction for $\nu > 1 + \gamma_0$. This can be seen by similar estimates as above. Consider two solutions u_∞, v_∞ found in the above fashion for $f,\ g \in H_{\nu,0} \otimes H_0 \subset H_{\nu,0} \otimes H_{-1}$,

$$\begin{aligned}
(\,w\,|\,\partial_0\,u_\infty\,)_{\nu,0,0} + (\,w\,|\,A\,(u_\infty)\,)_{\nu,0,0} &= (\,w\,|\,f\,)_{\nu,0,0}\,,\\
(\,w\,|\,\partial_0\,v_\infty\,)_{\nu,0,0} + (\,w\,|\,A\,(v_\infty)\,)_{\nu,0,0} &= (\,w\,|\,g\,)_{\nu,0,0}\,.
\end{aligned}$$

Subtracting yields

$$(w \mid \partial_0 (u_\infty - v_\infty))_{\nu,0,0} + (w \mid A (u_\infty) - A (v_\infty))_{\nu,0,0} = (w \mid f - g)_{\nu,0,0} ,$$

for arbitrary $w \in H_{\nu,0} \otimes H_1$. Letting $w = u_\infty - v_\infty$, taking real parts and estimating (by using coerciveness again) we obtain

$$(\nu - \gamma_0) \| u_\infty - v_\infty \|^2_{\nu,0,0} + \varepsilon_0 \| u_\infty - v_\infty \|^2_{\nu,0,1} \leq \| u_\infty - v_\infty \|_{\nu,0,0} \| f - g \|_{\nu,0,0} ,$$

and so

$$(\nu - \gamma_0) \| u_\infty - v_\infty \|_{\nu,0,0} \leq \| f - g \|_{\nu,0,0} .$$

The latter implies the above contraction property.

This contraction estimate implies that if the right-hand side of the evolution problem is replaced by a Lipschitz continuous mapping $f : H_{\nu,0} \otimes H_0 \longrightarrow H_{\nu,0} \otimes H_0$ with Lipschitz constant Λ, then

$$(\partial_0 + A)^{-1} \circ f : H_{\nu,0} \otimes H_0 \longrightarrow H_{\nu,0} \otimes H_0 ,$$

is a contraction for $\nu > \Lambda + \gamma_0$. From this the unique existence of a solution in $H_{\nu,0} \otimes H_1$ is immediate also in this case, since $(\partial_0 + A)^{-1}$ maps into $H_{\nu,0} \otimes H_1$.

3.3. **Continuous dependence.** The continuous dependence of the solution in $H_{\nu,0} \otimes H_1$ from the right–hand side is also apparent from the estimate (73). The solution of the original problem (57) is (written in the original notation) given by

$$(\partial_0 + \tilde{A})^{-1} (f) + h \otimes u_0 \in H_{\nu,0} \otimes H_0 ,$$

which depends continuously upon the right–hand side $f \in H_{\nu,0} \otimes H_0$ in the sense of $H_{\nu,0} \otimes H_1$ provided the initial data satisfy $u_0 \in H_1$. This result can be refined in the following way. Let $u,\ v \in H_{\nu,0} \otimes H_1$ be the uniquely existing solution of

$$(\partial_0 + A) (u) = \delta \otimes u_0 + f , \tag{74}$$

$$(\partial_0 + A) (v) = \delta \otimes v_0 + g , \tag{75}$$

respectively. Here $f,\ g \in H_{\nu,0} \otimes H_{-1}$ and $u_0,\ v_0 \in H_1$. Subtracting (74), (75) and multiplying by an arbitrary $\varphi \in H_{\nu,1} \otimes H_1$ we obtain

$$\begin{aligned} &(\partial_0^* \varphi \mid u - v)_{\nu,0,0} + (\varphi \mid A (u) - A (v))_{\nu,0,0} \\ &\qquad = (\varphi(0) \mid u_0 - v_0)_0 + (\varphi \mid f - g)_{\nu,0,0} . \end{aligned}$$

Taking $\varphi \equiv \varphi_n = \Pi\,(D_\nu\,;\,]-n,n])\,(u-v) \in H_{\nu,1}\otimes H_1$, we obtain

$$\begin{aligned}(\,\partial_0^*\,\varphi_n\,|\,\varphi_n\,)_{\nu,0,0} + (\,\varphi_n\,|\,A\,(u) - A\,(v)\,)_{\nu,0,0}\\ = (\,\varphi_n(0)\,|\,u_0 - v_0\,)_0 + (\,\varphi_n\,|\,f - g\,)_{\nu,0,0}\,.\end{aligned}$$

With $u_n = \Pi\,(D_\nu\,;\,]-n,n])\,u$ and $v_n = \Pi\,(D_\nu\,;\,]-n,n])\,v$ this yields

$$\begin{aligned}&(\,\partial_0^*\,\varphi_n\,|\,\varphi_n\,)_{\nu,0,0} + (\,u_n - v_n\,|\,A\,(u_n) - A\,(v_n)\,)_{\nu,0,0}\\ &= (\,\varphi_n(0)\,|\,u_0 - v_0\,)_0 + (\,\varphi_n\,|\,f - g\,)_{\nu,0,0} - (\,\varphi_n\,|\,A\,(u) - A\,(u_n)\,)_{\nu,0,0}\\ &\quad + (\,\varphi_n\,|\,A\,(v) - A\,(v_n)\,)_{\nu,0,0}\,.\end{aligned}$$

Focusing on real parts this yields

$$\begin{aligned}&\nu\,(\varphi_n\,|\,\varphi_n\,)_{\nu,0,0} + \mathrm{Re}\,(u_n - v_n\,|\,A\,(u_n) - A\,(v_n)\,)_{\nu,0,0}\\ &= \mathrm{Re}\,(\,\varphi_n(0)\,|\,u_0 - v_0\,)_0\; - \mathrm{Re}\,(\,\varphi_n\,|\,f - g\,)_{\nu,0,0}\\ &\quad - \mathrm{Re}\,(\,\varphi_n\,|\,A\,(u) - A\,(u_n)\,)_{\nu,0,0} + \mathrm{Re}\,(\,\varphi_n\,|\,A\,(v) - A\,(v_n)\,)_{\nu,0,0}\,.\end{aligned}$$

Using coerciveness and Lipschitz continuity of A we estimate

$$\begin{aligned}&(\nu - \gamma_0)\,\|\,\varphi_n\,\|^2_{\nu,0,0} + \varepsilon_0\,\|\,\varphi_n\,\|^2_{\nu,0,1} \le \mathrm{Re}\,(\varphi_n(0)\,|\,u_0 - v_0\,)_0\\ &+ \|\,\varphi_n\,\|_{\nu,0,1}\,\|\,f - g\,\|_{\nu,0,-1} + M\,\|\,\varphi_n\,\|_{\nu,0,1}\,(\|\,u - u_n\,\|_{\nu,0,1} + \|\,v - v_n\,\|_{\nu,0,1})\,.\end{aligned}$$

Letting $n \to \infty$ this leads to

$$\begin{aligned}(\nu - \gamma_0\,)\,\|\,u - v\,\|^2_{\nu,0,0} + \frac{\varepsilon_0}{2}\,\|\,u - v\,\|^2_{\nu,0,1}\\ \le \lim_{n\to\infty} \mathrm{Re}\,(\,\varphi_n(0)\,|\,u_0 - v_0\,)_0 + \frac{1}{2\,\varepsilon_0}\,\|\,f - g\,\|^2_{\nu,0,-1}\,.\end{aligned} \tag{76}$$

We shall now show that

$$\lim_{n\to\infty} \varphi_n(0) = \frac{1}{2}\,(u_0 - v_0) \text{ in } H_{-1}\,.$$

By the argument leading up to (59) we see first that

$$\begin{aligned}\lim_{n\to\infty} \varphi_n(0) &= \lim_{n\to\infty} \Pi\,(D_\nu\,;\,]-n,n])\,(\,(u-v) - h\otimes(u_0 - v_0)\,)\,(0)\\ &\quad + \lim_{n\to\infty}(\,\Pi\,(D_\nu\,;\,]-n,n])\,h\,)\,(0)\,(u_0 - v_0)\,,\\ &= \lim_{n\to\infty}(\,\Pi\,(D_\nu\,;\,]-n,n])\,h\,)\,(0)\,(u_0 - v_0)\,.\end{aligned}$$

We calculate explicitly that

$$\lim_{n\to\infty}(\,\Pi\,(D_\nu\,;\,]-n,n])\,h\,)\,(0) = \frac{1}{2}\,.$$

Thus, we get from (76)

$$(77)\quad (\nu-\gamma_0)\,\|\,u-v\,\|^2_{\nu,0,0}+\frac{\varepsilon_0}{2}\,\|\,u-v\,\|^2_{\nu,0,1} \leq \frac{1}{2}\,\|\,u_0-v_0\,\|_0^2+\frac{1}{2\,\varepsilon_0}\,\|\,f-g\,\|^2_{\nu,0,-1}\,.$$

This shows not only the Lipschitz continuous dependence of the solution from the data, but also that we may extend by continuous extension to initial data which are merely in H_0. This leads to the following final result:

Theorem 3. *The mapping* (∂_0+A) *is invertible as an operator in* $H_{\nu,-1}\otimes H_{-1}$ *with domain* $H_{\nu,0}\otimes H_1$*. In particular,*

$$(\,\partial_0+A\,)^{-1}\,:\,\delta\otimes H_0+H_{\nu,0}\otimes H_{-1}\longrightarrow H_{\nu,0}\otimes H_1\,,$$

is the well-defined solution operator of problem (57). Moreover, $(\,\partial_0+A\,)^{-1}$ *is Lipschitz continuous in the sense that*

$$(78)\quad \|\,(\partial_0+A)^{-1}\,(\delta\otimes u_0+f)-(\partial_0+A)^{-1}\,(\delta\otimes u_0+g)\,\|^2_{\nu,0,1} \leq \frac{1}{\varepsilon_0}\,\|\,u_0-v_0\,\|_0^2+\frac{1}{\varepsilon_0}\,\|\,f-g\,\|^2_{\nu,0,-1}\,.$$

Moreover, we have in the norm of $H_{\nu,1}\otimes H_{-1}$*:*

$$(79)\quad \|\,h\otimes(v_0-u_0)+(\partial_0+A)^{-1}\,(\delta\otimes u_0+f)-(\partial_0+A)^{-1}\,(\delta\otimes v_0+g)\,\|_{\nu,1,-1} \leq \frac{M}{\sqrt{\varepsilon_0}}\,\|\,u_0-v_0\,\|_0+(\,1+\frac{M}{\varepsilon_0}\,)\,\|\,f-g\,\|_{\nu,0,-1}\,.$$

Proof. The first part of the theorem summarizes the previous findings. The latter claim follows from (74), (75) using the initial idea employed to remove the initial condition. These equations transform into

$$(80)\quad \partial_0\,(u-h\otimes u_0)+A\,(u)\;=\;f\,,$$

$$(81)\quad \partial_0\,(v-h\otimes v_0)+A\,(v)\;=\;g\,.$$

Subtraction results in

$$\partial_0\,(\,u-h\otimes u_0-v+h\otimes v_0\,)=f-g-A\,(u)+A\,(v)\,.$$

Taking norms on both sides we obtain

$$\begin{aligned}\|\,u-h\otimes u_0-v+h\otimes v_0\,\|_{\nu,0,-1} &= \|\,\partial_0\,(u-h\otimes u_0-v+h\otimes v_0\,)\,\|_{\nu,1,-1}\\ &\leq \|\,f-g\,\|_{\nu,0,-1}+\|\,A\,(u)-A\,(v)\,\|_{\nu,0,-1}\,.\end{aligned}$$

To estimate the last term further we use the Lipschitz continuity of A, thus for $\nu > \gamma_0$ we get from (77)

$$\begin{aligned}\| u - h\otimes u_0 - v + h\otimes v_0 \|_{\nu,0,-1} &= \| \partial_0 (u - h\otimes u_0 - v + h\otimes v_0) \|_{\nu,1,-1} \\ &\leq \| f - g \|_{\nu,0,-1} + M \| u - v \|_{\nu,0,1} , \\ &\leq \frac{M}{\sqrt{\varepsilon_0}} \| u_0 - v_0 \|_0 + (1 + \frac{M}{\varepsilon_0}) \| f - g \|_{\nu,0,-1} .\end{aligned}$$

This is the desired estimate (79). $\square$

4. Continuous dependence for the electromagnetic field. The solution theory developed in Section 2 not only proves the needed existence and uniqueness result utilized in the proof of Theorem 1, but also provides the means to show a continuous dependence result for the original problem. We conclude with the derivation and formulation of the continuous dependence of solutions to the original semi–static problem on its initial data.

Theorem 4. *Let E_1, B_1 and E_2, B_2 be two solutions of the original problem, corresponding to initial values $B_{0\,1}$ and $B_{0\,2}$ and data ω_{β_1} and ω_{β_2}. Let*

$$B_0 = B_{0\,1} - B_{0\,2} \, , \ \omega_\beta = \omega_{\beta_1} - \omega_{\beta_2} \, , \ B = B_1 - B_2 \, , \ E = E_1 - E_2 \, .$$

Then the following estimate holds:

$$\begin{aligned}&\| E \|^2_{\nu,0,-1} + \| E \|^2_{\nu,-1,1} + \| B \|^2_{\nu,0,0} + \| B - h\otimes B_0 \|^2_{\nu,1,-2} + \| H \|^2_{\nu,0,0} \\ &\quad \leq C_0 \Big(\sum_{\beta\in\mathcal{B}} \| \omega_\beta - h \, (B_0 \, | \, \beta \,)_0 \|^2_{\nu,1} + \| B_0 - \sum_{\beta\in\mathcal{B}} (B_0 \, | \, \beta \,)_0 \, \beta \, \|^2_{-1} \Big) ,\end{aligned}$$

for some positive constant C_0 depending only on σ, ν, L, ε_0.

Proof. Let

$$\begin{aligned} j_k &:= -\partial_0 \Big(\operatorname{curl}^{-1} \Big(\sum_{\beta\in\mathcal{B}} (\omega_{\beta_k} - \omega_{\beta_k} (0+) \,) \, \beta \Big) \Big) , \\ \psi_k &:= \operatorname{curl}^{-1} \Big(B_k - \sum_{\beta\in\mathcal{B}} \omega_{\beta_k} \, \beta \Big) = -\partial_0^{-1} E_k + h \otimes \psi_{k,0} + \partial_0^{-1} j_k \, , \quad k = 1,2 \, ,\end{aligned}$$

then

$$\psi \equiv \psi_1 - \psi_2 \;:=\; \operatorname{curl}^{-1}\Big(B - \sum_{\beta\in\mathcal{B}} \omega_\beta\, \beta\Big)$$

$$= \; -\,\partial_0^{-1} E + h \otimes \psi_0 + \partial_0^{-1} j \in H_{\nu,0} \otimes \mathcal{H}_1 \,.$$

In particular

$$\operatorname{curl}\, \psi = B - \sum_{\beta\in\mathcal{B}} \omega_\beta \otimes \beta \in H_{\nu,0} \otimes \mathcal{H}_0 \,,$$

and

$$\psi_{k,0} := \operatorname{curl}^{-1}\Big(B_{k,0} - \sum_{\beta\in\mathcal{B}} \omega_{\beta_k}\,(0+)\,\beta\Big)\,.$$

Since ψ_i, $i = 1, 2$, solves the problem (**PB**) with the above specializations and corresponding data we have for ψ a stability estimate of the form

$$\|\, \psi \,\|_{\nu,0,1}^2 \le C\,(\, \|\, \psi_0 \,\|_0^2 + \|\, j \,\|_{\nu,0,-1}^2 \,)\,, \tag{82}$$

where

$$j := j_1 - j_2 \,,$$

and

$$\psi_0 := \psi_{1,0} - \psi_{2,0} \,.$$

This implies the following estimates for (E, B).

$$\begin{aligned}\|\, B - \sum_{\beta\in\mathcal{B}} \omega_\beta \otimes \beta \,\|_{\nu,0,0}^2 &\le C\Bigg(\Big\|\operatorname{curl}^{-1}\Big(B_0 - \sum_{\beta\in\mathcal{B}} \omega_\beta\,(0+)\,\beta\Big)\Big\|_0^2 \\ &\quad + \Big\|\,\partial_0\Big(\operatorname{curl}^{-1}\Big(\sum_{\beta\in\mathcal{B}} (\,\omega_\beta - \omega_\beta\,(0+)\, h\,) \otimes \beta\Big)\Big)\,\Big\|_{\nu,0,-1}^2\Bigg) \\ &\le C\Bigg(\,\Big\|\, B_0 - \sum_{\beta\in\mathcal{B}} \omega_\beta(0+)\,\beta\Big\|_{-1}^2 + \Big\|\,\sum_{\beta\in\mathcal{B}} (\omega_\beta - \omega_\beta\,(0+)\, h) \otimes \beta\Big\|_{\nu,1,-2}^2\Bigg)\,.\end{aligned}$$

This implies (with a generic constant $C > 0$), using Sobolev's inequality, and re–utilizing the notation $\|\,\cdot\,\|_{\nu,0,0}$ as norm of $H_{\nu,0} \otimes L_2(\Omega)$,

$$\|\, B \,\|_{\nu,0,0}^2 \le C\Bigg(\,\sum_{\beta\in\mathcal{B}} \|\, \omega_\beta - \omega_\beta\,(0+)\, h \,\|_{\nu,1}^2 + \Big\|\, B_0 - \sum_{\beta\in\mathcal{B}} \omega_\beta\,(0+)\,\beta\Big\|_{-1}^2\Bigg)\,.$$

Recalling the compatibility condition we get

$$\| B \|_{\nu,0,0}^2 \le C\Bigg(\sum_{\beta\in\mathcal{B}} \| \omega_\beta - h\,(B_0 \,|\, \beta\,)_0 \|_{\nu,1}^2 \qquad (83)$$
$$+ \Big\| B_0 - \sum_{\beta\in\mathcal{B}} (B_0 \,|\, \beta_0\,)\,\beta \Big\|_{-1}^2 \Bigg).$$

Similarly we can argue for the electric field E (re-using here the notation $\| \cdot \|_{\nu,-1,1}$ as norm of $H_{\nu,-1} \otimes H(\mathrm{curl}\,,\Omega)$):

$$\begin{aligned}
\| E \|_{\nu,-1,1} &\equiv \| \partial_0^{-1} \,\mathrm{curl}\; E \|_{\nu,0,0} \\
&\le \| h \otimes \mathrm{curl}\; \psi_0 + \partial_0^{-1}\,\mathrm{curl}\; j \|_{\nu,0,0} \\
&\quad + \| - \partial_0^{-1}\,\mathrm{curl}\; E + h \otimes \mathrm{curl}\; \psi_0 + \partial_0^{-1}\,\mathrm{curl}\; j \|_{\nu,0,0} \\
&= \| h \otimes \mathrm{curl}\; \psi_0 + \partial_0^{-1}\,\mathrm{curl}\; j \|_{\nu,0,0} + \Big\| B - \sum_{\beta\in\mathcal{B}} \omega_\beta \otimes \beta \Big\|_{\nu,0,0}^2 , \\
&\le \| h \otimes \mathrm{curl}\; \psi_0 + \partial_0^{-1}\mathrm{curl}\; j \|_{\nu,0,0} \\
&\quad + C\Bigg(\sum_{\beta\in\mathcal{B}} \| \omega_\beta - h\,(B_0 \,|\, \beta\,)_0 \|_{\nu,1}^2 + \Big\| B_0 - \sum_{\beta\in\mathcal{B}} (B_0 \,|\, \beta\,)_0\, \beta \Big\|_{-1}^2 \Bigg), \\
&\le C\Bigg(\sum_{\beta\in\mathcal{B}} \| \omega_\beta - h\,(B_0 \,|\, \beta\,)_0 \|_{\nu,1}^2 + \Big\| B_0 - \sum_{\beta\in\mathcal{B}} (B_0 \,|\, \beta\,)_0\, \beta \Big\|_{-1}^2 \Bigg).
\end{aligned}$$

From $H = \zeta\,(B_1) - \zeta\,(B_2)$ we obtain the corresponding estimate for H. Since $E = \sigma^{-1}\,\mathrm{curl}\; H = \sigma^{-1}\,\mathrm{curl}\;(N\,H)$, we have $N\,H = \sigma\,\mathrm{curl}^{-1}\,E$, and so

$$\| E \|_{\nu,0,-1}^2 = \| \mathrm{curl}^{-1}\, E \|_{\nu,0,0}^2 = \| N\,H \|_{\nu,0,0}^2 \le \| H \|_{\nu,0,0}^2 \,. \qquad (84)$$

Finally, $B - h \otimes B_0 = \partial_0^{-1}\,\mathrm{curl}\; E$ leads to the remaining estimates, since

$$\| B - h \otimes B_0 \|_{\nu,1,-2}^2 = \| \partial_0^{-1}\,\mathrm{curl}\; E \|_{\nu,1,-2}^2 = \| E \|_{\nu,0,-1}^2 \,.$$

□

This concludes our presentation of the solution theory for our initial value problem modeling the magnetohydrodynamic limit case of Maxwell's equations. It should be obvious that Lipschitz continuous external forcing terms could easily be included. We have dealt with a pure initial boundary value problem just for sake of simplicity of presentation.

Appendix: A Sobolev type estimate. For completeness of reasoning let us consider finally the Sobolev estimate used in the above. We observe that for any $\varphi \in \overset{\circ}{C}_\infty\,(\,[0,1)\,,\,\mathcal{H}_0\,)$

$$
\begin{aligned}
|\,\varphi\,(0)\,| &\leq \int_0^1 |\,\partial_0\,\varphi(s)\,|\,\mathrm{d}s\,,\\
&\leq \sqrt{\int_0^1 |\,\partial_0\,\varphi(s)\,|^2\,e^{-2\,\nu\,s}\,\mathrm{d}s}\,\sqrt{\int_0^1 e^{2\,\nu\,s}\,\mathrm{d}s}\,,\\
&\leq \frac{e^\nu}{\sqrt{2\,\nu}}\sqrt{\int_0^1 |\,\partial_0\,\varphi(s)\,|^2\,e^{-2\,\nu\,s}\,\mathrm{d}s}
\end{aligned}
$$

Let now $\alpha \in \overset{\circ}{C}_\infty\,(\,\overline{\mathbb{R}^+}\,)$ with support of α contained in $[0,1)$ and $\alpha\,(0) = 1$. Apply this estimate to $\alpha\,\varphi$ with $\varphi \in \overset{\circ}{C}_\infty\,(\,[0,\infty)\,,\,\mathcal{H}_0)$ then

$$
\begin{aligned}
|\,\varphi\,(0)\,| &\leq \int_0^1 |\,(\,\partial_0\,(\alpha\,\varphi)\,)\,(s)\,|\,\mathrm{d}s\,,\\
&\leq C\,(\alpha,\nu)\sqrt{\int_0^\infty |\,\partial_0\,\varphi\,(s)\,|^2\,e^{-\,2\,\nu\,s}\,\mathrm{d}s + \int_0^\infty |\,\varphi\,(s)\,|^2\,e^{-2\,\nu\,s}\,\mathrm{d}s}\,,\\
&\leq C'\,(\alpha,\nu)\sqrt{\int_0^\infty |\,\partial_0\,\varphi\,(s)\,|^2\,e^{-\,2\,\nu\,s}\,\mathrm{d}s}\,,\\
&\leq C'\,(\alpha,\nu)\sqrt{\int_0^\infty |\,\partial_0\,(\,\varphi - \varphi\,(0)\,h\,)\,(s)\,|^2\,e^{-\,2\,\nu\,s}\,\mathrm{d}s}\,.
\end{aligned}
$$

Thus we obtain (by a density argument)

$$
|\,\varphi\,(0+)\,| \leq C'\,(\alpha,\nu)\,\|\,\varphi - \varphi_0\,h\,\|_{\nu,1}\,,
$$

for all $\varphi \in H_{\nu,0}$ with $\operatorname{supp}\varphi \subset \overline{\mathbb{R}^+}$ satisfying $\varphi - \varphi_0\,h \in H_{\nu,1}$ for some $\varphi_0 \in \mathbb{C}$, which is the desired estimate.

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